Math 429, Extra Credit Problems

These problems are hard and interesting. The solutions should be presents orally before April 16. It might improve your score, but should be used for fun. Only the first solution will be graded.

0. Find a mistake or misprint in the lecture notes. (The score depends on the type of mistake.)

~~1.~~ *(solved)* Find three disjoint open sets on the real line that have the same nonempty boundary.

~~2.~~ *(solved)* Construct a continuous function $f\colon [0,1] \to [0,1]$ such that f takes every value in $[0,1]$ an infinite number of times.

3. Describe all the homeomorphisms from the Sierpinski triangle to it self.

~~4.~~ *(solved)* Prove that $\mathbb{R}^3\backslash\mathbb{S}^1$ is homeomorphic to $\mathbb{R}^3\backslash(\ell \cup \{p\})$, where ℓ is a straight line and $p \notin \ell$ is a point in $\mathbb{R}^3$.

5. Show that any nonempty open star-shaped set in $\mathbb{R}^2$ is homeomorphic to the open disc.

6. Let P and Q be two countable everywhere dense subsets in $\mathbb{R}^2$. Show that $\mathbb{R}^2\backslash P \simeq \mathbb{R}^2\backslash Q$.

~~7.~~ *(solved)* Construct two functions $\mathbb{R} \to \mathbb{R}$, one is closed but not continuous, and the other is open but not continuous.

8. Classify topological spaces (up to homeomorphism) containing a unique nowhere dense subset.

~~9.~~ *(solved)* Find two compact sets with noncompact intersection.

~~10.~~ *(solved)* Let K be a compact space. Show that a function $f\colon X \to K$ is continuous if its graph $\{\, (x, f(x)) \in X \times K \mid x \in X \,\}$ is closed.

11. Find two compact subsets $A, B \subset \mathbb{R}^2$ such that A is not homeomorphic to B but $A \times [0,1]$ is homeomorphic to $B \times [0,1]$.

12. Construct a continuous injective map $s\colon \mathbb{R} \to \mathbb{R}^2$ such that $|s(n)| > |n|$ for any integer n and the complement $\mathbb{R}^2 \backslash s(\mathbb{R})$ is connected.

13. Construct a bounded open set Ω in $\mathbb{R}^2$ such that its boundary $\partial\Omega$ is totally path-diconnected; that is, no pair of distinct points $x, y \in \partial\Omega$ can be connected by a path in $\partial\Omega$.

14. Recall that $\mathbb{Q}$ denotes the set of rational numbers. Consider the following sets in the plane:

$$A = \{\, (x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q} \,\} \quad \text{and} \quad B = \{\, (x, y) \in \mathbb{R}^2 \mid x, y \notin \mathbb{Q} \,\}.$$

Show that $A \cup B$ is path connected.

~~15.~~ *(solved)* Show that any connected finite space is path connected.

~~16.~~ *(solved)* Prove that any two spaces with the same homotopy type can be embedded as deformation retracts in the same topological space.

17. Let X be a path connected space, consider space

$$Z = X \times X / \sim,$$

where $(x, y) \sim (y, x)$. Prove that $\pi_1(Z)$ is abelian.

18. Let V, W be two open subsets in a topological space X. Prove that if $V \cup W$ and $V \cap W$ are simply connected, then so are V and W.

19. Let a topological space X be the union of two open path-connected sets U and V. Prove that if $U \cap V$ has at least three connected components, then the fundamental group of X is nonabelian.

20. Construct a finite topological space with fundamental group isomorphic to $\mathbb{Z}_2$.

21. Construct two covering maps $f\colon X \to Y$ and $g\colon Y \to Z$ such that the composition $g \circ f\colon X \to Z$ is not a covering map.

t.b.c.