Solutions

Quiz 1

1. Discuss the existence and uniqueness of the following initial value problem:

$$\dot{x} = x^{1/3}; \quad x(0) = 0.$$

Solution. The function $x \mapsto x^{1/3}$ is continuous therefore equation has local solution for any initial data.

Note that $x(t) = 0$ is a solution and

$$x(t) = \begin{cases} 0 & \text{if } t \le 0 \\ (\frac{2}{3} \cdot t)^{3/2} & \text{if } t > 0 \end{cases}$$

is also a solution. The latter is evident for $t < 0$ and

$$\dot{x} = \tfrac{3}{2} \cdot (\tfrac{2}{3} \cdot t)^{1/2} \cdot \tfrac{2}{3} = x^{1/3}$$

for $t \ge 0$.

Hence there is no uniqueness.

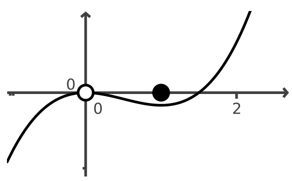
Comment. The function $x \mapsto x^{1/3}$ is not Lipschitz at 0, otherwise local uniqueness would follow we could use Picard's theorem.

Quiz 2

1. For the following vector field, plot the potential function $V(x)$ and identify all the equilibrium points and their stability.

$$\dot{x} = x(1 - x).$$

Solution.



Quiz 3

1. Consider the equation $\dot{x} = rx + x^3$, where $r > 0$ is fixed. Show that $x(t) \to \pm\infty$ in finite time, starting from any initial condition $x_0 \ne 0$.

Solution. Since $f(x) = rx + x^3$ is an odd function, it is sufficient to consider the case $x_0 > 0$.

Since $r > 0$, we have $f(x) > x^3 > 0$ for $x > 0$. Therefore it is sufficient to show that the solution of

$$\dot{x} = x^3$$

escapes to ∞ in finite time for any initial condition $x_0 > 0$.

Solving the equation we get

$$x(t) = \frac{1}{(\frac{1}{x_0^2} - 2t)^{1/2}};$$

the solution approach ∞ as $t \to \frac{2}{x_0^2}$

Quiz 4

1. For the following flow on the circle

$$\dot{\theta} = \mu \cos\theta + \sin(2 \cdot \theta),$$

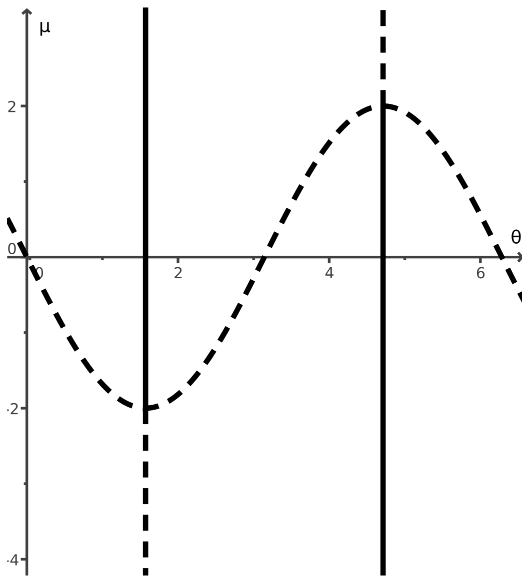
draw the phase portrait, classify the bifurcations that occur as μ varies, and find all the bifurcation values of μ.

Solution.

$$\mu \cos\theta + \sin(2 \cdot \theta) = 0$$

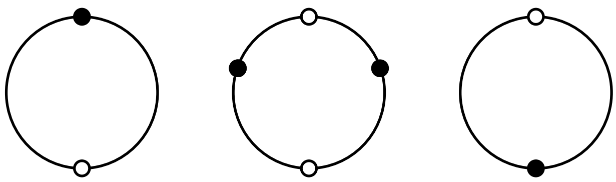
if and only if

$$\theta = \pm\tfrac{\pi}{2} \pmod{\pi} \quad \text{or} \quad \mu = -2\sin\theta.$$



We have two (subcritical) pitchfork bifurcations at $\mu = \pm 2$.

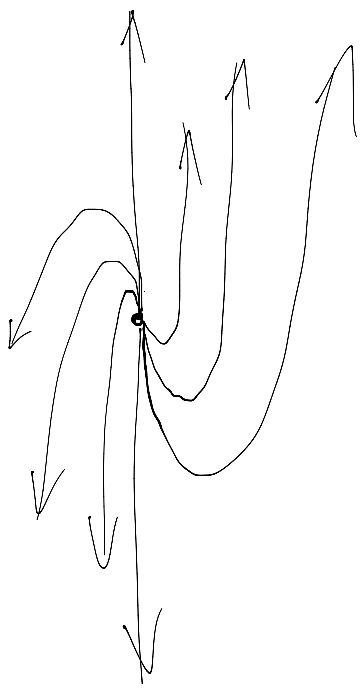
The following diagram shows the behavior of the flow fro $\mu \geq 2$, $|\mu| < 2$ and $\mu \leq -2$ correspondingly.



2. Sketch some typical trajectories of the linear system

$$\begin{cases} \dot{x} = x, \\ \dot{y} = x + y. \end{cases}$$

Solution. The matrix is $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$; both eigenvalues are 1 and it has only one eigenvector $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. So, it is unstable degenerate node. Typical trajectories should be go like this:



(General solution is $x(t) = x_0 \cdot e^t$, $y(t) = (x_0 \cdot t + y_0) \cdot e^t$, but you do not need it.)

Quiz 5

1. For the following system, find the fixed points, classify them, sketch the neighboring trajectories.

$$\begin{cases} \dot{x} = y + x - x^3, \\ \dot{y} = -y. \end{cases}$$

Solution. The Jacobian is

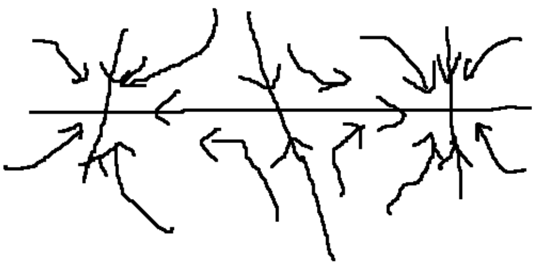
$$\begin{pmatrix} 1 - 3x^2 & 1 \\ 0 & -1 \end{pmatrix}.$$

The system

$$\begin{cases} y + x - x^3 = 0, \\ -y = 0. \end{cases}$$

has 3 solutions: $(-1, 0)$, $(0, 0)$ and $(1, 0)$

- for $(-1, 0)$ we have $\Delta = 2$, $\tau = -3$ — stable node
- for $(0, 0)$ we have $\Delta = -1$ — saddle.
- for $(1, 0)$ we have $\Delta = 2$, $\tau = -3$ — stable node



Quiz 6

1. Find a conserved quantity for the system

$$\begin{cases} \dot{x} = x \cdot (1 - y), \\ \dot{y} = \mu \cdot y \cdot (x - 1). \end{cases}$$

Solution.

$$\frac{dy}{dx} = \mu \cdot \frac{y \cdot (x-1)}{x \cdot (1-y)},$$

$$dy \cdot \tfrac{1-y}{y} = dx \cdot \mu \cdot \tfrac{x-1}{x},$$

$$\ln|y| - y = \mu \cdot (x - \ln|x|) + C.$$

So the

$$V(x,y) = \ln|y| - y - \mu \cdot (x - \ln|x|)$$

is a conserved quantity.

Quiz 7

1. Show that each of the following systems is reversible; sketch its phase portrait.

$$\begin{cases} \dot{x} = y \cdot (4 - x^2), \\ \dot{y} = 1 - y^2. \end{cases}$$

Solution. The map $(x, y, t) \mapsto (x, -y, -t)$ sends a solution to a solution; indeed in this case $\dot{x} \mapsto -\dot{x}$; $\dot{y} \mapsto \dot{y}$, $y \cdot (4 - x^2)$ is odd in y and $1 - y^2$ is even in y.

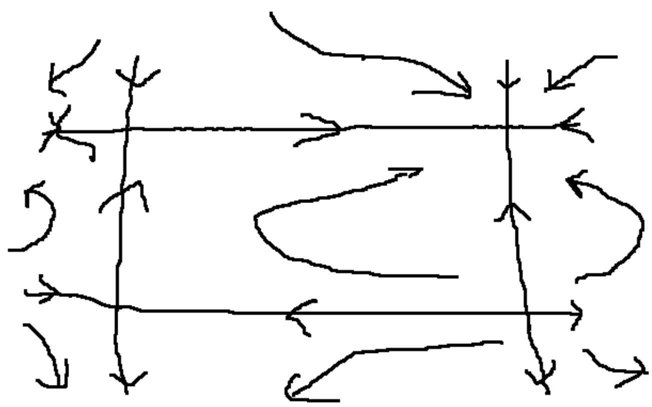
The system has 4 fixed points $(2, 1)$, $(-2, 1)$, $(2, -1)$, $(-2, -1)$. The lines $x = \pm 2$ and $y = \pm 1$ are invariant. Jacobian is

$$\begin{pmatrix} -2xy & 4 - x^2 \\ 0 & -2y \end{pmatrix}$$

So the eigenvalues are $-2xy$ and $-2y$. Therefore

- $(-2, 1)$ is a saddle,
- $(-2, -1)$ is a saddle,
- $(2, 1)$ is a stable node,
- $(2, -1)$ is an unstable node,

The reflection in x-axis should revert the orientation of the trajectories:



Quiz 8

1. Is the origin a nonlinear center for the system

$$\begin{cases} \dot{x} = y - x^2, \\ \dot{y} = x. \end{cases}$$

Solution. The origin is a fixed point; its Jacobian is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

the determinant is -1 so it is a saddle point, can not be a nonlinear center.

(By the way, this system is reversible — $f(x,y) = y - x^2$ is odd in y and $g(x,y) = x$ is even in y, therefore the map $(x,y,t) \mapsto (x,-y,-t)$ sends any solution to a solution.)

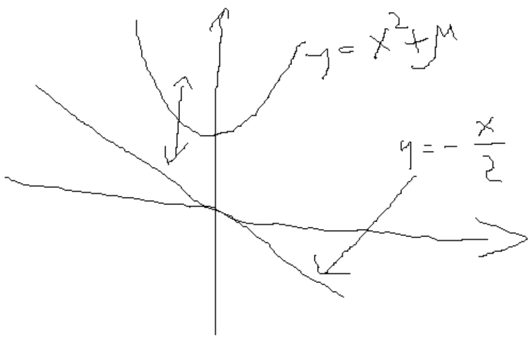
Quiz 9

1. Consider the system

$$\begin{cases} \dot{x} = x + 2y, \\ \dot{y} = \mu + x^2 - y. \end{cases}$$

a) Sketch the nullclines.

Solution.

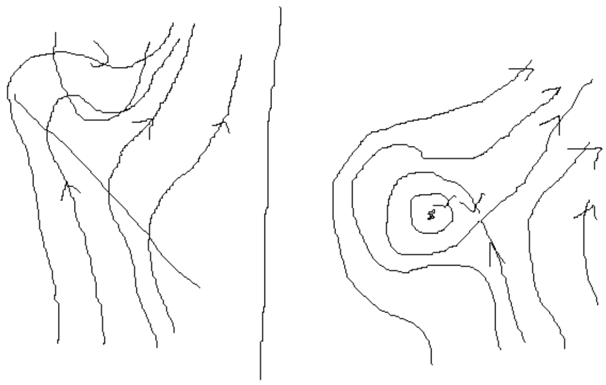


b) Find and classify the bifurcation that occur as μ varies.

Solution. Saddle node bifurcation happens at $\mu = \frac{1}{16}$ — at the moment when parabola is tangent to the line, its derivative is $-\frac{1}{2}$ so $2 \cdot x_0 = -\frac{1}{2}$, $y_0 = -\frac{1}{2} \cdot x_0$ and $y_0 = x_0^2 + \mu$. Hence $x_0 = -\frac{1}{4}$, $y_0 = \frac{1}{8}$ and $\mu = y_0 - x_0^2 = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$.

c) Sketch the phase portraits before and after the bifurcation.

Solution. After the bifuration two new fixed points appear — a saddle a center. The divergence vanish therefore all orbits near the second fixed points are closed.



Quiz 10

Set
$$f(x) = \frac{x\cdot(x-1)\cdot(x-2)}{(x+100)\cdot(x+101)\cdot(x+102)}.$$

1. A flow on the plane has only one fixed point at the origin and
$$P(x) = x + f(x)$$
is its Poincare map which is defined for the positive x-axis. Classify the fixed point; how many cycles the system has; classify each. Explain why there is no more cycles.

Solution. There are two positive fixed points of the Poincare map for: 1 and 2. each corresponds to a cycle.

By index theory, any cycle surrounds a fixed point. Since the origin is the only fixed point, any cycle must surround the origin. Therefore it must cross the positive part of x-axis. Hence any cycle corresponds to a fixed point of P — we have exactly two cycles.

Quiz 11

$$\begin{cases} \dot{x} = \sigma(y-x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = xy - bz. \end{cases}$$

1. Show that there is a certain ellipsoidal region E of the form
$$rx^2 + \sigma y^2 + \sigma(z-2r)^2 \le C$$
such that all trajectories of the Lorenz equations (see above) eventually enter E and stay in there forever.

Solution. Set
$$V(x,y,z) = rx^2 + \sigma y^2 + \sigma(z-2r)^2.$$

Then

$$\begin{aligned}
\dot{V} &= 2rx\dot{x} + 2\sigma y\dot{y} + 2\sigma(z-2r)\dot{z} = \\
&= 2rx \cdot \sigma(y-x) + 2\sigma y \cdot (rx - y - xz) + 2\sigma(z-2r) \cdot (xy - bz) = \\
&= -2rx^2 - 2\sigma y^2 - 2\sigma bz^2 - 4rbz.
\end{aligned}$$

If $2rx^2 + 2\sigma y^2 + 2\sigma bz^2$ is sufficiently large (which happens outside of a bounded set, denote it by B) then it it larger then the linear term $4rbz$. Therefore $\dot{V} < -1$ outside of a bounded set. We can choose the value C so the the ellipsoid E contains the bounded set B. Therefore $\dot{V} < -1$ on any trajectory outside of E; hence in finite time it gets the value C — at that moment it meets the ellipsoid and it can not leave it.