Quiz 1.

1. Let $x, y \in F$, where F is an ordered field. Suppose $0 < x < y$. Show that

$$x^2 < y^2$$

Solution: Since $x, y > 0$, multiplying $x < y$ by x and y, we get

$$x^2 < yx \quad \text{and} \quad xy < y^2.$$

Therefore

$$x^2 < y^2.$$

2. Show that the following identity holds in any field

$$(-x)y = -xy.$$

Solution: In addition to the properties of the field (Definition 1.1.5) we will use the identity $0y = 0$ which was proved already.

$$xy + (-xy) = 0 = 0y = (x + (-x))y = xy + (-x)y.$$

Therefore

$$xy + (-xy) = xy + (-x)y.$$

Adding $(-xy)$ to both sides, we get

$$(-xy) = (-x)y.$$

Quiz 2.

1. Let $S \subset \mathbb{R}$ be a nonempty set, bounded from above. Show that for every $\varepsilon > 0$ there exists $x \in S$ such that

$$\sup S - \varepsilon < x \leq \sup S.$$

Solution: The second inequality holds for any $x \in S$ since $\sup S$ is an upper bound of S.

If there is no $x \in S$ such that $\sup S - \varepsilon < x$ then $x \leq \sup S - \varepsilon$ for any $x \in S$. That is, $\sup S - \varepsilon$ is an upper bound of S. In particular $\sup S$ is not the least upper bound, a contradiction.

2. Let $A, B \subset \mathbb{R}$ be bounded nonempty sets. Assume for any $a \in A$ there is $b \in B$ such that $a \leq b$. Show that $\sup A \leq \sup B$.

Solution: Since for any $a \in A$ there is $b \in B$ such that $a \leq b$, any upper bound for B is an upper bound for A. In particular, $\sup B$ is an upper bound for A. Therefore $\sup A \leq \sup B$.

Quiz 3.

1. Let A and B be two nonempty bounded sets of real numbers. Let

$$C := \{a + b : a \in A, b \in B\}.$$

Show that C is a bounded set and that

$$\sup\ C = \sup\ A + \sup\ B.$$

Solution: Since $a \le \sup A$ for any $a \in A$ and $b \le \sup B$ for any $b \in B$, we have

$$a + b \le \sup\ A + \sup\ B$$

for any $a \le \sup A$ and $b \in B$. That is $\sup\ A + \sup\ B$ is an upper bound for C.

Note that for any $\varepsilon > 0$ there is $a \in A$ such that $a > \sup A - \frac{\varepsilon}{2}$ and $b \in B$ such that $b > \sup B - \frac{\varepsilon}{2}$. Therefore $a+b > \sup A + \sup B - \varepsilon$. That is $\sup A + \sup B - \varepsilon$ is not an upper bound for any $\varepsilon > 0$; hence the statement follows.

2. Give a definition of absolute value.

Solution:

$$|x| = \left[\begin{array}{rl} x & \text{if } x \ge 0; \\ -x & \text{if } x < 0. \end{array}\right.$$

Use it to prove that $|-x| = |x|$ for any $x \in \mathbb{R}$.

Solution:

$$\begin{array}{llllll} \text{If } x = 0 \text{ then } x = -x & & \Rightarrow & & & |x| = |-x|; \\ \text{If } x > 0 \text{ then } -x < 0 & \Rightarrow & |x| = x \text{ and } |-x| = x & \Rightarrow & & |x| = |-x|; \\ \text{If } x < 0 \text{ then } -x > 0 & \Rightarrow & |x| = -x \text{ and } |x| = -x & \Rightarrow & & |x| = |-x|. \end{array}$$

Quiz 4.

1. Suppose S is a set of disjoint open intervals in $\mathbb{R}$. That is, if $(a, b) \in S$ and $(c, d) \in S$, then either $(a, b) = (c, d)$ or $(a, b) \cap (c, d) = \emptyset$.

Prove S is a countable set.

Solution: Since the set of rationals is dense in $\mathbb{R}$, we can choose a rational number $q \in \mathbb{Q}$ in each interval from S; that is, there is an bijection from S to a subset of $\mathbb{Q}$.

Since $\mathbb{Q}$ is countable, the statement follows.

2. Show that the set of irrational numbers is uncountable.

Solution: Arguing by contradiction, assume the set irrational numbers $\mathbb{I} = \mathbb{R} \backslash \mathbb{Q}$ is countable. In this case the $\mathbb{R}$ can be presented as a union of two countable sets $\mathbb{I}$ and $\mathbb{Q}$. Therefore $\mathbb{R}$ is countable. The latter contradicts Cantor's theorem.

Quiz 5.

1. Let $\{x_n\}$ be a sequence.

a) Show that $\lim x_n = 0$ (that is, the limit exists and is zero) if and only if $\lim |x_n| = 0$.

Solution: $\lim x_n = 0 \Leftrightarrow$ "for any $\varepsilon > 0$ there is $M \in \mathbb{N}$ such that $|x_n - 0| < \varepsilon$ for any $n \geq M$". Since

$$||x_n| - 0| = |x_n| = |x_n - 0|.$$

This statement is equivalent to the following "for any $\varepsilon > 0$ there is $M \in \mathbb{N}$ such that $||x_n| - 0| < \varepsilon$ for any $n \geq M$". The latter means that $|x_n| \to 0$.

b) Find an example such that $\{|x_n|\}$ converges and $\{x_n\}$ diverges.

Solution: $x_n = (-1)^n$.

2. Prove that any convergent sequence has a unique limit.

See Proposition 2.1.6.