Nash's theorem via Günther's trick: three lectures

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These are polished versions of my lecture notes. The reader is assumed to be familiar with smooth manifolds (charts, atlases, coverings, partitions of unity, and tensor fields). We also use Schauder estimates, Whitney embedding theorem, and the existence of fine triangulations of smooth manifolds, but these results may be treated as a black box. I would like to thank Mikhael Gromov, Fedor Nazarov, and Deane Yang for their help.

Contents

Lecture 1

Formulation and approximation

Riemannian geometry has two fundamental theorems. One is about the existence and uniqueness of the Levi-Civita connection, and the other is a result of John Nash [10], which states that *any Riemannian manifold is isometric to a smooth submanifold of Euclidean space.* In other words, Riemannian manifolds can be defined as submanifolds of Euclidean space.

We present a simplified proof discovered by Matthias Günther [3], with minor improvements by Deane Yang, Terence Tao, and Ralph Howard [5, 13, 16]. Günther's approach has also been used in books by Michael Taylor [14] and by Qing Han and Jia-Xing Hong [4], but our presentation should be more accessible to students.

A Induced metric

Given a smooth n-dimensional manifold Ω, we denote by $\mathrm{T}\Omega$ and $\mathrm{T}_p\Omega$ its tangent bundle and the tangent space at a point $p \in \Omega$, respectively.

Recall that a metric tensor, say g, on Ω is a smooth field of symmetric bilinear forms on $\mathrm{T}\Omega$. Here and further, *smooth* means C^∞.

Let $\bar{\Omega}$ be another smooth manifold, and let $w\colon \Omega \to \bar{\Omega}$ be a smooth map. Recall that the derivative of w in the direction of a tangent vector $\mathrm{x} \in \mathrm{T}_p\Omega$ is denoted by $\mathrm{x}w$, and

$$\mathrm{x}w = dw(\mathrm{x});$$

that is, the derivative of w in the direction x is the differential

$dw\colon \mathrm{T}\Omega \to \mathrm{T}\bar\Omega$ of w, evaluated at X. Note that $\mathrm{X}w \in \mathrm{T}_{w(p)}\bar\Omega$.

Suppose that $\bar\Omega$ is equipped with a metric tensor $\bar g$. Then the metric tensor g on Ω defined by

$$g(\mathrm{X},\mathrm{Y}) = \bar g(\mathrm{X}w,\mathrm{Y}w)$$

is called the induced metric tensor, or the pullback of $\bar g$ to Ω. This relation can also be written as $g = w^*\bar g$. Equivalently, we may say that w is an isometric map from (Ω,g) to $(\bar\Omega,\bar g)$.

A metric tensor g is called Riemannian if it is *positive*; that is, $g(\mathrm{X},\mathrm{X}) > 0$ for any nonzero tangent vector X. A smooth manifold equipped with a Riemannian metric is called a Riemannian manifold.

Note that if $g = w^*\bar g$ is a Riemannian metric, then w must be regular; that is, the differential d_pw has rank n at every point $p\in\Omega$. In particular, w is an immersion.

We consider the space $\mathbb{R}^d$ equipped with the metric tensor defined by the standard scalar product:

$$\bar g(\mathrm{X},\mathrm{Y}) := \langle x,y\rangle = x_1{\cdot}y_1 + \dots + x_d{\cdot}y_d,$$

where x_i and y_i denote the coordinates of vectors $x,y\in\mathbb{R}^d$. Note that $\bar g$ is Riemannian.

1.1. Main theorem. *Any n-dimensional Riemannian manifold (Ω,g) admits an isometric embedding into a Euclidean space $\mathbb{R}^d$ for some d. Moreover, d can be bounded in terms of n.*

We will not make any effort to optimize the bound on d in terms of n. Tracing the estimates in our argument yields $d\leqslant 10{\cdot}n^3$. The lower bound given in the following exercise may be optimal; known upper bounds are not much larger.

1.2. Exercise. *Show that d in the main theorem must be at least $n{\cdot}(n+1)/2$.*

1.3. Exercise. *Construct a metric tensor g on the torus $\mathbb{T}^2=\mathbb{S}^1\times\mathbb{S}^1$ such that $g(\mathrm{X},\mathrm{X})\geqslant 0$ for all tangent vectors X, but no smooth map $w\colon\mathbb{T}^2\to\mathbb{R}^{10}$ induces g.*

1.4. Exercise. *Let w be a smooth map from a smooth Riemannian manifold (Ω,g) to $\mathbb{R}^d$. Show that w is isometric if and only if it is length-preserving; that is, if*

$$\operatorname{length}\gamma = \operatorname{length}(w\circ\gamma)$$

for every curve γ in Ω.

1.5. Exercise. *Show that there is no length-preserving map $\mathbb{R}^2\to\mathbb{R}$.*

B Q-form

Recall that Ω is a smooth n-dimensional manifold.

We define a symmetric bilinear form Q that takes two C^1-smooth maps $v, w\colon \Omega \to \mathbb{R}^d$ and returns the following metric tensor on Ω:

$$g(\mathrm{X},\mathrm{Y}) \equiv \tfrac{1}{2}\cdot[\langle \mathrm{X}v, \mathrm{Y}w\rangle + \langle \mathrm{X}w, \mathrm{Y}v\rangle].$$

Note that if the maps v and w are C^k-smooth, then $g = Q(v,w)$ is C^{k-1}-smooth.

1.6. Observation. *A metric tensor g is induced by a smooth map $w\colon \Omega \to \mathbb{R}^d$ if and only if $g = Q(w,w)$.*

The components of $g = Q(w,w)$ in a local chart can be written as

$$g_{ij} = Q_{ij}(w,w) = \langle \partial_i w, \partial_j w\rangle.$$

Let Ω be a smooth manifold. The direct sum of two maps $v_1 : \Omega \to \to \mathbb{R}^{d_1}$ and $v_2\colon \Omega \to \mathbb{R}^{d_2}$ will be denoted by $w = v_1 \oplus v_2$; it is the map

$$w\colon \Omega \to \mathbb{R}^{d_1+d_2} = \mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2}$$

defined by $w\colon p \mapsto (v_1(p), v_2(p))$.

1.7. Exercise. *Show that*

$$Q(v_1 \oplus v_2, v_1 \oplus v_2) = Q(v_1,v_1) + Q(v_2,v_2)$$

for any two smooth maps $v_1 : \Omega \to \mathbb{R}^{d_1}$ and $v_2 : \Omega \to \mathbb{R}^{d_2}$.

Recall that the support of a map is the closure of the set where it takes nonzero values.

1.8. Exercise. *Let $v_1, v_2 : \Omega \to \mathbb{R}^d$ be smooth maps defined on a smooth manifold Ω. Suppose that v_1 and v_2 have disjoint supports. Show that*

$$Q(v_1 + v_2, v_1 + v_2) = Q(v_1,v_1) + Q(v_2,v_2).$$

C Disc covering

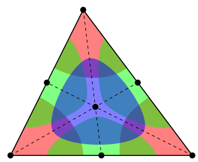
Let Ω be a smooth n-dimensional manifold. A subset $D \subset \Omega$ is called a smooth n-disc if there exists a neighborhood N of D and a smooth embedding $N \to \mathbb{R}^n$ that sends D to the unit ball.

1.9. Claim. *Let $\{U_\alpha\}$ be an open cover of a smooth n-dimensional manifold Ω. Then there exists a countable collection of smooth n-discs $\{D_\mathrm{i}\}$ in Ω such that the following conditions hold:*

(a) *The interiors $D_\mathfrak{i}^\circ$ cover the entire manifold Ω.*
(b) *For each $D_\mathfrak{i}$, there exists α such that $D_\mathfrak{i} \subset U_\alpha$.*
(c) *The collection $\{D_\mathfrak{i}\}$ can be colored with $n+1$ colors such that any two distinct n-discs of the same color are disjoint. More precisely, the index set $\mathfrak{I}$ of $\{D_\mathfrak{i}\}$ can be partitioned into $n+1$ subsets $\mathfrak{I}_0, \dots, \mathfrak{I}_n$ such that for any fixed k and any distinct $\mathfrak{i}, \mathfrak{j} \in \mathfrak{I}_k$, we have $D_\mathfrak{i} \cap D_\mathfrak{j} = \varnothing$.*

Sketch of proof. Recall that the star of a vertex $\mathfrak{i}$ in a simplicial complex is the union of all open simplices that have $\mathfrak{i}$ as a vertex.

Choose a triangulation τ of Ω such that the star of each vertex of τ lies in some U_α. (To prove the existence, we may use that any smooth manifold admits a triangulation and then apply [15, Theorem 35].) Let τ' be the barycentric subdivision of τ.



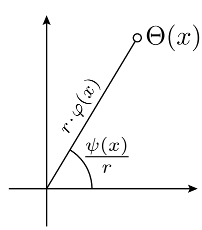
Colored discs in a triangle of τ.

For each vertex $\mathfrak{i}$ of τ', observe that its star can be smoothly parametrized by an open n-disc. By choosing a slightly smaller disc within this parametrization, we obtain a smooth n-disc $D_\mathfrak{i}$. This construction can be done so that the interiors $D_\mathfrak{i}^\circ$, for all vertices $\mathfrak{i}$ of τ', cover Ω. Since each $D_\mathfrak{i}$ lies within the star of $\mathfrak{i}$, condition *(b)* is satisfied.

Recall that each vertex of τ' corresponds to a simplex in τ. We now color the vertices of τ' with $n+1$ colors labeled $0, \dots, n$, assigning color k to each vertex corresponding to a k-simplex of τ. Observe that this coloring induces a coloring of the n-discs that meets condition *(c)*. $\square$

Our sketch relied on the existence of a triangulation of a smooth manifold, which has a tedious proof. Alternatively, one can choose a Riemannian metric on Ω, inscribe a covering of small, almost Euclidean balls into $\{U_\alpha\}$, and then argue as in the Besicovitch covering lemma. This approach yields a version of the claim with q colors instead of $n+1$, where q depends only on n. It serves as a replacement for Claim 1.9 in the remainder of our argument.

D Nash's twist



Let φ and ψ be smooth functions on a smooth manifold Ω. Given $r > 0$, denote by $\mathbb{S}^1_r$ the circle of radius r in $\mathbb{R}^2$. Nash's twist $\Theta\colon \Omega \to \mathbb{R}^2$ of the triple (r, φ, ψ) is defined as the composition

$$\Omega \xrightarrow{\psi} \mathbb{R} \xrightarrow{\ell_r} \mathbb{S}^1_r \xrightarrow{\times\varphi} \mathbb{R}^2,$$

where $\ell_r\colon \mathbb{R}\to\mathbb{S}^1_r$ is a length-preserving covering map, say

$$\ell_r(x) = (r\cdot\cos\tfrac{x}{r}, r\cdot\sin\tfrac{x}{r}),$$

and $\times\varphi$ denotes multiplication by φ; so

$$\Theta(x) := \varphi(x)\cdot(\ell_r\circ\psi(x)).$$

Let us denote by $(d\psi)^2$ the metric tensor induced by $\psi\colon\Omega\to\mathbb{R}$; that is,

$$(d\psi)^2(\mathrm{X},\mathrm{Y}) := d\psi(\mathrm{X})\cdot d\psi(\mathrm{Y}) = (\mathrm{X}\psi)\cdot(\mathrm{Y}\psi).$$

1.10. Claim. *Nash's twist Θ for the triple (r,φ,ψ) induces the metric*

$$g = \varphi^2\cdot(d\psi)^2 + r^2\cdot(d\varphi)^2.$$

Computations. Suppose that X is a tangent vector on Ω. Then,

$$\begin{aligned}\mathrm{X}\Theta &= \mathrm{X}(\varphi\cdot\ell_r\circ\psi) = \\ &= (\mathrm{X}\varphi)\cdot(\ell_r\circ\psi) + \varphi\cdot(\ell_r'\circ\psi)\cdot(\mathrm{X}\psi).\end{aligned}$$

Observe that $|\ell_r| = r$, $|\ell_r'| = 1$, and $\ell_r\perp\ell_r'$. Therefore,

$$g(\mathrm{X},\mathrm{Y}) = \langle\mathrm{X}\Theta,\mathrm{Y}\Theta\rangle = r^2\cdot(\mathrm{X}\varphi)\cdot(\mathrm{Y}\varphi) + \varphi^2\cdot(\mathrm{X}\psi)\cdot(\mathrm{Y}\psi). \qquad \square$$

E Approximate version

The following statement can be regarded as an approximate version of the main theorem (1.1).

1.11. Proposition. *Let g be a Riemannian metric on a smooth n-manifold Ω. Then there exists a metric tensor h on Ω and a one-parameter family of smooth maps $w_t\colon\Omega\to\mathbb{R}^d$ such that*

$$Q(w_t,w_t) = g + t\cdot h$$

for any $t>0$.

Note that $Q(w_t,w_t)$ converges to g as $t\to 0$. Therefore, one might be tempted to take the limit of w_t as $t\to 0$. However, as we will see, the maps w_t constructed in the proof converge to a constant; thus, the limit does not solve 1.1.

Let $w\colon\Omega\to\mathbb{R}^d$ be a smooth map. Suppose that $w_1,\dots,w_d$ are the coordinate functions of w. Then the induced metric tensor g can be written as

❶ $$g = Q(v,w) = (dw_1)^2 + \dots + (dw_d)^2.$$

In the main theorem, we need to find functions $w_1, \dots, w_d \colon \Omega \to \mathbb{R}$ that satisfy equation ❶ for a given Riemannian metric g. The weaker form ❷ of this equition will play a key role in the proof of 1.11.

1.12. Proposition. *Let (Ω, g) be a compact n-dimensional Riemannian manifold. Then there are smooth functions $\varphi_1, \dots, \varphi_d, \psi_1, \dots, \psi_d$ on Ω such that*

❷ $$g = \varphi_1^2 \cdot (d\psi_1)^2 + \dots + \varphi_d^2 \cdot (d\psi_d)^2.$$

Moreover, we can assume that $d \leqslant 2 \cdot (n+1) \cdot n^2$.

Proof. The metric tensor g can be written in local coordinates $(x_1, \dots, x_n)$ as

❸ $$g = \sum_{i,j} g_{ij} \cdot dx_i \cdot dx_j,$$

where $g_{ij} = g_{ji}$ are smooth functions of $(x_1, \dots, x_n)$.

Since g is Riemannian, at any point $p \in \Omega$, we can choose a chart so that the vectors ∂_i are orthonormal at p; that is, $g_{ii} = 1$ and $g_{ij} = 0$ at p for all $i \neq j$. Because the functions g_{ij} are smooth, for any $\varepsilon > 0$, we can find a neighborhood $U \ni p$ such that

❹ $$g_{ii} \lesseqgtr 1 \pm \varepsilon \quad \text{and} \quad g_{ij} \lesseqgtr \pm \varepsilon$$

for all $i \neq j$ and any point in U.

Observe that

$$\pm dx_i \cdot dx_j = \tfrac{1}{2} \cdot (dx_i \pm dx_j)^2 - \tfrac{1}{2} \cdot (dx_i)^2 - \tfrac{1}{2} \cdot (dx_j)^2.$$

Let us plug this formula in ❸ and take ❹ into account. Assuming that ε is small, we will get that g is a linear combination with positive coefficents of the metric tensors $(dx_i)^2$ and $(dx_i \pm dx_j)^2$ for all i and j. In other words, we can take the functions x_i and $x_i \pm x_j$ for all $i < j$ as our functions $\psi_1, \dots, \psi_d$ (these should be extended smoothly to the whole manifold) and find $\varphi_1, \dots, \varphi_d$ such that ❷ holds in a neighborhood of p.

Applying a partition-of-unity argument, we get the global statement.

It suffices to take n^2 functions in each chart. So, if Ω is covered by m charts, then we may take $d = n^2 \cdot m$. Moreover, the same bound applies if the charts can be colored using m colors such that charts of the same color are disjoint. Therefore, Claim 1.9 implies that $d = n^2 \cdot (n+1)$ suffices. □

Proof of 1.11. Suppose that $\varphi_1, \dots, \varphi_d, \psi_1, \dots, \psi_d$ are provided by 1.12. Let $r = \sqrt{t}$; denote by Θ_i the Nash's twist for the triple (r, φ_i, ψ_i). By 1.10,

$$w_t = \Theta_1 \oplus \dots \oplus \Theta_d$$

is the needed map with $h = (d\varphi_1)^2 + \dots + (d\varphi_d)^2$. □

F A pseudoeuclidean degression

Let us denote by $\mathbb{R}^r \ominus \mathbb{R}^s$ the pseudoeuclidean space with signature (r, s); that is, the space $\mathbb{R}^{r+s}$ with the scalar product

$$\langle x, y\rangle = x_1 \cdot y_1 + \dots + x_s \cdot y_s - x_{s+1} \cdot y_{s+1} - \dots - x_{s+r} \cdot y_{s+r},$$

where x_i and y_i denote the coordinates of vectors x and y in $\mathbb{R}^{r+s}$.

1.13. Problem. *Show that any metric g on a smooth manifold Ω is induced by a smooth embedding $\Omega \to \mathbb{R}^r \ominus \mathbb{R}^s$ for some large r and s.*

Solution based on Nash's theorem. Note that any metric g can be written as the difference $g = g_1 - g_2$ of two Riemannian metrics g_1 and g_2 on Ω. By Nash's theorem, we can find two smooth embeddings $w_1, w_2 \colon \Omega \to \mathbb{R}^d$ whose induced metrics are g_1 and g_2, respectively. It remains to observe that g is induced by

$$w_1 \ominus w_2 \colon p \mapsto (w_1(p), w_2(p)) \in \mathbb{R}^d \ominus \mathbb{R}^d.$$

□

The above solution uses the main theorem, which we have not yet proved. The following solution is more direct and relies only on Nash's twist.

Simpler solution. Assume g is Riemannian, and let $\varphi_1, \dots, \varphi_d, \psi_1, \dots$ $\dots, \psi_d$ be provided by 1.12. Let Θ_i be Nash's twist for the triple $(1, \varphi_i, \psi_i)$, and

$$h = (d\varphi_1)^2 + \dots + (d\varphi_d)^2,$$
$$\Theta = \Theta_1 \oplus \dots \oplus \Theta_d \colon \Omega \to \mathbb{R}^{2\cdot d},$$
$$\varphi = \varphi_1 \oplus \dots \oplus \varphi_d \colon \Omega \to \mathbb{R}^d.$$

According to 1.10, the map Θ induces $g + h$. Note that φ induces h. Therefore $w = \Theta \ominus \varphi \colon \Omega \to \mathbb{R}^{2\cdot d} \ominus \mathbb{R}^d$ induces g.

To handle the general case, express the metric as a difference of two Riemannian metrics: $g = g_1 - g_2$. As shown above, there exist smooth embeddings w_1 and w_2 into pseudoeuclidean spaces inducing g_1 and g_2, respectively. It follows that $w_1 \ominus w_2$ induces g. $\square$

1.14. Exercise. *Let ψ be a smooth isometric immersion of a Riemannian manifold (Ω, g) into the sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, and let $\varphi\colon \Omega \to \mathbb{R}$ be a smooth function. Show that $\varphi^2 \cdot g$ is induced by the map $\varphi \cdot \psi \ominus \varphi$.*

Lecture 2

Reductions

In this lecture, we reduce the main theorem (1.1) to its restricted version (2.1), which is an immersion theorem for very special metrics on tori. The reduction proceeds in the following steps:

$$\begin{array}{c}
\text{Restricted theorem (2.1)}\\
\Big\Downarrow \text{ 2B}\\
\text{Immersion theorem for tori (2.2)}\\
\Big\Downarrow \text{ 2C}\\
\text{Embeding theorem for tori (2.3)}\\
\Big\Downarrow \text{ 2D}\\
\text{Embeding theorem for compact manifolds (2.4)}\\
\Big\Downarrow \text{ 2E}\\
\text{The main theorem (1.1)}
\end{array}$$

A Restricted theorem

Let us denote by $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional torus.

The torus is a smooth manifold; it comes with global periodic coordinates, which can be used to define the partial derivatives $\partial_1, \dots, \partial_n$. A metric tensor on $\mathbb{T}^n$ can be written in these coordinates; therefore, we can (and will) regard a metric tensor as a smooth map $g\colon \mathbb{T}^n \to \mathbb{R}^N$, where $N := \frac{n(n+1)}{2}$. These observations make it easier to work with the torus than with a general manifold.

2.1. Restricted theorem. *There exists a Riemannian metric g_0 on $\mathbb{T}^n$ such that, for any metric tensor h on $\mathbb{T}^n$, the metric $g_0 + t{\cdot}h$ is* *induced by a smooth map* $\mathbb{T}^n \to \mathbb{R}^d$ *for all sufficiently small* $t > 0$. *Moreover, the dimension* d *can be bounded in terms of* n.

This theorem will be proved in the next lecture; see Section 3C.

B General metric on the torus

2.2. Claim. *Given a Riemannian metric* g *on* $\mathbb{T}^n$*, there exists an isometric immersion* $(\mathbb{T}^n, g) \looparrowright \mathbb{R}^d$ *for some* d *depending only on* n.

Reduction of 2.2 to 2.1. Let g_0 be the metric from 2.1. After rescaling if necessary, we may assume that $g_0 < g$; that is, $g_0(\mathrm{x}, \mathrm{x}) < g(\mathrm{x}, \mathrm{x})$ for any nonzero tangent vector x. In particular, the difference $g_1 = g - g_0$ is a Riemannian metric.

By 1.11, there exists a metric tensor h such that for any $t > 0$, there is a smooth map $w_1 \colon \mathbb{T}^n \to \mathbb{R}^{d_1}$ that induces $g_1 + t\cdot h$.

By 2.1, for any small $t > 0$, there is a smooth map $w_0 \colon \mathbb{T}^n \to \mathbb{R}^{d_0}$ that induces the metric $g_0 - t\cdot h$.

By 1.7, the metric tensor

$$g = g_0 - t\cdot h + g_1 + t\cdot h$$

is induced by the map $w_0 \oplus w_1$. $\square$

C From immersion to embedding

2.3. Claim. *Given a Riemannian metric* g *on* $\mathbb{T}^n$*, there is an isometric embedding* $(\mathbb{T}^n, g) \hookrightarrow \mathbb{R}^d$ *for some* d *depending only on* n.

Reduction of 2.3 to 2.2. Choose your favorite smooth embedding $w_1 \colon \mathbb{T}^n \to \mathbb{R}^{n+1}$. Let $g_1 = Q(w_1, w_1)$; in other words, g_1 is the metric induced by w_1. Since w_1 is an embedding, g_1 must be Riemannian.

We can assume that $g > g_1$; in other words the metric $g_2 = g - g_1$ is Riemannian. If this is not the case, replace w_1 with its rescaling $\varepsilon\cdot w_1$ for sufficiently small $\varepsilon > 0$.

By 2.2, we have a smooth isometric immersion $w_2 \colon (\mathbb{T}^n, g_2) \to \mathbb{R}^d$. Note that

$$w = w_1 \oplus w_2 \colon (\mathbb{T}^n, g = g_1 + g_2) \to \mathbb{R}^{d+2\cdot n+1}$$

is an isometric embedding. Indeed, w is isometric by 1.7, and it is an embedding since w_1 is. $\square$

D From torus to compact

2.4. Claim. *Any compact n-dimensional Riemannian manifold (Ω, g) admits an isometric embedding into $\mathbb{R}^d$ for some d that depends only on n.*

This step in the reduction relies on the following exercise:

2.5. Exercise. *Let Ω be a closed submanifold of the torus $\mathbb{T}^n$. Show that any Riemannian metric on Ω is the restriction of a Riemannian metric on $\mathbb{T}^n$.*

Reduction of 2.4 to 2.3. Since Ω is compact, it can be smoothly embedded into $\mathbb{T}^{2n+1}$.

Indeed, by the Whitney embedding theorem, there is a smooth embedding $\iota\colon \Omega \to \mathbb{R}^{2\cdot n+1}$. Let g_1 be the metric on Ω induced by ι. Since Ω is compact, we can assume (after rescaling if necessary) that $g > g_1$; that is, $g - g_1$ is Riemannian, and the image lies in a small ball. Composing ι with the covering map $\mathbb{R}^{2\cdot n+1} \to \mathbb{T}^{2\cdot n+1}$, we obtain the required embedding into the torus.

Thus, we may assume that Ω is a compact submanifold of $\mathbb{T}^{2\cdot n+1}$. By 2.5, any Riemannian metric g on Ω is the restriction of a Riemannian metric, say $\bar{g}$, on $\mathbb{T}^{2\cdot n+1}$.

Let $w\colon (\mathbb{T}^{2\cdot n+1}, \bar{g}) \to \mathbb{R}^d$ be an isometric smooth embedding provided by 2.3. Then the composition $f = w \circ \iota\colon (\Omega, g) \to \mathbb{R}^d$ is an isometric embedding. $\square$

E From compact to noncompact

This step is known as Nash's reduction; it appears in [10, Part D]. Let us begin with two exercises.

2.6. Exercise. *Let Ω be a connected noncompact manifold. Show that it admits a smooth, proper, bounded embedding into $\mathbb{R}^d \setminus \{0\}$. Moreover, given a Riemannian metric g on Ω, we can assume that the induced metric of this embedding is smaller than g.*

2.7. Exercise. *Suppose $\{D_\mathfrak{i}\}$ is a collection of smooth n-discs in a smooth n-dimensional manifold Ω provided by 1.9. Show that for any Riemannian metric g on Ω, there exists a collection of Riemannian metrics $h_\mathfrak{i}$ on $\mathbb{S}^n$ and smooth maps $r_\mathfrak{i}\colon \Omega \to \mathbb{S}^n$ such that:*

(a) The restriction $r_\mathfrak{i}|_{D_\mathfrak{i}^\circ}$ is a smooth embedding.
(b) $r_\mathfrak{i}$ maps $\Omega \setminus D_\mathfrak{i}^\circ$ to the south pole of $\mathbb{S}^n$.
(c) $g = \sum_\mathfrak{i} r_\mathfrak{i}^* h_\mathfrak{i}$.

Reduction of 1.1 to 2.4. Let $w\colon \Omega\to\mathbb{R}^d$ be an embedding provided by 2.6 for the metric $\frac{1}{2}\cdot g$. Let g_0 be the metric induced by w; note that $g_1=g-g_0$ is Riemannian.

Apply 2.7 to (Ω,g_1); let $r_\mathfrak{i}$, $h_\mathfrak{i}$, and $D_\mathfrak{i}$ be the resulting maps, metrics, and smooth n-discs. Since $\mathbb{S}^n$ is compact, there exists an isometric embedding $w_\mathfrak{i}\colon(\mathbb{S}^n,h_\mathfrak{i})\to\mathbb{R}^d$; we may assume that $w_\mathfrak{i}\circ r_\mathfrak{i}$ maps all points in $\Omega\setminus D_\mathfrak{i}^\circ$ to the origin in $\mathbb{R}^d$. In particular, each composition $w_\mathfrak{i}\circ r_\mathfrak{i}$ has compact support in $D_\mathfrak{i}^\circ$.

Let $\mathfrak{I}_0,\dots,\mathfrak{I}_n$ be the coloring of the index set provided by 1.9. Set

$$s_i=\sum_{\mathfrak{i}\in\mathfrak{I}_i} w_\mathfrak{i}\circ r_\mathfrak{i}\quad\text{and}\quad s=s_1\oplus\dots\oplus s_{n+1}.$$

By 1.8 and 1.7,

$$\begin{aligned}Q(w\oplus s,w\oplus s)&=Q(w,w)+\sum_i Q(s_i,s_i)=\\&=g_0+\sum_\mathfrak{i} Q(w_\mathfrak{i}\circ r_\mathfrak{i},w_\mathfrak{i}\circ r_\mathfrak{i})=\\&=g_0+\sum_\mathfrak{i} r_\mathfrak{i}^*h_j\\&=g_0+g_1=g.\end{aligned}$$

Since w is an embedding, so is $w\oplus s$, and the result follows. $\square$

Lecture 3

Perturbation

In this lecture, we formulate and prove three lemmas that imply the restricted version of the main theorem (2.1), and therefore the main theorem (1.1).

A Free Maps

As before, let Ω be a smooth n-dimensional manifold.

Recall that a C^1-map $f\colon\Omega\to\mathbb{R}^d$ is called a regular map if its differential df has rank n at every point. In other words, for any point $p\in\Omega$ and for some (and therefore any) choice of local coordinates at p, the first-order partial derivatives $\partial_1 f,\dots,\partial_n f$ are linearly independent.

A C^2 map $f\colon\Omega\to\mathbb{R}^d$ is called a free map (or two-regular map) if an analogous property holds for both first- and second-order partial derivatives: that is, for any point $p\in\Omega$ and for some (and therefore any) local coordinates at p, the collection of $n+\frac{n(n+1)}{2}$ partial derivatives $\partial_i f$, $\partial_{ij} f$ for $i\leqslant j$ are linearly independent.

3.1. Exercise. *Show that the freeness of a map $f\colon\Omega\to\mathbb{R}^d$ does not depend on the choice of charts on Ω.*

3.2. Exercise. *Show that there is no free map $\mathbb{S}^2\to\mathbb{R}^4$.*

3.3. Exercise. *Let $f\colon\Omega\to\mathbb{R}^{d_1}$ and $w\colon\Omega\to\mathbb{R}^{d_2}$ be smooth maps defined on a smooth manifold Ω. Show that if f is free, then so is the map $f\oplus w$.*

Let ψ be a smooth function on Ω. Define the map $\Theta_\psi\colon\Omega\to\mathbb{R}^2$ by

$$\Theta_\psi(x)=(\cos[\psi(x)],\sin[\psi(x)]).$$

In other words, Θ_ψ is the composition of two maps

$$\Omega \xrightarrow{\psi} \mathbb{R} \xrightarrow{\ell} \mathbb{S}^1 \subset \mathbb{R}^2,$$

where $\ell\colon \mathbb{R} \to \mathbb{S}^1$ is the natural covering map $y \mapsto (\cos y, \sin y)$. Note that Θ_ψ is Nash's twist for the triple $(1, 1, \psi)$.

3.4. Exercise. *Consider the (x, y)-plane $\mathbb{R}^2$. Show that the map*

$$\Theta_x \oplus \Theta_y \oplus \Theta_{x+y}\colon \mathbb{R}^2 \to \mathbb{R}^6 = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$$

is free.

Generalize this statement to maps $\mathbb{R}^n \to \mathbb{R}^{n(n+1)}$.

3.5. Exercise. *Let $w\colon \Omega \to \mathbb{R}^s$ be a regular smooth map, and let $f\colon \mathbb{R}^s \to \mathbb{R}^d$ be a free map. Show that the composition $f \circ w\colon \Omega \to \mathbb{R}^d$ is free.*

Apply the Whitney embedding theorem and 3.4 to conclude that any smooth manifold admits a free embedding into a Euclidean space.

B Key lemmas

Recall that our torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is equipped with periodic coordinates and globally defined partial derivatives $\partial_1, \dots, \partial_n$. Furthermore, a metric tensor on $\mathbb{T}^n$ can be described by a smooth map $g\colon \mathbb{T}^n \to \mathbb{R}^N$, where $N := \frac{n\cdot(n+1)}{2}$.

Fix $\alpha \in (0, 1)$, say $\alpha = \frac{1}{2}$. Denote by $|x - y|_{\mathbb{T}^n}$ the standard distance between points $x, y \in \mathbb{T}^n$. Let us recall the definition of the Hölder norms of a function $u\colon \mathbb{T}^n \to \mathbb{R}$

$$\begin{aligned}
\|u\|_{0,\alpha} &:= \sup_{x\in\mathbb{T}^n} \{\, |u(x)| \,\} + \sup_{x,y\in\mathbb{T}^n} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha_{\mathbb{T}^n}} \right\},\\
\|u\|_{k,\alpha} &:= \max_{|\mathcal{I}|\leqslant k} \{\, \|\partial_{\mathcal{I}} u\|_{0,\alpha} \,\},
\end{aligned}$$

where $\mathcal{I} = (i_1, \dots, i_n)$ denotes multi-index; so, $\partial_{\mathcal{I}} := \partial_1^{i_1} \dots \partial_n^{i_n}$ and $|\mathcal{I}| := i_1 + \dots + i_n$. The space of functions with finite $\|\ \|_{k,\alpha}$-norm is called the (k, α)-Hölder space and is denoted by $C^{k,\alpha}(\mathbb{T}^n, \mathbb{R})$.

3.6. Exercise. *Let $v \in C^\infty(\mathbb{T}^n, \mathbb{R})$. Show that there is a constant c such that*

$$\|u\cdot v\|_{k,\alpha} \leqslant c\cdot \|u\|_{k,\alpha}$$

for any $u \in C^{k,\alpha}(\mathbb{T}^n, \mathbb{R})$.

Now define $C^{k,\alpha}(\mathbb{T}^n,\mathbb{R}^d)$ as the space of functions $u\colon \mathbb{T}^n \to \mathbb{R}^d$ whose coordinate functions $u_1,\dots,u_d$ belong to $C^{k,\alpha}(\mathbb{T}^n,\mathbb{R})$, and set

$$\|u\|_{k,\alpha} := \max_i\{\, \|u_1\|_{k,\alpha},\dots,\|u_d\|_{k,\alpha}\,\}.$$

From now on, fix a smooth free map $f\colon \mathbb{T}^n \to \mathbb{R}^d$; it exists by 3.5. Furthermore, define

$$L(w) := 2{\cdot}Q(f,w).$$

3.7. Nash's lemma. *The linear operator*

$$L\colon C^\infty(\mathbb{T}^n,\mathbb{R}^d) \to C^\infty(\mathbb{T}^n,\mathbb{R}^N)$$

admits a right inverse

$$M\colon C^\infty(\mathbb{T}^n,\mathbb{R}^N) \to C^\infty(\mathbb{T}^n,\mathbb{R}^d)$$

such that

$$\|Mh\|_{2,\alpha} \leqslant a{\cdot}\|h\|_{2,\alpha}$$

for some constant a.

3.8. Günther's lemma. *There exists a symmetric bilinear form*

$$\tilde{Q}\colon C^\infty(\mathbb{T}^n,\mathbb{R}^d)\times C^\infty(\mathbb{T}^n,\mathbb{R}^d) \to C^\infty(\mathbb{T}^n,\mathbb{R}^d)$$

such that

$$\begin{aligned} L\tilde{Q} &= Q,\\ \|\tilde{Q}(w,w)\|_{2,\alpha} &\leqslant b_2{\cdot}\|w\|_{2,\alpha}^2,\\ \|\tilde{Q}(w,w)\|_{k,\alpha} &\leqslant b_2{\cdot}\|w\|_{2,\alpha}{\cdot}\|w\|_{k,\alpha} + b_k{\cdot}\|w\|_{k-1,\alpha}^2 \end{aligned}$$

for any integer $k \geqslant 3$ and for some fixed constants $b_2, b_3, \dots$

C Perturbation

We now complete the proof of Nash's theorem, modulo the two lemmas stated in the previous section. Recall that $f\colon \mathbb{T}^n \to \mathbb{R}^d$ is a fixed smooth free map and $L(w) := 2{\cdot}Q(f,w)$.

3.9. Perturbation lemma. *There exists $r > 0$ such that the equation*

❶ $$L(w) + Q(w,w) = h$$

admits a smooth solution $w\colon \mathbb{T}^n \to \mathbb{R}^d$ *for any smooth metric* h *with* $\|h\|_{2,\alpha} < r$. *Moreover, we can assume that*

$$\|w\|_{2,\alpha} \leqslant c\cdot\|h\|_{2,\alpha}$$

for some constant c.

In the previous lecture, we showed that the restricted theorem (2.1) implies the main theorem (1.1). Now we will show that the perturbation lemma implies the restricted theorem (2.1), and therefore the main theorem (1.1).

Proof of 2.1 modulo 3.9. Let g be the metric tensor induced by f. Note that the equation

$$Q(f+w, f+w) = g + h$$

is equivalent to ❶. Indeed,

$$\begin{aligned} Q(f+w, f+w) &= Q(f,f) + 2\cdot Q(f,w) + Q(w,w) \\ &= g + L(w) + Q(w,w). \end{aligned}$$

Hence, the restricted theorem (2.1) follows. □

In the proof of the perturbation lemma, we will follow the argument of the classical inverse function theorem in Banach spaces; see, for example, [8].

Proof of 3.9. Recall that M, $\tilde{Q}$, a, and b_2 are provided by 3.7 and 3.8. Set $R = \frac{1}{10\cdot b_2}$ and $r = \frac{R}{10\cdot a}$. Assume $\|h\|_{2,\alpha} \leqslant r$. By 3.8,

$$\Phi\colon w \mapsto Mh - \tilde{Q}(w,w)$$

is a contraction on the closed ball $\overline{\mathrm{B}}[0,R] \subset C^{2,\alpha}(\mathbb{T}^n, \mathbb{R}^d)$.

Consider the sequence of maps $w_0, w_1, \ldots \colon \mathbb{T}^n \to \mathbb{R}^d$ defined by $w_0 = 0$ and $w_{n+1} = \Phi(w_n)$ for all n. Note that each w_n is smooth. Since Φ is a contraction, w_n converges in $C^{2,\alpha}(\mathbb{T}^n, \mathbb{R}^d)$ as $n \to \infty$; denote its limit by w. Then

$$\|w\|_{2,\alpha} \leqslant R \qquad \text{and} \qquad w = Mh - \tilde{Q}(w,w).$$

Since $LM = \mathrm{id}$ and $L\tilde{Q} = Q$, applying L to both sides yields that w solves ❶.

To finish the proof, it remains to show the following claim:

❷ *The sequence* $\|w_n\|_{k,\alpha}$ *is bounded for every integer* $k \geqslant 2$.

Indeed, since

$$\|w\|_{k,\alpha} \leqslant \varliminf_{n\to\infty} \|w_n\|_{k,\alpha}$$

it follows that $w\in C^{k,\alpha}$ for every k; hence $w\in C^\infty$.

We prove ❷ by induction on k. The base case $k=2$ is proved already.

By the induction hypothesis, $\|w_n\|_{k-1,\alpha}$ is bounded. In particular, there is a constant c such that

$$\|Mh\|_{k-1,\alpha}+b_k\cdot\|w_n\|^2_{k-1,\alpha}\leqslant c$$

for all n; here b_k is the constant from 3.8.

Using the second estimate in 3.8, we get

$$\begin{aligned}\|w_{n+1}\|_{k,\alpha}&\leqslant\|Mh\|+b_2\cdot\|w_n\|_{2,\alpha}\cdot\|w_n\|_{k,\alpha}+b_k\cdot\|w_n\|^2_{k-1,\alpha}\leqslant\\&\leqslant c+b_2\cdot R\cdot\|w_n\|_{k,\alpha}\leqslant\\&\leqslant c+\tfrac{1}{10}\cdot\|w_n\|_{k,\alpha}.\end{aligned}$$

In particular,

$$\|w_n\|_{k,\alpha}\leqslant 2\cdot c\qquad\Longrightarrow\qquad\|w_{n+1}\|_{k,\alpha}\leqslant 2\cdot c$$

for all n. Since $w_0=0$, Claim ❷ follows. □

3.10. Exercise. *Show that any n-dimensional Riemannian manifold admits an isometric embedding into an arbitrarily small ball in $\mathbb{R}^d$ for some d depending only on n.*

D Proof of Nash's lemma

Recall that $f\colon\mathbb{T}^n\to\mathbb{R}^d$ is a fixed free map; in particular, f is regular and smooth.

The tangent space T_p is the n-dimensional space spanned by the first-order partial derivatives $\partial_1 f,\dots,\partial_n f$ at p. Let $\mathrm{T}^\perp_p$ denote the orthogonal complement of T_p in $\mathbb{R}^d$. Further, define the osculating space T^2_p as the span of all first- and second-order partial derivatives $\partial_i f$ and $\partial_{ij}f$ at p, and set $\mathrm{N}_p:=\mathrm{T}^2_p\cap\mathrm{T}^\perp_p$.

Since f is free, the dimensions of the spaces T_p, $\mathrm{T}^\perp_p$, T^2_p, and N_p are n, $d-n$, $n+\frac{n(n+1)}{2}$, and $\frac{n(n+1)}{2}$, respectively. We will write T, $\mathrm{T}^\perp$, T^2, and N for the corresponding vector bundles over $\mathbb{T}^n$.

Proof of 3.7. Recall that $\mathbb{T}^n$ has global periodic coordinates.

Choose a metric tensor h on $\mathbb{T}^n$. Since all $\partial_i f$, $\partial_{ij}f$ are linearly independent, the equations $\langle\partial_{ij}f,w\rangle=h_{ij}$ for all i and j uniquely define a N-vector field w. Equivalently, w can be defined as a linear combination of $\partial_i f$, $\partial_{ij}f$ such that

❶ $$\langle\partial_i f,w\rangle=0,\quad -2\cdot\langle\partial_{ij}f,w\rangle=h_{ij}$$

at each point.

Note that

$$\begin{aligned} L_{ij}(w) &= 2{\cdot}Q_{ij}(f,w) = \\ &= \langle \partial_i f, \partial_j w\rangle + \langle \partial_j f, \partial_i w\rangle = \\ &= \partial_j\langle \partial_i f, w\rangle + \partial_i\langle \partial_j f, w\rangle - 2{\cdot}\langle \partial_{ij} f, w\rangle. \end{aligned}$$

By ❶, the operator $M\colon h\mapsto w$ serves as the desired right inverse. □

This proof implies the following.

3.11. Observation. *The inverse M in Nash's lemma (3.7) can be chosen so that Mh is an* N*-field; that is, $Mh(p)\in \mathrm{N}_p$ for all $p\in\mathbb{T}^n$.*

The following exercise provides a key to proof of Günther's lemma.

3.12. Exercise. *Let $f\colon \mathbb{T}^n\to\mathbb{R}^d$ be a free map. Given a tangent vector field v on $\mathbb{T}^n$ and a metric tensor h on $\mathbb{T}^n$, show that there is a unique* N*-field w such that*

$$L(v+w)=h.$$

In other words, given v, there is a right inverse $M_v\colon h\mapsto v+w$ of L, where w is an N*-field.*

E Proof of Günther's lemma

There are many symmetric bilinear forms that solve the equation $L\tilde Q = Q$. In particular, one can take $\tilde Q = MQ$, where M is provided by Nash's lemma (3.7). However, this choice of $\tilde Q$ does not satisfy the inequalities required by Günther's lemma. These inequalities are essential; they enable the use of the contraction mapping theorem in the proof of the perturbation lemma (3.9).

The actual construction of $\tilde Q$ relies on Exercise 3.12, which states that, given a tangent vector field v, we can choose M such that $w = Mh - v$ is an N-field. The choice of v provides additional freedom, which might seem unnecessary at first.

Indeed, moving the map in a tangential direction does not affect the Riemannian metric to first order; it only changes the parametrization.[1] However, as we will see, v can be used to absorb high-frequency terms (see ❷ and ❼ below), which desolves the loss-of-derivatives problem—the main difficulty in Nash's original proof.

[1] Compare this to the DeTurck trick used in the Ricci flow.

The Laplacian is defined by

$$\Delta u = \sum_s \partial_{ss} u,$$

where $\partial_1, \dots, \partial_n$ are our standard partial derivatives on $\mathbb{T}^n$.

Let $k \geqslant 2$ be an integer and let $v \in C^{k-2,\alpha}(\mathbb{T}^n, \mathbb{R})$. Fourier analysis guarantees the existence of a weak solution u to the equation $\Delta u - u = v$. By the Schauder estimate, it follows that $u \in C^{k,\alpha}(\mathbb{T}^n, \mathbb{R})$, and we obtain the following proposition.[2]

3.13. Proposition. *Recall that $\alpha \in (0,1)$ is a fixed constant, say $\alpha = \frac{1}{2}$. Let $k \geqslant 2$ be an integer. Then the operator*

$$(\Delta - 1)\colon C^{k,\alpha}(\mathbb{T}^n, \mathbb{R}) \to C^{k-2,\alpha}(\mathbb{T}^n, \mathbb{R}).$$

is bi-Lipschitz. In particular, the linear operator

$$(\Delta - 1)\colon u \mapsto \Delta u - u$$

has a Lipschitz inverse

❶ $$(\Delta - 1)^{-1}\colon C^{k-2,\alpha}(\mathbb{T}^n, \mathbb{R}) \to C^{k,\alpha}(\mathbb{T}^n, \mathbb{R})$$

Note that $(\Delta - 1)^{-1}$ is a smoothing operator; it swallows a $C^{k-2,\alpha}$-function and spits a $C^{k,\alpha}$-function. Moreover, $(\Delta - 1)^{-1}$ commutes with partial derivatives; that is,

$$(\Delta - 1)^{-1}\partial_i = \partial_i(\Delta - 1)^{-1}.$$

These two properties of $(\Delta - 1)^{-1}$ will play a key role in the proof that follows.

Proof of 3.8. Recall that $Q_{ij}(w,w) = \langle \partial_i w, \partial_j w \rangle$. Let us show that

❷ $$(\Delta - 1)Q_{ij}(w,w) = A_{ij}(w) + \partial_i A_j(w) + \partial_j A_i(w),$$

where each A_{ij}, A_i and A_j is a linear combination of the following quadratic forms

❸ $$w \mapsto \langle \partial_a w, \partial_b w \rangle, \quad w \mapsto \langle \partial_{ab} w, \partial_c w \rangle, \quad w \mapsto \langle \partial_{ab} w, \partial_{cd} w \rangle.$$

for all indices a, b, c, d.

[2] A proof of the Schauder estimates can be found in [7, Chapter 3]; some parts can be omitted in our case.

Indeed,

$$\begin{aligned}(\Delta-1)Q_{ij}(w,w) &= -\langle\partial_i w,\partial_j w\rangle+\\ &\quad+\sum_s\left[2\cdot\langle\partial_{is}w,\partial_{js}w\rangle+\langle\partial_{iss}w,\partial_j w\rangle+\langle\partial_i w,\partial_{jss}w\rangle\right]=\\ &= -\langle\partial_i w,\partial_j w\rangle+2\cdot\sum_s\langle\partial_{is}w,\partial_{js}w\rangle-2\cdot\langle\Delta w,\partial_{ij}w\rangle+\\ &\quad+\partial_i\langle\Delta w,\partial_j w\rangle+\partial_j\langle\Delta w,\partial_i w\rangle.\end{aligned}$$

This proves ❷.

Now choose $A=A_i$ or $A=A_{ij}$ for some i and j. Note that

❹ $$\begin{aligned}\|A(w)\|_{0,\alpha}&\leqslant c_2\cdot\|w\|^2_{2,\alpha},\\ \|A(w)\|_{k-2,\alpha}&\leqslant c_2\cdot\|w\|_{k,\alpha}\cdot\|w\|_{2,\alpha}+c_k\cdot\|w\|^2_{k-1,\alpha}.\end{aligned}$$

for some fixed constants $c_2,c_3,\dots$ These inequalities follow by checking each form in ❸, which is straightforward.

Applying ❶, we get

❺ $$\begin{aligned}&\|(1-\Delta)^{-1}A(w)\|_{2,\alpha}\leqslant c_2\cdot\|w\|^2_{2,\alpha},\\ &\|(1-\Delta)^{-1}A(w)\|_{k,\alpha}\leqslant c_2\cdot\|w\|_{k,\alpha}\cdot\|w\|_{2,\alpha}+c_k\cdot\|w\|^2_{k-1,\alpha};\end{aligned}$$

these constants $c_2,c_3,\dots$ here may differ from those above, but each depends only on the map f.

Recall also that

❻ $$L_{ij}(w)=-2\cdot\langle\partial_{ij}f,w\rangle+\partial_i\langle\partial_j f,w\rangle+\partial_j\langle\partial_i f,w\rangle;$$

see Section 3D. Let us define $\tilde Q$ as a linear combination of $\partial_i f$ and $\partial_{ij}f$ for all $i\leqslant j$ such that

❼ $$\begin{aligned}-2\cdot\langle\partial_{ij}f,\tilde Q(w,w)\rangle&=(\Delta-1)^{-1}A_{ij}(w),\\ \langle\partial_i f,\tilde Q(w,w)\rangle&=(\Delta-1)^{-1}A_i(w)\end{aligned}$$

for all i and j. Since f is free, $\tilde Q$ is well-defined. Combining ❻, ❼, and ❷, we get

$$L\tilde Q=Q.$$

Let $\{e_i,e_{ij}\}_{i\leqslant j}$ be the dual frame to $\{\partial_i f,\partial_{ij}f\}_{i\leqslant j}$ in osculating bundle T^2. Note that

$$\tilde Q(w,w)=\sum_i(\Delta-1)^{-1}A_i(w)\cdot e_i-\tfrac12\cdot\sum_{i\leqslant j}(\Delta-1)^{-1}A_{ij}(w)\cdot e_{ij}.$$

Since the maps $e_i,e_{ij}\colon\mathbb{T}^n\to\mathbb{R}^d$ are smooth, ❺ together with 3.6 imply the final inequality in the lemma for all integers $k\geqslant 3$. □

Semisolutions

1.2. Choose a point $p \in \Omega$ and a chart at p.

Assume that the main theorem holds for some dimension d. Then the components $g_{ij} = g(\partial_i, \partial_j)$ of any Riemannian metric can be written as

$$\text{❽} \qquad \langle \partial_i w, \partial_j w \rangle = g_{ij},$$

for some smooth map $w\colon \Omega \to \mathbb{R}^d$.

There are $\binom{n+1}{2} = n\cdot(n+1)/2$ components g_{ij}, each of which is a smooth function on Ω. The equation ❽ has d unknowns; namely, the coordinate functions of w. This suggests that d should be at least $\binom{n+1}{2}$. The remaining proof is very standard, we reduce the problem to an algebraic system between the jet spaces of w and g at the point p; the jet space corresponds to the space of Taylor polynomials of a given degree at p.

Denote by $J^m(g)$ the degree-m Taylor polynomial of g at p. This polynomial has $\binom{m+n}{n}$ coefficients in an $n\cdot(n+1)/2$-dimensional space. (Each coefficient corresponds to a certain partial derivative of g of order at most m at p.) Such polynomials form an open subset of a $\binom{n+1}{2}\cdot\binom{m+n}{n}$-dimensional space.

Similarly, let $J^m(w)$ be the degree-m Taylor polynomial of w at p. This polynomial has $\binom{m+n}{n}$ coefficients in a d-dimensional space; hence these polynomials form a space of dimension $d\cdot\binom{m+n}{n}$.

The equations ❽ define a smooth map $J^{m+1}(w) \mapsto J^m(g)$. In other words, any partial derivative of g of degree at most m can be expressed in terms of partial derivatives of w of degree at most $m+1$. Since the system ❽ admits a solution for any metric g, there exists a $J^{m+1}(w)$ mapping to any given $J^m(g)$. By Sard's lemma,

$$d\cdot\binom{m+n+1}{n} \geqslant \binom{n+1}{2}\cdot\binom{m+n}{n}.$$

Since $\binom{m+n+1}{n}/\binom{m+n}{n} \to 1$ as $m \to \infty$, we conclude that $d \geqslant \binom{n+1}{2}$.

Remark. The proof is taken from [2, Appendix I] and it works for general system of partial differential equations.

Equation ❽ makes perfect sense for C^1-smooth maps w, but the proof essentially uses that w is C^m-smooth for sufficiently large m. Without this assumption, the statement does not hold.

Indeed, the Nash–Kuiper theorem [6, 9] states that equation ❽ admits a C^1-smooth solution w if $d = n + 1$. This bound is optimal, since if $d = n$, then the metric g must be flat, which is not the case in general. Note that for $d = n + 1$, the system ❽ is overdetermined, yet still admits a C^1-smooth solution.

It might happen that the optimal dimension for C^2-embeddings is smaller than $\binom{n+1}{2}$; at least the our argument gives a worse bound in this case. This question mentioned by Mikhael Gromov and Vladimir Rokhlin [2].[3]

1.3. Consider square of a parallel 1-form α on $\mathbb{T}^2$ with kernel in irrational direction.

1.4. The only-if part is trivial.

Apply the length-preserving property to all arcs of curves in the coordinate directions to show that the equality

$$g_{ii} = \langle \partial_i w, \partial_i w \rangle$$

holds almost everywhere, and therefore everywhere.

Observe that

$$2{\cdot}g_{ij} = g(\partial_i + \partial_j, \partial_i + \partial_j) - g(\partial_i, \partial_i) - g(\partial_j, \partial_j).$$

Using this formula together with the argument above for curves in the direction ∂_i, ∂_j and $\partial_i + \partial_j$, show that the equality

$$g_{ij} = \langle \partial_i w, \partial_j w \rangle$$

holds almost everywhere, and therefore everywhere.

1.5. Assume the contrary; let $w\colon \mathbb{R}^2 \to \mathbb{R}$ be a length-preserving map.

Note that w is Lipschitz. By Rademacher's theorem [12], the differential $d_x w$ is defined for almost all x. Argue as in 1.4 to show that $|\partial_1 w| = |\partial_2 w| = 1$, but $\partial_1 w{\cdot}\partial_2 w = 0$ almost everywhere and arrive at a contradiction.

[3]Counting degrees of freedom of curvature tesor one can get $d \geqslant \frac{1}{6}{\cdot}n{\cdot}(n+5)$. Indeed the curvature nensor has $\frac{1}{12}{\cdot}n^2{\cdot}(n^2-1)$ idependent components, and each quadratic form has $n(n+1)$ idependent components. Applying Gauss formula, we get that the codimension of the embedding has to be at least $\frac{1}{6}{\cdot}n{\cdot}(n+1)$.

Remark. This argument also shows that there is no length-preserving map from an n-dimensional Riemannian manifold to $\mathbb{R}^{n-1}$. On the other hand, a theorem of Mikhael Gromov [1, Section 2.4.11] implies that *any n-dimensional Riemannian manifold admits a length-preserving map into $\mathbb{R}^n$*. This is a close relative of the Nash–Kuiper theorem [6, 9]. Another proof of Gromov's theorem is outlined in the previously mentioned lecture notes [11].

1.7+1.8. Spell out the definitions.

1.14. Straightforward calculations.

2.5. By the definition of a submanifold, we can cover Ω by rectangular charts in which Ω is expressed as the coordinate subspace of the first $k = \dim\Omega$ coordinates. Note that in each such chart, the metric on Ω can be written as the restriction of a Riemannian metric on the entire chart.

Extend this collection of charts to an atlas of $\mathbb{T}^n$ such that the images of the remaining charts do not intersect Ω. Equip these new charts with constant Riemannian metrics.

Choose a partition of unity subordinate to the chart covering, and use it to patch together the local metrics into a smooth Riemannian metric on $\mathbb{T}^n$.

2.6. Let $\iota\colon \Omega \to \mathbb{R}^d$ be an embedding provided by Whitney's theorem. Think of $\mathbb{R}^d$ as the affine subspace $\{x_{d+1} = 1\}$ in $\mathbb{R}^{d+1}$. The desired embedding can be found among maps of the form $x \mapsto \varphi{\cdot}\iota(x)$, where $\varphi\colon \Omega \to \mathbb{R}$ is a smooth positive function.

Try writing down the necessary conditions and convince yourself that such a function φ exists.

2.7. Let $\mathbb{D} \subset \mathbb{R}^n$ be the unit n-disc, and let $\mathbb{D}^\circ$ denote its interior. Construct a smooth map $r\colon \mathbb{R}^n \to \mathbb{S}^n$ such that $r|_{\mathbb{D}^\circ}$ is a smooth embedding, and the set $\mathbb{R}^n \setminus \mathbb{D}^\circ$ is mapped to the south pole.

Since each $D_\mathfrak{i}$ is a smooth n-disc, there exists a diffeomorphism $\iota_\mathfrak{i}\colon \mathbb{D} \to D_\mathfrak{i}$. Then the composition $r_\mathfrak{i} = r \circ \iota_\mathfrak{i}^{-1}$ satisfies the first two conditions.

Now, let us construct a metrics $h_\mathfrak{i}$ on $\mathbb{S}^n$ that meet the last condition. Start by choosing sufficiently small metrics $\hat h_\mathfrak{i}$ on $\mathbb{S}^n$ such that

$$g > \hat g = \sum_\mathfrak{i} r_\mathfrak{i}^* \hat h_\mathfrak{i},$$

and let $g_1 = g - \hat g$.

Choose a partition of unity $\varphi_\mathfrak{i}$ subordinate to the covering $\{D_\mathfrak{i}^\circ\}$; that is, $\sum \varphi_\mathfrak{i} = 1$, each $\varphi_\mathfrak{i} \geqslant 0$, and the support of $\varphi_\mathfrak{i}$ lies in $D_\mathfrak{i}^\circ$ for each $\mathfrak{i}$. Denote by $s_\mathfrak{i}$ the inverse of the restriction $r_\mathfrak{i}|_{D_\mathfrak{i}^\circ}$.

Observe that $s_{\mathfrak{i}}^*(\varphi_{\mathfrak{i}}\cdot g_1)$ is a nonnegative metric on $\mathbb{S}^n$ and $\varphi_{\mathfrak{i}}\cdot g_1 = r_{\mathfrak{i}}^* s_{\mathfrak{i}}^*(\varphi_{\mathfrak{i}}\cdot g_1)$ for each $\mathfrak{i}$. Therefore, the metric

$$h_{\mathfrak{i}} = \hat{h}_{\mathfrak{i}} + s_{\mathfrak{i}}^*(\varphi_{\mathfrak{i}}\cdot g_1)$$

is Riemannian. Verify that each $h_{\mathfrak{i}}$ satisfies the last condition.

3.1. Show and use that a smooth map $f\colon \Omega \to \mathbb{R}^d$ is free at $p \in \Omega$ if there is no affine subspace A of dimension less than $\frac{n\cdot(n+3)}{2}$ that has second-order contact with f at p; that is,

$$\rho \circ f(x) = o(|p - x|^2),$$

where $\rho(z)$ denotes the distance from z to A.

Alternatively, express the partial derivatives in one chart in terms of the partial derivatives in another chart and draw the conclusion.

3.2. The definition of a free map requires the linear independence of certain partial derivatives. Count them.

3.3. Spell the definition.

3.4. This can be done by computing the partial derivatives. Alternatively, one may use the observation about the order of contact with an affine subspace from 3.1.

In the higher-dimensional case, one may take

$$\bigoplus_{i\leqslant j} \Theta_{x_i+x_j},$$

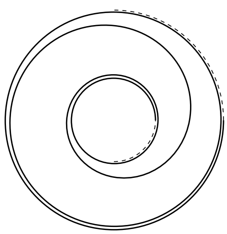
where x_i are coordinate functions. The same argument should work.

3.5. Use the definition of free map and 3.1.

3.6. Apply the product rule for derivatives.

3.10. Check that all constructions in the proof of the main theorem can be carried out in an arbitrarily small neighborhood of the origin.

Alternatively, construct an isometric embedding of $\mathbb{R}^d$ into a small neighborhood of the origin in $\mathbb{R}^{2\cdot d}$ (take product of embeddings $\mathbb{R} \hookrightarrow \mathbb{R}^2$ shown on the picture) and compose it with an isometric embedding $(\Omega, g) \hookrightarrow \mathbb{R}^d$ provided by the main theorem.



3.12. Note that the equation can be written as $L(w) = \hat{h}$, where $\hat{h} = h - L(v)$. Then argue as in Nash's lemma.

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