Curves

Cycloid. A cycloid is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage.

Assume the wheel is a unit circle, its rolls on the x-axis and in the upper half-plane with unit speed and the time $t = 0$ the point on the rim is in the origin. Let us find the position vector $\mathbf{f}(t)$ of the point on the rim at time t.

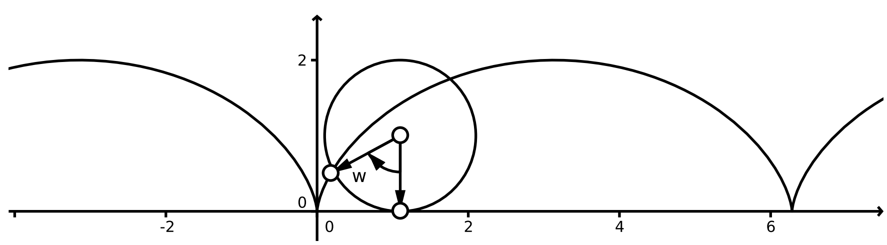
Note that coordinates of the center of the wheel at time t is $\mathbf{c}(t) = (t, 1)$.

Since there is no slippage the arc from the point $\mathbf{d}(t) = (t, 0)$ on the rim to the $\mathbf{f}(t)$ has length t. It follows that the angle the vectors $\mathbf{w}(t) = \mathbf{f}(t) - \mathbf{c}(t)$ is clockwise rotation of the vector $(0, -1) = \mathbf{d}(t) - \mathbf{c}(t)$ by angle t. Therefore

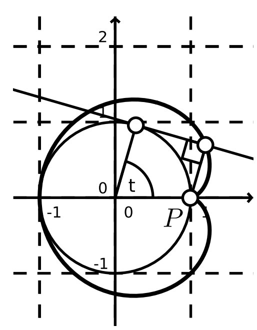
$$\mathbf{w}(t) = (-\sin t, -\cos t)$$

and

$$\mathbf{f}(t) = \mathbf{c}(t) + \mathbf{w}(t) = (t - \sin t, 1 - \cos t).$$



Pedal curve. The pedal curve is traced by the orthogonal projection of a fixed point P on the tangent lines of a given curve $\mathbf{f}(t)$.



Write a parametric expression $\mathbf{h}(t)$ for the pedal curve for the unit circle $\mathbf{f}(t) = (\cos(t), \sin t)$ and the point $P = (1, 0)$, so its position vector is $\mathbf{i}$.

Denote by $\mathbf{w}(t)$ the projection of $\mathbf{v}(t) = \mathbf{i} - \mathbf{f}(t)$ to the tangent line at $\mathbf{f}(t)$, so $\mathbf{h}(t) = \mathbf{f}(t) + \mathbf{w}(t)$.

The velocity vector $\mathbf{f}'(t) = (-\sin t, \cos t)$ is parallel to the tangent line at $\mathbf{f}(t)$. Note that $\|\mathbf{f}'(t)\| = 1$ for any

t. Therefore

$$\begin{aligned}\mathbf{w}(t) &= (\mathbf{f}'(t)\cdot\mathbf{v}(t))\mathbf{f}'(t)\\ &= (\mathbf{f}'(t)\cdot(\mathbf{i}-\mathbf{f}(t)))\mathbf{f}'(t) =\\ &= (\sin^2 t, -\sin t\cos t).\end{aligned}$$

and

$$\begin{aligned}\mathbf{h}(t) &= \mathbf{f}(t)+\mathbf{w}(t)\\ &= (\cos t+\sin^2 t, \sin t-\sin t\cos t).\end{aligned}$$

Involute. Involute is a curve obtained from another given curve by attaching an imaginary taut string to the given curve and tracing its free end as it is wound onto that given curve.

Write a parametric expression $\mathbf{h}(t)$ for the involute for the unit circle $\mathbf{f}(t) = (\cos(t), \sin t)$ starting at the point $P = (1, 0)$, so its position vector is $\mathbf{i}$.

Note that $\mathbf{f}'(t) = (-\sin t, \cos t)$ and $\|\mathbf{f}'(t)\| = 1$.

Note that the vector $\mathbf{w}(t) = \mathbf{h}(t) - \mathbf{f}(t)$ is tangent to the circle at $\mathbf{f}(t)$ and since the length of the string equals to the length of unwound we get

$$\mathbf{h}(t) - \mathbf{f}(t) = -t\mathbf{f}'(t).$$

Therefore

$$\begin{aligned}\mathbf{h}(t) &= \mathbf{f}(t) - t\mathbf{f}'(t)\\ &= (\cos t + t\sin t, \sin t - t\cos t).\end{aligned}$$