

# Extra credit problems

Math 485

0. Find a mistake or misprint in “Extra pearls”. (The score depends on the type of mistake.)

1. (*solved*) Assume  $d_1, \dots, d_p$  is a sequence of integers in a nonincreasing order. Show that it is multigraphic if and only if  $d_p \geq 0$ , the sum  $d_1 + \dots + d_p$  is even and

$$d_1 \leq d_2 + \dots + d_p.$$

(A sequence of integers  $d_1, \dots, d_p$  is called *multigraphic* if it appears as a sequence of degrees of a multigraph.)

2. (*solved*) Assume that a sequence  $d_1, \dots, d_p$  is graphic,  $d_i \geq 1$  for each  $i$  and

$$d_1 + \dots + d_p \geq 2 \cdot (p - 1).$$

Show that there is a *connected* graph  $G$  with the degree sequence  $d_1, \dots, d_p$ .

3. (*solved*) Show that in any connected graph  $G$  there is a vertex  $v$  such that  $G - v$  is connected.

4. (*solved*) Let  $G$  be a connected graph. Show that any two paths of maximum length in  $G$  have a common vertex.

6. (*solved*) Let  $G$  be a critical graph and  $\chi(G) = k + 1$ . Show that after removing any  $k - 1$  edges from  $G$  the obtained graph remains connected.

7. (*solved*) Assume both sequences  $d_1, \dots, d_p$  and  $d_1 - 1, \dots, d_p - 1$  are graphic. Show that there is a graph with a 1-factor and degree sequence  $d_1, \dots, d_p$ .

8. (*solved*) Show that any edge of cubic graph lies in an even number of Hamiltonian cycles.

9. (*solved*) Assume that the edges of a complete graph are colored in two colors. Show that there is a Hamiltonian cycle which either monochromatic or consists of two monochromatic paths.

10. Assume that  $G$  is a connected cubic graph such that for any two vertices  $v, w$  of  $G$  there is an isomorphism  $G \rightarrow G$  which sends  $v$  to  $w$ . Prove that  $G$  remains connected after removing any 2 edges.

11. (solved) Let  $G$  be a connected graph with no triangles. Given any two vertices  $v, w$  in  $G$ , denote by  $d(v, w)$  the length of a shortest path containing  $v$  to  $w$ .

Show that  $G$  is a tree if and only if

$$d(x, y) + d(v, w) = \max\{d(x, v) + d(y, w), d(x, w) + d(y, v)\}$$

for any 4 vertices  $x, y, v, w$  in  $G$ .

12. Understand the proof of Vizing's theorem (see the references for Theorem 2.2.2 in "Pearls" or find a proof elsewhere).

13. Suppose  $p = r(m, n)$  is the Ramsey number for  $m$  and  $n$ .

Assume that  $G$  results from  $K_p$  by deleting a single edge. Show that  $G$  has a red/blue edge coloring with no red  $K_m$  and no blue  $K_n$ .

14. Assume that the real intervals  $I_1, \dots, I_k$  have  $2 \cdot k$  different ends that are chosen randomly from  $\{1, 2, \dots, 2 \cdot k\}$ . Consider the graph  $G$  with  $k$  vertexes such that  $i$ th vertex is connected to  $j$ th vertex if and only if  $I_i$  overlaps with  $I_j$ . Show that  $G$  has diameter  $\leq 2$  with probability at least  $\frac{2}{3}$ .

15. Understand the proof of Erdős-Lovász theorem (see the references for Theorem 2.1.5 in "Pearls", or Theorem 2.14 in "Extra pearls", or find a proof elsewhere).

16. (solved) Let  $G$  be a graph with  $p$  vertexes. Show that  $G$  is a tree if and only if its chromatic polynomial is  $x \cdot (x - 1)^{p-1}$ .

17. Let  $T$  be a spanning tree in a graph  $G$  with weighted edges. Show that  $T$  has minimal total weight if and only if the weight of any edge  $(a, b)$  in  $G$  is at least as large as the weight of any edge on the path from  $a$  to  $b$  in  $T$ .

18. In a group of people, for some fixed  $s$  and any  $k$ , any  $k$  girls like at least  $k - s$  boys in total. Show that then all but  $s$  girls may get married on the boys they like.

19. Let  $M$  be a maximal planar map. Assume the vertexes of  $M$  are colored in 3 colors. Show that the number of the regions that get all three colors is even.

20. Let  $M$  be a map and  $G$  its dual. Show that  $M$  and  $G$  have the same number of spanning trees.

21. Show that any map has a vertex of degree at most 3 or a country with at most 3 sides.

22. Suppose that a toroidal graph has two connected components. Show that at least one of the components is planar.