Notes on smooth manifolds

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Lectures

1. Inverse function theorem, implicit function theorem.
   Topological n-manifold = second countable locally Euclidean Hausdorff space.
   Smooth manifolds = topological manifold + smooth structure.
   Chart, gluing map, atlas, maximal atlas, smooth structure.
   Tangent vectors as equivalence classes of smooth curves.
2. Sard's lemma.
3. Degree modulo 2, orientation, degree. Transversal intersections, Thom's theorem. Intersection number, vector fields, Euler's number.
4. Vector field as a differential operator. Lie bracket, Lie derivative, the straightening lemma.
5. Tensor fields. Lie derivative. Einstein notation. Examples of tensor fields (scalar product and cross product in polar coordinates).
6. Grassmann algebra, forms, forms as tensor fields, wedge product convention.
7. Pullback, exterior differential, internal differential, Cartan calculus.
8. Moser's trick. $H^n_{dR}(M^n) = \mathbb{R}$ for oriented closed connected M. De Rham cohomology, degree. Calculations via symmetry for products of $\mathbb{S}^n$ and maybe $\mathbb{C}\mathrm{P}^n$.
9. Homotopy invariance of De Rham cohomology, Poincaré's lemma, Mayer–Vietoris sequence.
10. Morse theory.

Moser's trick

Observe that the flow for a vector field defines a diffeomorphism.

Indeed, let v_t be a smooth time-dependent vector field on a smooth manifold M. Recall that the flow φ^s of the vector field v_t is defined as $\varphi^s\colon x(0)\mapsto x(s)$, where x is a solution of the following ordinary differential equation $x'(t)=\mathrm{v}_t(x(t))$. By the Picard theorem, the flow φ^s is smooth in its domain of definition. Moreover, the same holds for its inverse ψ^s; indeed, ψ^s is the flow of the vector field $-\mathrm{v}_{s-t}$. It follows that, φ^s is a diffeomorphism from its domain of definition to its image.

Moser's trick uses this source of diffeomorphism when it is needed to construct a diffeomorphism with a certain property. Thus, to find a diffeomorphism, we need to construct a vector field of a certain type. The latter problem is typically simpler.

Now we will illustrate this idea with several examples.

A Moser's theorem

Let M be an n-dimensional smooth manifold. An n-form ω on M is called a volume form if it does not vanish at any point. Note that if a manifold has a volume form, then it is orientable.

0.1. Theorem. *Let ω_0 and ω_1 be volume forms on a closed connected oriented smooth manifold M. Assume*

❶ $$\int\limits_M \omega_0=\int\limits_M \omega_1.$$

Then there is a diffeomorphism $\varphi\colon M\to M$ such that $\omega_0=\varphi^*\omega_1$. Moreover, we can assume that φ is isotopic to the identity map; that is, there is a smooth map $(x,t)\mapsto\varphi_t(x)$ such that $\varphi_0=\mathrm{id}$, $\varphi_1=\varphi$, and $x\mapsto\varphi_t(x)$ is a diffeomorphism for each t.

Proof. Observe that $\omega_t=(1-t)\cdot\omega_0+t\cdot\omega_1$ is a one-parameter family of volume forms on M. We plan to find a time-dependent vector field v_t

such that

❷ $$\mathcal{L}_{\mathrm{v}_t}\,\omega_t + \tfrac{d}{dt}\omega_t = 0.$$

If φ^t denotes the flow defined by v_t, then

$$\tfrac{d}{dt}(\varphi^{t*}\omega_t) = \mathcal{L}_{\mathrm{v}_t}\,\omega_t + \tfrac{d}{dt}\omega_t.$$

Therefore, ❷ implies that $\varphi^{t*}\omega_t$ does not depend on t. In particular,

$$\omega_0 = \varphi^{0*}\omega_0 = \varphi^{1*}\omega_1.$$

That is, $\varphi = \varphi^1$ does the trick; it only remains to prove ❷.

Suppose M is n-dimensional. Recall that $[\omega] \mapsto \int_M \omega$ defines an isomorphism $H^n_{dR}(M) \to \mathbb{R}$. By ❶, $[\omega_1 - \omega_0] = 0 \in H^n_{dR}(M)$. That is, $\omega_1 - \omega_0$ is exact; so, there is an $(n-1)$-form η on M such that

$$\omega_1 - \omega_0 = d\eta.$$

Note that $\tfrac{d}{dt}\omega_t + \eta = 0$.

Since ω_t does not vanish, there is a unique vector field v_t such that $i_{\mathrm{v}_t}\omega_t = \eta$. By the magic formula,

$$\mathcal{L}_{\mathrm{v}_t}\,\omega_t = d i_{\mathrm{v}_t}\omega_t + i_{\mathrm{v}_t}\cancelto{0}{d\omega_t} = d\eta = -\tfrac{d}{dt}\omega_t;$$

hence ❷ follows. □

B Open star-shaped domains

0.2. Exercise. *Any open star-shaped domain in $\Omega \subset \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.*

Extended hint. We can assume that the closed unit ball $\bar{B}$ lies in Ω. Let B be the interior of $\bar{B}$. It is sufficient to construct a diffeomorphism $\Omega \to B$

Construct a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f(\mathrm{x}) = 0$ if $\mathrm{x} \in \Omega$ and $f(\mathrm{x}) > 0$ otherwise. Further, construct a smooth function $\varphi\colon \mathbb{R} \to [0,1]$ such that $\varphi(x) = 1$ if $\leqslant 0$ and $\varphi(x) = 0$ if $x \geqslant 1$.

Consider the time-dependent vector field

$$\mathrm{v}_t(\mathrm{x}) = -\varphi((1+t)\cdot(1-|\mathrm{x}|))\cdot f(\mathrm{x})\cdot \mathrm{x}.$$

Let φ^s be the flow of v_t in the interval $[0,s]$.

Prove that given x, the value $\varphi^s(x)$ is constant for all sufficiently large s. In particular, $\varphi^s(x)$ converges as $s \to \infty$ for any x; denote its limit by $\varphi^\infty(x)$.

Show that φ^∞ is a diffeomorphism $\Omega \to B$; that is, show that φ^∞ is smooth in Ω, it has a smooth inverse, and $\varphi^\infty(\Omega) = B$. □

C Morse lemma

0.3. Lemma. *Let p be a nondegenerate critical point of index k of a smooth function f on a smooth n-dimensional manifold M. Then f can be wrtten as*

$$f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2.$$

in some local coordinates $(x_1, \dots, x_n)$ with the origin at p.

Proof. Without loss of generality we may assume that $f(p) = 0$. Choose local coordinates with the origin at p; suppose they are given by a chart $s\colon U \to M$ with domain $U \subset \mathbb{R}^n$.

Let A_x be the Hessian matrix of f at $\mathrm{x} \in U$; that is,

$$(\mathrm{v}\mathrm{w}f)(\mathrm{x}) = \langle A_\mathrm{x}\mathrm{v}, \mathrm{w}\rangle$$

for any $\mathrm{x} \in U$ and $\mathrm{v}, \mathrm{w} \in \mathbb{R}^n$, where $\langle\ ,\ \rangle$ denotes the standard scalar product on $\mathbb{R}^n$. (Here and further, we use the same letter (say v) for a vector in $\mathbb{R}^n$, its corresponding point, and the corresponding parallel vector field.)

Since p is a nondegenerate critical point, the matrix A_0 is invertible. Applying a linear transformation to $\mathbb{R}^n$, we can assume that A_0 is a diagonal matrix with the first k elements equal to -2 and the remaining $n - k$ elements equal to 2. Then the function $f_0(\mathrm{x}) := \frac{1}{2}\cdot\langle A_0\mathrm{x}, \mathrm{x}\rangle$ has a constant Hessian matrix A_0, and

❸ $$f_0(x_1, \dots, x_n) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2.$$

Set $h := f - f_0$, $f_t := f_0 + t\cdot h$, so $f = f_1$. We plan to find a vector field v_t such that for its flow φ^t the value $f_t \circ \varphi^t(x)$ does not depend on t (while it is defined) and $\mathrm{v}_t(0) = 0$ for any t. Once it is done, we have

$$f \circ \varphi^1 = f_0$$

in a neighborhood of $p = 0$. By ❸, $s \circ \varphi^1$ will define the needed chart.

Note that

$$\tfrac{d}{dt}(f_t \circ \varphi^t) = \mathrm{v}_t f_t + \tfrac{d}{dt} f_t = \mathrm{v}_t f_0 + t\cdot\mathrm{v}_t h + h.$$

Therefore, it is sufficient to find v_t such that

❹ $$\mathrm{v}_t f_0 + t\cdot\mathrm{v}_t h + h = 0$$

for any t.

Set

$$B_{\mathrm{x}} := \int_0^1 (A_{t\cdot\mathrm{x}} - A_0)\cdot dt \quad \text{and} \quad C_{\mathrm{x}} := \int_0^1 B_{t\cdot\mathrm{x}}\cdot dt.$$

Since A_{x} is symmetric, so are B_{x} and C_{x}. Observe that $B_0 = C_0 = 0$. Passing to a smaller subdomain of U, we can assume that B_{x} and C_{x} are sufficiently close to 0; in particular the matrix $t\cdot B_{\mathrm{x}} + A_0$ is invertible for any $\mathrm{x} \in U$ and $t \in [0,1]$.

Since $(\mathrm{w}h)(0) = 0$ for any $\mathrm{w} \in \mathbb{R}^n$,

$$(\mathrm{w}h)(\mathrm{x}) = \int_0^1 (\mathrm{x}\mathrm{w}(f - f_0))(t\cdot\mathrm{x})\cdot dt = \langle B_{\mathrm{x}}\mathrm{w}, \mathrm{x}\rangle.$$

Applying this formula for $\mathrm{w} = \mathrm{x}$, we get

$$h(\mathrm{x}) = \int_0^1 (\mathrm{x}h)(t\cdot\mathrm{x})\cdot dt = \langle C_{\mathrm{x}}\mathrm{x}, \mathrm{x}\rangle.$$

Therefore, ❹ can be rewritten as $\langle (A_0 + t\cdot B_{\mathrm{x}})\mathrm{v}_t, \mathrm{x}\rangle = \langle t\cdot C_{\mathrm{x}}\mathrm{x}, \mathrm{x}\rangle$, and $\mathrm{v}_t(\mathrm{x}) = (A_0 + t\cdot B_{\mathrm{x}})^{-1}(t\cdot C_{\mathrm{x}})\mathrm{x}$ provides the required solution. □

D Darboux's theorem

A nondegenerate closed 2-form is called symplectic. In other words, a 2-form ω on a smooth manifold M is symplectic if $d\omega = 0$ and $\mathrm{X} \mapsto i_{\mathrm{X}}\omega$ defines an isomorphism $\mathrm{T}_p \to \mathrm{T}_p^*$ at each point $p \in M$. A manifold equipped with a symplectic form is called symplectic.

0.4. Exercise. *Show that any 2-covector can be written as $\alpha_1 \wedge \beta_1 + \dots + \alpha_n \wedge \beta_n$ for linearly independent covectors $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$. Conclude that if a manifold admits a symplectic form, then it must have even dimension.*

0.5. Exercise. *Let ω be a closed 2-form on a smooth $2\cdot n$-dimensional manifold M. Show that ω is symplectic if and only if $\omega^{\wedge n}$ does not vanish.*

0.6. Theorem. *Let M be a smooth manifold with symplectic form ω. Then at any point $p \in M$ there are local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ such that $\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$*

Proof. Choose local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ with the origin at p. By the exercise, we may assume that the equality

$$\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$$

holds at the origin.

Consider the form $\omega_0 = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$ in the chart. Observe that ω_0 is symplectic. Passing to a domain of the chart, we may assume that $\omega_t = (1-t)\cdot\omega_0 + t\cdot\omega$ is a symplectic form for each $t \in [0,1]$. We plan to find a time-dependent vector field v_t such that $\mathrm{v}_t(0) = 0$ and for the flow φ^t of v_t we have

$$❺ \qquad \omega_0 = \varphi^{t*}\omega_t$$

for any $t \in [0,1]$. Once this is done, the diffeomorphism φ^1 gives the needed chart in a neighborhood of p.

Since $\omega_1 - \omega_0$ is closed, we can find a 1-form α such that $d\alpha = \omega_1 - \omega_0$. We can assume that $\alpha(0) = 0$.

By the magic formula,

$$\tfrac{d}{dt}(\varphi^{t*}\omega_t) = \mathcal{L}_{\mathrm{v}_t}\omega_t + \tfrac{d}{dt}\omega_t = di_{\mathrm{v}_t}\omega_t + i_{\mathrm{v}_t}\cancelto{0}{d\omega_t} + d\alpha.$$

Since ω_t is nondegenerate, there is a time-dependent vector field v_t such that $i_{\mathrm{v}_t}\omega = -\alpha$. Since $\alpha(0) = 0$, we have $\mathrm{v}_t(0) = 0$ for any t. □