Lectures on metric geometry

Anton Petrunin

These lecture notes for a graduate course given at Penn State, Spring 2020. Considerable part of the text is a compilation from [4, 5, 108, 112, 113] and its drafts.

I want to thank Sergei Ivanov, Urs Lang, Alexander Lytchak, Luigi Massacci, Rostislav Matveyev, Julien Melleray, and Sergio Zamora Barrera for their help.

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Part I

Pure metric geometry

Lecture 1

Definitions

In this lecture, we remind several definitions related to metric spaces and fix some conventions.

This lecture is self-contained, but it is written for students with some prior knowledge of metric spaces; an introduction to general topology is sufficient but not necessary. For a more detailed introduction, we recommend the first couple of chapters in the book by Dmitri Burago, Yuri Burago, and Sergei Ivanov [34].

A Metric spaces

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

Let us recall the definition of metric.

1.1. Definition. *A metric on a set $\mathcal{X}$ is a real-valued function $(x,y) \mapsto |x-y|_{\mathcal{X}}$ that satisfies the following conditions for any $x, y, z \in \in \mathcal{X}$:*

(a) $|x-y|_{\mathcal{X}} \geqslant 0$,

(b) $|x-y|_{\mathcal{X}} = 0 \iff x = y$,

(c) $|x-y|_{\mathcal{X}} = |y-x|_{\mathcal{X}}$,

(d) $|x-y|_{\mathcal{X}} + |y-z|_{\mathcal{X}} \geqslant |x-z|_{\mathcal{X}}$.

Recall that a metric space is a set with a metric on it. The elements of the set are called points. Most of the time we keep the same notation for the metric space and its underlying set; the latter can be denoted by $\underline{\mathcal{X}}$ if needed.

Given radius $R \in [0, \infty]$ and center $x \in \mathcal{X}$, the sets

$$\begin{aligned} \mathrm{B}(x, R) &= \{\, y \in \mathcal{X} \;:\; |x - y| < R \,\}, \\ \overline{\mathrm{B}}[x, R] &= \{\, y \in \mathcal{X} \;:\; |x - y| \leqslant R \,\} \end{aligned}$$

are called, respectively, the open and the closed balls. The notations $\mathrm{B}(x, R)_{\mathcal{X}}$ and $\overline{\mathrm{B}}[x, R]_{\mathcal{X}}$ might be used if we need to emphasize that these balls are taken in the metric space $\mathcal{X}$.

1.2. Exercise. *Show that the following inequality*

$$|p - q|_{\mathcal{X}} + |x - y|_{\mathcal{X}} \leqslant |p - x|_{\mathcal{X}} + |p - y|_{\mathcal{X}} + |q - x|_{\mathcal{X}} + |q - y|_{\mathcal{X}}$$

holds for any four points p, q, x, and y in a metric space $\mathcal{X}$.

B Topology

The standard calculus definitions of closed and open sets, continuous functions, and converging sequences admit straightforward generalizations in the context of metric spaces.

1.3. Exercise. *Let x be a point in a metric space $\mathcal{X}$. Show that the distance function* $\mathrm{dist}_x\colon \mathcal{X} \to \mathbb{R}$ *defined by*

$$\mathrm{dist}_x\colon y \mapsto |x - y|$$

is continuous.

1.4. Exercise. *Let A and B be two disjoint closed sets in a metric space $\mathcal{X}$. Construct a continuous function $f\colon \mathcal{X} \to [0, 1]$ such that $A = f^{-1}\{0\}$ and $B = f^{-1}\{1\}$.*

1.5. Advanced exercise. *Let $f\colon A \to \mathbb{R}$ be a continuous function defined on a closed set A in a metric space $\mathcal{X}$. Show that it admits a continuous extension to the whole space; that is, there is a continuous function $F\colon \mathcal{X} \to \mathbb{R}$ such that $F(a) = f(a)$ for any $a \in A$.*

C Variations

Pseudometris. A metric for which the distance between two distinct points can be zero is called a semimetric (also known as pseudometric). In other words, to define semimetric, we need to remove condition *(b)* from 1.1.

Assume $\mathcal{X}$ is a semimetric space. Consider an equivalence relation $\sim$ on $\mathcal{X}$ defined by

$$x \sim y \qquad \Longleftrightarrow \qquad |x - y| = 0.$$

Note that if $x \sim x'$, then $|y - x| = |y - x'|$ for any $y \in \mathcal{X}$. Thus, $|{*} - {*}|$ defines a metric on the quotient $\mathcal{X}/{\sim}$. The so-obtained metric space, say $\mathcal{X}'$, is called the corresponding metric space for the semimetric space $\mathcal{X}$.

This construction shows that nearly any question about semimetric spaces can be reduced to a question about genuine metric spaces. Often we do not distinguish between a semimetric space $\mathcal{X}$ and its corresponding metric space $\mathcal{X}'$.

∞-metrics. One may also consider metrics with values in $[0, \infty]$; that is, we allow infinite distance between points. We might call them ∞-metrics, but most of the time we use the term *metric*.

The following construction shows how to reduce questions about ∞-metrics to genuine metrics.

Let

$$x \approx y \qquad \Longleftrightarrow \qquad |x - y| < \infty;$$

it defines another equivalence relation on $\mathcal{X}$. The equivalence class of a point $x \in \mathcal{X}$ will be called the metric component of x; it will be denoted by $\mathcal{X}_x$. Note that

$$\mathcal{X}_x = \mathrm{B}(x, \infty)_{\mathcal{X}};$$

that is, the metric component of x is the open ball centered at x and radius ∞.

If $\{\mathcal{X}_\alpha\}$ is a collection of metric spaces, then disjoint union $\boldsymbol{X} = \bigsqcup_\alpha \mathcal{X}_\alpha$ will be considered with a natural metric defined by

$$|x - y|_{\boldsymbol{X}} := \begin{cases} |x - y|_{\mathcal{X}_\alpha} & \text{if} \quad x, y \in \mathcal{X}_\alpha \quad \text{for some} \quad \alpha, \\ \infty & \text{otherwise.} \end{cases}$$

It follows that any ∞-metric space is a disjoint union of genuine metric spaces — the metric components of the original ∞-metric space.

1.6. Exercise. *Given two sets A and B on the plane, set*

$$|A - B| = \mu(A \triangle B),$$

where μ denotes the Lebesgue measure and $\triangle$ denotes symmetric difference

$$A \triangle B := (A \cup B) \setminus (B \cap A) = (A \setminus B) \cup (B \setminus A).$$

(a) Show that $|{*}-{*}|$ is a semimetric on the set of bounded closed subsets.

(b) Show that $|{*}-{*}|$ is an ∞-metric on the set of all open subsets.

D Maximal metric and gluing

Maximal metric. Let $\{\,|\ -\ |_\alpha\,\}$ be a family of ∞-semimetrics on a fixed set. Observe that

$$|x-y| := \sup_\alpha\{\,|x-y|_\alpha\,\}$$

defines an ∞-semimetric; it is called the maximal metric of the family.

Gluing. Suppose $\sim$ is an equivalence relation on an ∞-semimetric space $\mathcal{X}$. Given $x \in \mathcal{X}$, denote by $[x]$ its equivalence class in the quotient $\mathcal{X}/\sim$. Consider all ∞-semimetrics $|\ -\ |_\alpha$ on $\mathcal{X}/\sim$ such that the maps $\mathcal{X} \to \mathcal{X}/\sim$ defined by $x \mapsto [x]$ are short; that is,

$$|[x]-[x']|_\alpha \leqslant |x-x'|_{\mathcal{X}}$$

for any $x, x' \in \mathcal{X}$. Let us equip $\mathcal{X}/\sim$ with the maximal metric of this family; in general, it is an ∞-semimetric. The space $\mathcal{Z}$ that corresponds to the obtained ∞-semimetric space is called gluing of $\mathcal{X}$ along $\sim$.

This definition can be applied to a disjoint union of spaces; this way we can glue an arbitrary collection of metric spaces.

Note that any partially defined map φ from $\mathcal{X}$ to $\mathcal{Y}$ defines a minimal equivalence relation on $\mathcal{X} \sqcup \mathcal{Y}$ such that $x \sim \varphi(x)$; the corresponding gluing space is called gluing along φ.

The following exercise shows that metric gluing and the corresponding topological gluing might have different topologies.

1.7. Exercise. *Construct a homeomorphism $\varphi\colon [0,1] \to [0,1]$ such that gluing of two unit intervals $[0,1]$ along φ is a one-point metric space.*

E Completeness

A metric space $\mathcal{X}$ is called complete if every Cauchy sequence of points in $\mathcal{X}$ converges in $\mathcal{X}$.

1.8. Exercise. *Suppose that ρ is a positive continuous function on a complete metric space $\mathcal{X}$ and $\varepsilon > 0$. Show that there is a point $x \in \mathcal{X}$ such that*

$$\rho(x) < (1+\varepsilon)\cdot\rho(y)$$

for any point $y \in \mathrm{B}(x, \rho(x))$.

Most of the time we will assume that a metric space is complete. The following construction produces a complete metric space $\bar{\mathcal{X}}$ for any given metric space $\mathcal{X}$.

Completion. Given a metric space $\mathcal{X}$, consider the set $\mathcal{C}$ of all Cauchy sequences in $\mathcal{X}$. Note that for any two Cauchy sequences (x_n) and (y_n) the right-hand side in ❶ is defined; moreover, it defines a semimetric on $\mathcal{C}$

❶ $$|(x_n) - (y_n)|_{\mathcal{C}} := \lim_{n\to\infty} |x_n - y_n|_{\mathcal{X}}.$$

The corresponding metric space is called the completion of $\mathcal{X}$; it will be denoted by $\bar{\mathcal{X}}$.

For each point $x \in \mathcal{X}$, one can consider a constant sequence $x_n = x$ which is Cauchy. It defines a natural inclusion map $\mathcal{X} \hookrightarrow \bar{\mathcal{X}}$. It is easy to check that this map is distance-preserving. In particular, we can (and will) consider $\mathcal{X}$ as a subset of $\bar{\mathcal{X}}$.

Note that $\mathcal{X}$ is a dense subset in its completion $\bar{\mathcal{X}}$.

1.9. Exercise. *Show that the completion of a metric space is complete.*

F G-delta sets

1.10. Baire's theorem. *For any sequence $\Omega_1, \Omega_2, \ldots$ of open dense subsets in a complete metric space, the intersection $\bigcap_{n\in\mathbb{N}} \Omega_n$ is dense.*

A subset is called a G-delta if it can be presented as an intersection of a countable number of open subsets. Note that by Baire's theorem, a countable intersection of dense G-delta sets is a dense G-delta set — in particular it is nonempty. Therefore we are allowed to say that a dense G-delta set contains *most* of the points in a complete metric space.

Proof. We may assume that the space is nonempty; otherwise, there is nothing to prove.

Given a closed ball $\overline{\mathrm{B}}[p_0, R_0]$, let us apply induction to construct a nested sequence of closed balls

$$\overline{\mathrm{B}}[p_0, R_0] \supset \overline{\mathrm{B}}[p_1, R_1] \supset \overline{\mathrm{B}}[p_2, R_2] \supset \ldots$$

such that $\overline{\mathrm{B}}[p_n, R_n] \subset \Omega_n$ and $R_n > 0$ for each $n \geqslant 1$. Assume $\overline{\mathrm{B}}[p_{n-1}, R_{n-1}]$ is already constructed. Since Ω_n is dense we can choose a closed ball $\overline{\mathrm{B}}[p_n, R_n] \subset \Omega_n \cap \overline{\mathrm{B}}[p_{n-1}, R_{n-1}]$.

Note that we can assume that $R_n < \frac{1}{n}$ for each $n \geqslant 1$. In this case, the sequence $p_1, p_2, \dots$ is Cauchy; therefore, it is converging. Observe that its limit p_∞ belongs to each Ω_n. It follows that any closed ball $\overline{\mathrm{B}}[p_0, R_0]$ contains a point in $\bigcap_{n\in\mathbb{N}} \Omega_n$, hence the result. □

G Compact spaces

Let us recall a few statements about compact metric spaces.

1.11. Definition. *A metric space $\mathcal{K}$ is compact if and only if one of the following equivalent conditions holds:*

(a) Every open cover of $\mathcal{K}$ has a finite subcover.

(b) Every sequence of points in $\mathcal{K}$ has a subsequence that converges in $\mathcal{K}$.

(c) The space $\mathcal{K}$ is complete and totally bounded; that is, for any $\varepsilon > 0$, the space $\mathcal{K}$ admits a finite cover by open ε-balls.

Totally bounded spaces are also called precompact. Note that a *space is precompact if and only if its completion is compact.*

1.12. Lebesgue lemma. *Let $\mathcal{K}$ be a compact metric space. Then for any open cover of $\mathcal{K}$, there is $\varepsilon > 0$ such that any ε-ball in $\mathcal{K}$ lies in an element of the cover.*

The value ε is called a Lebesgue number of the covering.

A subset N of a metric space $\mathcal{K}$ is called ε-net if any point $x \in \mathcal{K}$ lies at the distance less than ε from a point in N. More generally, a subset N is called an ε-net of a subset $S \subset \mathcal{K}$ if any point $x \in S$ lies at the distance less than ε from a point in N.

Note that totally bounded spaces can be defined as spaces that admit a finite ε-net for any $\varepsilon > 0$.

1.13. Exercise. *Show that a space $\mathcal{K}$ is totally bounded if and only if it contains a compact ε-net for any $\varepsilon > 0$.*

Let $\operatorname{pack}_\varepsilon \mathcal{X}$ be the exact upper bound on the number of points $x_1, \dots, x_n \in \mathcal{X}$ such that $|x_i - x_j| \geqslant \varepsilon$ if $i \neq j$.

If $n = \operatorname{pack}_\varepsilon \mathcal{X} < \infty$, then the collection of points $x_1, \dots, x_n$ is called a maximal ε-packing. If $\mathcal{X}$ is a length space (see Section 1L), then n is the maximal number of disjoint open $\frac{\varepsilon}{2}$-balls in $\mathcal{X}$.

1.14. Exercise. *Show that any maximal ε-packing is an ε-net. Conclude that a complete space $\mathcal{X}$ is compact if and only if* $\operatorname{pack}_\varepsilon \mathcal{X} < \infty$ *for any $\varepsilon > 0$.*

1.15. Exercise. *Let $\mathcal{K}$ be a compact metric space and $f\colon \mathcal{K} \to \mathcal{K}$ be a distance-noncontracting map. Prove that f is an isometry; that is, f is a distance-preserving bijection.*

A metric space $\mathcal{X}$ is called locally compact if any point in $\mathcal{X}$ admits a compact neighborhood; equivalently, for any point $x \in \mathcal{X}$, a closed ball $\overline{\mathrm{B}}[x, r]$ is compact for some $r > 0$.

H Proper spaces

A metric space $\mathcal{X}$ is called proper if all closed bounded sets in $\mathcal{X}$ are compact. Note that $\mathcal{X}$ is proper if for some (and therefore any) point $p \in \mathcal{X}$ and any $R < \infty$, the closed ball $\overline{\mathrm{B}}[p, R]_{\mathcal{X}}$ is compact.

Recall that a function $f\colon \mathcal{X} \to \mathbb{R}$ is proper if, for any compact set $K \subset \mathbb{R}$, its inverse image $f^{-1}(K)$ is compact. Observe that $\mathcal{X}$ is proper if and only if the function $\mathrm{dist}_p\colon \mathcal{X} \to \mathbb{R}$ is proper for some (and therefore any) point $p \in \mathcal{X}$.

1.16. Exercise. *Give an example of a metric space that is locally compact but not proper.*

I Geodesics

Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ a real interval. A distance-preserving map $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a geodesic[1]; in other words, $\gamma\colon \mathbb{I} \to \mathcal{X}$ is a geodesic if

$$|\gamma(s) - \gamma(t)|_{\mathcal{X}} = |s - t|$$

for any pair $s, t \in \mathbb{I}$.

If $\gamma\colon [a, b] \to \mathcal{X}$ is a geodesic such that $p = \gamma(a)$, $q = \gamma(b)$, then we say that γ is a geodesic from p to q. In this case, the image of γ is denoted by $[pq]$, and, with abuse of notations, we also call it a geodesic. We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$.

In general, a geodesic from p to q need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we assume that we have chosen such geodesic.

A geodesic path is a geodesic with constant-speed parametrization by the unit interval $[0, 1]$.

A metric space is called geodesic if any pair of its points can be joined by a geodesic.

[1]Others call it differently: *shortest path*, *minimizing geodesic*. Also, note that the meaning of the term *geodesic* is different from what is used in Riemannian geometry, altho they are closely related.

An ∞-metric space $\mathcal{X}$ is called geodesic if each metric component of $\mathcal{X}$ is geodesic.

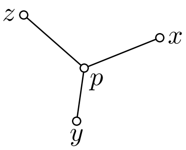
1.17. Exercise. *Let f be a centrally symmetric positive continuous function on $\mathbb{S}^2$. Given two points $x, y \in \mathbb{S}^2$, set*

$$\|x - y\| = \int\limits_{B(x,\frac{\pi}{2})\backslash B(y,\frac{\pi}{2})} f.$$

Show that $(\mathbb{S}^2, \|{*} - {*}\|)$ is a geodesic space, and the geodesics in $(\mathbb{S}^2, \|{*} - {*}\|)$ run along great circles of $\mathbb{S}^2$.

J Metric trees

A geodesic space $\mathcal{T}$ is called a metric tree if any two points in $\mathcal{T}$ are connected by a unique geodesic, and the union of any two geodesics $[xy]_{\mathcal{T}}$, and $[yz]_{\mathcal{T}}$ contains the geodesic $[xz]_{\mathcal{T}}$.



The latter means that any triangle in $\mathcal{T}$ is a tripod; that is, for any three points x, y, and z there is a point p such that

$$[xy] \cup [yz] \cup [zx] = [px] \cup [py] \cup [pz].$$

1.18. Exercise. *Let p, x, y, and z be points in a metric tree.*

(a) Consider three numbers

$$\begin{aligned} a &= |p - x| + |y - z|, \\ b &= |p - y| + |z - x|, \\ c &= |p - z| + |x - y|. \end{aligned}$$

Suppose that $a \leqslant b \leqslant c$. Show that $b = c$.

(b) Consider three numbers

$$\begin{aligned} \alpha &= \tfrac{1}{2}{\cdot}(|p - y| + |p - z| - |y - z|), \\ \beta &= \tfrac{1}{2}{\cdot}(|p - x| + |p - z| - |x - z|), \\ \gamma &= \tfrac{1}{2}{\cdot}(|p - x| + |p - y| - |x - y|). \end{aligned}$$

Suppose that $\alpha \leqslant \beta \leqslant \gamma$. Show that $\alpha = \beta$.

The set

$$S(p, r)_{\mathcal{X}} = \{\, x \in \mathcal{X} \;:\; |p - x|_{\mathcal{X}} = r \,\}$$

will be called a sphere with center p and radius r in a metric space $\mathcal{X}$.

1.19. Exercise. *Show that spheres in metric trees are ultrametric spaces. That is,*

$$|x-z| \leqslant \max\{\,|x-y|, |y-z|\,\}$$

for any $x, y, z \in S(p, r)_{\mathcal{T}}$.

K Length

A curve is defined as a continuous map from a real interval $\mathbb{I}$ to a metric space. If $\mathbb{I} = [0,1]$, then the curve is called a path.

1.20. Definition. *Let $\mathcal{X}$ be a metric space and $\alpha\colon \mathbb{I} \to \mathcal{X}$ be a curve. We define the length of α as*

$$\operatorname{length} \alpha := \sup_{t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n} \sum_i |\alpha(t_i) - \alpha(t_{i-1})|.$$

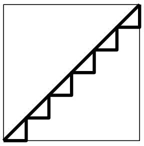
A curve α is called rectifiable if $\operatorname{length}\alpha < \infty$.

1.21. Theorem. *Length is a lower semi-continuous with respect to the pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\gamma_n\colon \mathbb{I} \to \mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty\colon \mathbb{I} \to \mathcal{X}$; that is, for any fixed $t \in \mathbb{I}$ we have $\gamma_n(t) \to \gamma_\infty(t)$ as $n \to \infty$. Then

❶
$$\varliminf_{n\to\infty} \operatorname{length}\gamma_n \geqslant \operatorname{length}\gamma_\infty.$$

Note that the inequality ❶ might be strict. For example, the diagonal γ_∞ of the unit square can be approximated by stairs-like polygonal curves γ_n with sides parallel to the sides of the square (γ_6 is on the picture). In this case



$$\operatorname{length}\gamma_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length}\gamma_n = 2$$

for any n.

Proof. Fix a sequence $t_0 \leqslant \ldots \leqslant t_k$ in $\mathbb{I}$. Set

$$\begin{aligned}\Sigma_n &:= |\gamma_n(t_0) - \gamma_n(t_1)| + \dots + |\gamma_n(t_{k-1}) - \gamma_n(t_k)|.\\ \Sigma_\infty &:= |\gamma_\infty(t_0) - \gamma_\infty(t_1)| + \dots + |\gamma_\infty(t_{k-1}) - \gamma_\infty(t_k)|.\end{aligned}$$

Note that for each i we have

$$|\gamma_n(t_{i-1}) - \gamma_n(t_i)| \to |\gamma_\infty(t_{i-1}) - \gamma_\infty(t_i)|$$

and therefore

$$\Sigma_n \to \Sigma_\infty$$

as $n \to \infty$. Note that

$$\Sigma_n \leqslant \operatorname{length} \gamma_n$$

for each n. Hence,

$$\varliminf_{n\to\infty} \operatorname{length} \gamma_n \geqslant \Sigma_\infty.$$

Since the partition was arbitrary, applying the definition of length, we get ❶. $\square$

1.22. Exercise. *Show that most of 1-Lipschitz paths in the plane have length 1.*

More precisely, consider the space $\mathcal{P}$ of 1-Lipschitz paths in the plane; that is, all paths $a\colon [0,1] \to \mathbb{R}^2$ such that $|a(t_0)-a(t_1)| \leqslant |t_0-t_1|$ for any t_0 and t_1. Equip $\mathcal{P}$ with the metric defined by

$$|a-b| := \sup\{\, |a(t)-b(t)| \;:\; t \in [0,1] \,\}.$$

Show that a dense G-delta set of paths in $\mathcal{P}$ has length 1.

L Length spaces

Let $\mathcal{X}$ be a metric space. If for any $\varepsilon > 0$ and any pair of points $x, y \in \mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length} \alpha < |x-y| + \varepsilon,$$

then $\mathcal{X}$ is called a length space and the metric on $\mathcal{X}$ is called a length metric.

An ∞-metric space is a length space if each of its metric components is a length space. In other words, if $\mathcal{X}$ is an ∞-metric space, then in the above definition we assume in addition that $|x-y|_{\mathcal{X}} < \infty$.

Note that any geodesic space is a length space. The following example shows that the converse does not hold.

1.23. Example. *Set $\mathbb{I}_n = [0, 1+\frac{1}{n}]$ for every natural n. Suppose a space $\mathcal{X}$ is obtained by gluing intervals $\{\mathbb{I}_n\}$, where the left ends are glued to p and the right ends to q.*

Observe that the space $\mathcal{X}$ carries a natural complete length metric with respect to which $|p-q|_{\mathcal{X}} = 1$, but there is no geodesic connecting p to q.

1.24. Exercise. *Give an example of a complete length space $\mathcal{X}$ such that no pair of distinct points in $\mathcal{X}$ can be joined by a geodesic.*

Directly from the definition, it follows that if $\alpha\colon [0,1] \to \mathcal{X}$ is a path from x to y (that is, $\alpha(0) = x$ and $\alpha(1) = y$), then

$$\operatorname{length}\alpha \geqslant |x - y|.$$

Set

$$\|x - y\| = \inf\{\operatorname{length}\alpha\}$$

where the greatest lower bound is taken for all paths from x to y. It is straightforward to check that $(x, y) \mapsto \|x - y\|$ is an ∞-metric; moreover, $(\mathcal{X}, \|{*} - {*}\|)$ is a length space. The metric $\|{*} - {*}\|$ is called the induced length metric.

1.25. Exercise. *Let $\mathcal{X}$ be a complete length space. Show that for any compact subset $K \subset \mathcal{X}$ there is a compact path-connected subset $K' \subset \mathcal{X}$ that contains K.*

1.26. Exercise. *Suppose $(\mathcal{X}, |{*}-{*}|)$ is a complete metric space. Show that $(\mathcal{X}, \|{*} - {*}\|)$ is complete.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x - y|_A = \inf_\alpha\{\operatorname{length}\alpha\},$$

where the greatest lower bound is taken for all paths α from x to y in A. In other words, $|{*} - {*}|_A$ denotes the induced length metric on the subspace A. (The notation $|{*} - {*}|_A$ conflicts with the previously defined notation for distance $|x - y|_\mathcal{X}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.)

Let x and y be points in a metric space $\mathcal{X}$.

(i) A point $z \in \mathcal{X}$ is called a midpoint between x and y if

$$|x - z| = |y - z| = \tfrac{1}{2}\cdot|x - y|.$$

(ii) Assume $\varepsilon \geqslant 0$. A point $z \in \mathcal{X}$ is called an ε-midpoint between x and y if

$$|x - z| \leqslant \tfrac{1}{2}\cdot|x - y| + \varepsilon \quad\text{and}\quad |y - z| \leqslant \tfrac{1}{2}\cdot|x - y| + \varepsilon.$$

Note that a 0-midpoint is the same as a midpoint.

1.27. Menger's lemma. *Assume $\mathcal{X}$ is a complete metric space.*

(a) Suppose that for any two points in $\mathcal{X}$, and any positive ε, there is an ε-midpoint. Then $\mathcal{X}$ is a length space.

(b) *Suppose that for any two points in $\mathcal{X}$, there is a midpoint. Then $\mathcal{X}$ is a geodesic space.*

The second part of this lemma was proved by Karl Menger [97, Section 6].

Proof; (a). Choose $x, y \in \mathcal{X}$; set $\varepsilon_n = \frac{\varepsilon}{4^n}$, $\alpha(0) = x$, and $\alpha(1) = y$.

Let $\alpha(\frac{1}{2})$ be an ε_1-midpoint between $\alpha(0)$ and $\alpha(1)$. Further, let $\alpha(\frac{1}{4})$ and $\alpha(\frac{3}{4})$ be ε_2-midpoints between the pairs $(\alpha(0), \alpha(\frac{1}{2}))$ and $(\alpha(\frac{1}{2}), \alpha(1))$ respectively. Continue the above procedure; on the n-th step, we define $\alpha(\frac{k}{2^n})$, for every odd integer k such that $0 < \frac{k}{2^n} < 1$, as an ε_n-midpoint of the already defined $\alpha(\frac{k-1}{2^n})$ and $\alpha(\frac{k+1}{2^n})$.

This way we define $\alpha(t)$ for all dyadic rationals t in $[0, 1]$. Moreover, α has Lipschitz constant $|x-y|+\varepsilon$. Since $\mathcal{X}$ is complete, the map α can be extended $(|x-y|+\varepsilon)$-Lipschitz map $\alpha\colon [0, 1] \to \mathcal{X}$. In particular

$$\text{❶} \qquad \operatorname{length} \alpha \leqslant |x-y| + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get *(a)*.

(b). Apply the same argument with midpoints instead of ε_n-midpoints. In this case, ❶ holds for $\varepsilon_n = \varepsilon = 0$. □

1.28. Exercise. *Suppose that for any two distinct points x and y in a complete metric space $\mathcal{X}$ there is yet another point z such that*

$$|x-z| + |y-z| = |x-y|.$$

Show that $\mathcal{X}$ is geodesic.

In a compact space, a sequence of $\frac{1}{n}$-midpoints z_n contains a convergent subsequence. Therefore Menger's lemma (1.27) implies the following.

1.29. Proposition. *Any proper length space is geodesic.*

1.30. Hopf–Rinow theorem. *Any complete, locally compact length space is proper.*

Before reading the proof, it is instructive to solve 1.16. In the proof, we will use the following exercise.

1.31. Exercise. *Let $\mathcal{X}$ be a length space. Show that $\mathrm{B}(x, R+\varepsilon)_{\mathcal{X}}$ is the ε-neighborhood of $\mathrm{B}(x, R)_{\mathcal{X}}$.*

Proof. Choose a point x in a locally compact length space $\mathcal{X}$. Let

$$\rho(x) := \sup \left\{\, R : \overline{\mathrm{B}}[x, R] \text{ is compact} \,\right\}.$$

Since $\mathcal{X}$ is locally compact,

❷ $$\rho(x) > 0 \quad \text{for any} \quad x \in \mathcal{X}.$$

It is sufficient to show that $\rho(x) = \infty$ for some (and therefore any) point $x \in \mathcal{X}$.

❸ *If $\rho(x) < \infty$, then $B = \overline{\mathrm{B}}[x, \rho(x)]$ is compact.*

Suppose $\rho(x) > \varepsilon > 0$; by 1.31, the set $\overline{\mathrm{B}}[x, \rho(x) - \varepsilon]$ is a compact $2{\cdot}\varepsilon$-net in B. Since B is closed and hence complete, it must be compact; see 1.11*c* and 1.13. △

❹ $|\rho(x) - \rho(y)| \leqslant |x - y|_{\mathcal{X}}$ *for any $x, y \in \mathcal{X}$; in particular, $\rho\colon \mathcal{X} \to \mathbb{R}$ is a continuous function.*

Suppose $\rho(x) + |x - y| < \rho(y)$ for some $x, y \in \mathcal{X}$. Then $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$ is a closed subset of $\overline{\mathrm{B}}[y, \rho(y)]$ for some $\varepsilon > 0$. Since $\overline{\mathrm{B}}[y, \rho(y)]$ is compact, so is $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$ — a contradiction. △

Let $\varepsilon = \min\{\, \rho(y) : y \in B \,\}$; the minimum is defined since B is compact and ρ is continuous. By ❷, we have $\varepsilon > 0$.

Choose a finite $\tfrac{\varepsilon}{10}$-net $\{a_1, a_2, \dots, a_n\}$ in $B = \overline{\mathrm{B}}[x, \rho(x)]$. The union W of the closed balls $\overline{\mathrm{B}}[a_i, \varepsilon]$ is compact. By 1.31, $\overline{\mathrm{B}}[x, \rho(x) + \tfrac{\varepsilon}{10}] \subset W$. Therefore, $\overline{\mathrm{B}}[x, \rho(x) + \tfrac{\varepsilon}{10}]$ is compact, a contradiction. □

1.32. Exercise. *Construct a geodesic space $\mathcal{X}$ that is locally compact, but whose completion $\bar{\mathcal{X}}$ is neither geodesic nor locally compact.*

1.33. Exercise. *Let $\mathcal{X}$ be a locally compact geodesic space. Suppose that any geodesic $\gamma\colon (a, b) \to \mathcal{X}$ can be extended to $[a, b]$. Show that $\mathcal{X}$ is proper.*

1.34. Advanced exercise. *Show that for any compact connected space $\mathcal{X}$ there is a number ℓ such that for any finite collection of points there is a point z that lies on average distance ℓ from the collection; that is, for any $x_1, \dots, x_n \in \mathcal{X}$ there is $z \in \mathcal{X}$ such that*

$$\tfrac{1}{n} \cdot \sum_i |x_i - z|_{\mathcal{X}} = \ell.$$

Lecture 2

Universal spaces

The Urysohn space is the main hero of this lecture. It shares some fundamental properties with classical spaces (spheres, Euclidean, and Lobachevsky spaces), but also has many counterintuitive properties.

This space often serves as a counterexample to plausible conjectures; so it is worth knowing it. In addition, this space is beautiful.

A Embedding in a normed space

Recall that a function $v \mapsto |v|$ on a vector space $\mathcal{V}$ is called norm if it satisfies the following condition for any two vectors $v, w \in \mathcal{V}$ and a scalar α:

- ◇ $|v| \geqslant 0$;
- ◇ $|\alpha{\cdot}v| = |\alpha|{\cdot}|v|$;
- ◇ $|v| + |w| \geqslant |v + w|$.

As an example, consider ℓ^∞ — the space of real sequences equipped with sup-norm; that is, the norm of $\boldsymbol{a} = (a_1, a_2, \dots)$ is defined by

$$|\boldsymbol{a}|_{\ell^\infty} := \sup_n \{\, |a_n| \,\}.$$

It is straightforward to check that for any normed space the function $(v, w) \mapsto |v - w|$ defines a metric on it. Therefore, any normed space is an example of metric space; moreover, it is a geodesic space. Often we do not distinguish normed space from the corresponding metric space. (By the Mazur–Ulam theorem, the metric remembers the affine structure of the space; so, to recover the original normed space we only need to specify the origin. A slick proof of this theorem was given by Jussi Väisälä [133].)

Recall that diameter of a metric space $\mathcal{X}$ (briefly $\operatorname{diam}\mathcal{X}$) is defined as the least upper bound on the distances between pairs of its points; that is,

$$\operatorname{diam}\mathcal{X} := \sup\{\, |x-y|_{\mathcal{X}} \;:\; x, y \in \mathcal{X}\,\}.$$

If $\operatorname{diam}\mathcal{X} < \infty$, then the space $\mathcal{X}$ is called bounded.

2.1. Lemma. *Suppose $\mathcal{X}$ is a bounded separable metric space; that is, $\mathcal{X}$ contains a countable, dense set, say $\{w_n\}$. Given $x \in \mathcal{X}$, set $a_n(x) = |w_n - x|_{\mathcal{X}}$. Then*

$$\iota\colon x \mapsto (a_1(x), a_2(x), \dots)$$

defines a distance-preserving embedding $\iota\colon \mathcal{X} \hookrightarrow \ell^\infty$.

Proof. By the triangle inequality

❶ $$|a_n(x) - a_n(y)| \leqslant |x-y|_{\mathcal{X}}.$$

Therefore, ι is short (in other words, ι is distance-nonexpanding).

Again by the triangle inequality we have

$$|a_n(x) - a_n(y)| \geqslant |x-y|_{\mathcal{X}} - 2{\cdot}|w_n - x|_{\mathcal{X}}.$$

Since the set $\{w_n\}$ is dense, we can choose w_n arbitrarily close to x. Whence

❷ $$\sup_n\{\, |a_n(x) - a_n(y)|\,\} \geqslant |x-y|_{\mathcal{X}};$$

that is, ι is distance-noncontracting.

Finally, observe that ❶ and ❷ imply the lemma. □

2.2. Exercise. *Show that any compact metric space $\mathcal{K}$ is isometric to a subspace of a compact geodesic space.*

The following exercise generalizes the lemma to arbitrary separable spaces.

2.3. Exercise. *Suppose $\{w_n\}$ is a countable, dense set in a metric space $\mathcal{X}$. Choose $x_0 \in \mathcal{X}$; given $x \in \mathcal{X}$, set*

$$a_n(x) = |w_n - x|_{\mathcal{X}} - |w_n - x_0|_{\mathcal{X}}.$$

Show that $\iota\colon x \mapsto (a_1(x), a_2(x), \dots)$ defines a distance-preserving embedding $\iota\colon \mathcal{X} \hookrightarrow \ell^\infty$.

Conclude that any separable metric space $\mathcal{X}$ admits a distance-preserving embedding $\iota\colon \mathcal{X} \hookrightarrow \ell^\infty$.

The following lemma implies that *any metric space is isometric to a subset of a normed vector space*; its proof is nearly identical to the proof of 2.3. Given a set $\mathcal{X}$, denote by $\ell^\infty(\mathcal{X})$ the space of all bounded functions on $\mathcal{X}$ equipped with sup-norm; that is,

$$|f - g|_{\ell^\infty} = \sup\{\, |f(x) - f(x)\ :\ x \in \mathcal{X}\,\}.$$

2.4. Lemma. *Let x_0 be a point in a metric space $\mathcal{X}$. Then the map $\iota\colon \mathcal{X} \to \ell^\infty(\mathcal{X})$ defined by*

$$\iota\colon x \mapsto (\mathrm{dist}_x - \mathrm{dist}_{x_0})$$

is distance-preserving.

In particular, any metric space $\mathcal{X}$ admits a distance-preserving into $\ell^\infty(\mathcal{X})$.

B Extension property

If a metric space $\mathcal{X}$ is a subspace of a semimetric space $\mathcal{X}'$, then we say that $\mathcal{X}'$ is an extension of $\mathcal{X}$. If in addition, $\operatorname{diam}\mathcal{X}' \leqslant d$, then we say that $\mathcal{X}'$ is a d-extension.

If the complement $\mathcal{X}' \setminus \mathcal{X}$ contains a single point, say p, then $\mathcal{X}'$ is called a one-point extension of $\mathcal{X}$. In this case, to define a metric on $\mathcal{X}'$, it is sufficient to specify the distance function from p; that is, a function $f\colon \mathcal{X} \to \mathbb{R}$ defined by

$$f(x) := |p - x|_{\mathcal{X}'}.$$

Any function f of that type will be called an extension function or d-extension function respectively.

The extension function f cannot be taken arbitrarily — the triangle inequality implies that

$$f(x) + f(y) \geqslant |x - y|_{\mathcal{X}} \geqslant |f(x) - f(y)|$$

for any $x, y \in \mathcal{X}$. In particular, f is a nonnegative 1-Lipschitz function on $\mathcal{X}$. For a d-extension, we need to assume in addition that $\operatorname{diam}\mathcal{X} \leqslant$ $\leqslant d$ and $f(x) \leqslant d$ for any $x \in \mathcal{X}$. A straightforward check shows that these conditions are necessary and sufficient.

2.5. Exercise. *Let $\mathcal{X}$ be a subspace of metric space $\mathcal{Y}$. Assume f is an extension function on $\mathcal{X}$.*

(a) *Show that*

$$\bar{f}(y) := \inf_{x \in \mathcal{X}} \{\, f(x) + |x - y|_{\mathcal{Y}} \,\}$$

defines an extension function on $\mathcal{Y}$.

(b) *Assume that* $\operatorname{diam} \mathcal{Y} \leqslant d$ *and* $f(x) \leqslant d$ *for any* $x \in \mathcal{X}$. *Show that*

$$\bar{f}_d := \min\{\, \bar{f}, d \,\}$$

is a d*-extension function on* $\mathcal{Y}$.

The functions $\bar{f}$ and $\bar{f}_d$ in the above exercise are called Katětov extensions of f and the minimal possible $\mathcal{X}$ is called its support, briefly $\operatorname{supp} \bar{f} = \mathcal{X}$.

2.6. Definition. *A metric space* $\mathcal{U}$ *meets the extension property if for any finite subspace* $\mathcal{F} \subset \mathcal{U}$ *and any extension function* $f \colon \mathcal{F} \to \mathbb{R}$ *there is a point* $p \in \mathcal{U}$ *such that* $|p - x| = f(x)$ *for any* $x \in \mathcal{F}$.

If we assume in addition that $\operatorname{diam} \mathcal{U} \leqslant d$ *and instead of extension functions we consider only* d*-extension functions, then it defines the* d*-extension property.*

Furhter, if, in addition, $\mathcal{U}$ *is separable and complete, then it is called Urysohn space or* d*-Urysohn space, respectively.*

2.7. Proposition. *There is a separable metric space with the (d-) extension property (for any* $d \geqslant 0$*).*

Proof. Choose $d \geqslant 0$. Let us construct a separable metric space with the d-extension property.

Let $\mathcal{X}$ be a metric space such that $\operatorname{diam} \mathcal{X} \leqslant d$. Denote by $\mathcal{X}^d$ the space of all d-extension functions on $\mathcal{X}$ equipped with the metric defined by the sup-norm. Note that the map $\mathcal{X} \to \mathcal{X}^d$ defined by $x \mapsto \operatorname{dist}_x$ is a distance-preserving embedding, so we can (and will) treat $\mathcal{X}$ as a subspace of $\mathcal{X}^d$; equivalently, $\mathcal{X}^d$ is an extension of $\mathcal{X}$.

Let us iterate this construction. Start with a one-point space $\mathcal{X}_0$ and consider a sequence of spaces $(\mathcal{X}_n)$ defined by $\mathcal{X}_{n+1} := \mathcal{X}_n^d$. Note that the sequence is nested; that is, $\mathcal{X}_0 \subset \mathcal{X}_1 \subset \dots$ and the union

$$\mathcal{X}_\infty = \bigcup_n \mathcal{X}_n;$$

comes with metric such that $|x - y|_{\mathcal{X}_\infty} = |x - y|_{\mathcal{X}_n}$ if $x, y \in \mathcal{X}_n$.

Note that if $\mathcal{X}$ is compact, then so is $\mathcal{X}^d$. It follows that each space $\mathcal{X}_n$ is compact. In particular, $\mathcal{X}_\infty$ is a countable union of compact spaces; therefore, $\mathcal{X}_\infty$ is separable.

Any finite subspace $\mathcal{F}$ of $\mathcal{X}_\infty$ lies in some $\mathcal{X}_n$ for $n < \infty$. By construction, given an extension function $f \colon \mathcal{F} \to \mathbb{R}$, there is a point

$p \in \mathcal{X}_{n+1}$ that meets the condition in 2.6. That is, $\mathcal{X}_\infty$ has the d-extension property.

The construction of a separable metric space with the extension property requires only two changes. First, the sequence should be defined by $\mathcal{X}_{n+1} := \mathcal{X}_n^{d_n}$, where d_n is an increasing sequence such that $d_n \to \infty$. Second, the point p should be taken in $\mathcal{X}_{n+k}$ for sufficiently large k, so that $d_{n+k} > \max\{f(x)\}$ (here one has to apply 2.5a).

(Alternatively, one can start with any separable space $\mathcal{X}_0$ and consider a nested sequence $\mathcal{X}_0 \subset \mathcal{X}_1 \subset \dots$ where $\mathcal{X}_{n+1}$ is the space of all extension functions on $\mathcal{X}_n$ with at most $n+1$ points in its support. The last condition is needed to keep $\mathcal{X}_n$ separable.) □

Given a metric space $\mathcal{X}$, denote by $\mathcal{X}^\infty$ the space of all extension functions on $\mathcal{X}$ equipped with the metric defined by the sup-norm.

2.8. Exercise. *Construct a proper length space $\mathcal{X}$ such that $\mathcal{X}^\infty$ is not separable.*

2.9. Proposition. *If a metric space $\mathcal{V}$ meets the (d-) extension property, then so does its completion.*

Proof. Let us assume $\mathcal{V}$ meets the extension property. We will show that its completion $\mathcal{U} = \bar{\mathcal{V}}$ meets the extension property as well. The d-extension case can be proved along the same lines.

Note that $\mathcal{V}$ is a dense subset in a complete space $\mathcal{U}$. Observe that $\mathcal{U}$ has the approximate extension property; that is, if $\mathcal{F} \subset \mathcal{U}$ is a finite set, $\varepsilon > 0$, and $f\colon \mathcal{F} \to \mathbb{R}$ is an extension function, then there exists $p \in \mathcal{U}$ such that

❶ $$|p - x| \lesssim f(x) \pm \varepsilon$$

for any $x \in \mathcal{F}$. Indeed, consider the Katětov extension $\bar{f}\colon \mathcal{U} \to \mathbb{R}$ of f. Since $\mathcal{V}$ is dense in $\mathcal{U}$, we can choose a finite set $\mathcal{F}' \in \mathcal{V}$ such that for any $x \in \mathcal{F}$ there is $x' \in \mathcal{F}'$ with $|x - x'| < \frac{\varepsilon}{2}$. Let p be the point provided by the extension property for the restriction $\bar{f}|_{\mathcal{F}'}$. It remains to observe p meets ❶.

It follows that there is a sequence of points $p_n \in \mathcal{U}$ such that for any $x \in \mathcal{F}$,

$$|p_n - x| \lesssim f(x) \pm \tfrac{1}{2^n}.$$

Moreover, we can assume that

❷ $$|p_n - p_{n+1}| < \tfrac{1}{2^n}$$

for all large n. Indeed, consider the sets $\mathcal{F}_n = \mathcal{F} \cup \{p_n\}$ and the functions $f_n\colon \mathcal{F}_n \to \mathbb{R}$ defined by $f_n(x) := f(x)$ and

$$f_n(p_n) := \max\left\{\, \big||p_n - x| - f(x)\big| \;:\; x \in \mathcal{F}\,\right\}$$

if $x \neq p_n$. Observe that f_n is an extension function for large n and $f_n(p_n) < \frac{1}{2^n}$. Therefore, applying the approximate extension property recursively, we get ❷.

Therefore, the sequence p_n is Cauchy. Note that its limit meets the condition in the definition of extension property (2.6). □

Note that 2.7 and 2.9 imply the following:

2.10. Theorem. *Urysohn space and d-Urysohn space exist for any $d > 0$.*

Here is a slightly stronger statement:

2.11. Theorem. *Any separable metric space $\mathcal{X}$ admits a distance-preserving embedding into an Urysohn space $\mathcal{U}$ such that any isometry of $\mathcal{X}$ can be extended to an isometry of $\mathcal{U}$.*

Sketch of proof. Start with $\mathcal{X}_0 = \mathcal{X}$ and construct a nested sequence of spaces $\mathcal{X}_0 \subset \mathcal{X}_1 \subset \dots$ as at the alternative end of the proof of 2.7. Note that any isometry $\mathcal{X}_n \to \mathcal{X}_n$ can be extended to a unique isometry $\mathcal{X}_{n+1} \to \mathcal{X}_{n+1}$. It follows that any isometry of $\mathcal{X}$ can be extended to an isometry of $\mathcal{X}' = \bigcup_n \mathcal{X}_n$.

Now, consider a new nested sequence $\mathcal{X} \subset \mathcal{X}' \subset \mathcal{X}'' \subset \dots$; denote its union by $\mathcal{Y}$. Arguing as in 2.7 and 2.9 we get that the completion of $\mathcal{Y}$ is an Urysohn space, say $\mathcal{U}$, that comes with a distance-preserving inclusion $\mathcal{X} \hookrightarrow \mathcal{U}$.

From above, any isometry of $\mathcal{X}$ can be extended to isometries of $\mathcal{X}'$, $\mathcal{X}''$, and so on. They all define an isometry of $\mathcal{Y}$; passing to its continuous extension, we get an isometry of $\mathcal{U}$. □

C Universality

A metric space will be called u n i v e r s a l if it has a subspace isometric to any given separable metric space. In 2.3, we proved that ℓ^∞ is a universal space. The following proposition shows that an Urysohn space is universal as well. Unlike ℓ^∞, Urysohn spaces are separable; so it might be considered as a *better* universal space. Theorem 2.20 will give another reason why Urysohn spaces are better.

2.12. Proposition. *An Urysohn space is universal. That is, if $\mathcal{U}$ is an Urysohn space, then any separable metric space $\mathcal{S}$ admits a distance-preserving embedding $\mathcal{S} \hookrightarrow \mathcal{U}$.*

Moreover, for any finite subspace $\mathcal{F} \subset \mathcal{S}$, any distance-preserving embedding $\mathcal{F} \hookrightarrow \mathcal{U}$ can be extended to a distance-preserving embedding $\mathcal{S} \hookrightarrow \mathcal{U}$.

A d-Urysohn space is d-universal; that is, the above statements hold provided that $\operatorname{diam} \mathcal{S} \leqslant d$*.*

Proof. We will prove the second statement; the first statement is its partial case for $\mathcal{F} = \varnothing$.

The required isometry will be denoted by $x \mapsto x'$.

Choose a dense sequence of points $s_1, s_2, \ldots \in \mathcal{S}$. We may assume that $\mathcal{F} = \{s_1, \ldots, s_n\}$, so $s_i' \in \mathcal{U}$ are defined for $i \leqslant n$.

The sequence s_i' for $i > n$ can be defined recursively using the extension property in $\mathcal{U}$. Namely, suppose that $s_1', \ldots, s_{i-1}'$ are already defined. Since $\mathcal{U}$ meets the extension property, there is a point $s_i' \in \mathcal{U}$ such that

$$|s_i' - s_j'|_{\mathcal{U}} = |s_i - s_j|_{\mathcal{S}}$$

for any $j < i$.

The constructed map $s_i \mapsto s_i'$ is distance-preserving. Therefore it can be continuously extended to the whole $\mathcal{S}$. It remains to observe that the constructed map $\mathcal{S} \hookrightarrow \mathcal{U}$ is distance-preserving. $\square$

2.13. Exercise. *Show that any two distinct points in an Urysohn space can be joined by an infinite number of distinct geodesics.*

2.14. Exercise. *Modify the proofs of 2.9 and 2.12 to prove the following theorem.*

2.15. Theorem. *Let K be a compact set in a separable space $\mathcal{S}$. Then any distance-preserving map from K to an Urysohn space can be extended to a distance-preserving map of the whole $\mathcal{S}$.*

2.16. Exercise. *Show that (d-) Urysohn space is simply-connected.*

D Uniqueness and homogeneity

2.17. Theorem. *Suppose $\mathcal{F} \subset \mathcal{U}$ and $\mathcal{F}' \subset \mathcal{U}'$ be finite isometric subspaces in a pair of (d-)Urysohn spaces $\mathcal{U}$ and $\mathcal{U}'$. Then any isometry $\iota\colon \mathcal{F} \leftrightarrow \mathcal{F}'$ can be extended to an isometry $\mathcal{U} \leftrightarrow \mathcal{U}'$.*

In particular, (d-)Urysohn space is unique up to isometry.

Note that 2.12 implies that there are distance-preserving maps $\mathcal{U} \to \mathcal{U}'$ and $\mathcal{U}' \to \mathcal{U}$. The next exercise shows that it does not solely imply the existence of an isometry $\mathcal{U} \leftrightarrow \mathcal{U}'$.

2.18. Exercise. *Construct two metric spaces $\mathcal{X}$ and $\mathcal{Y}$ such that there are distance-preserving maps $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{X}$, but no isometry $\mathcal{X} \leftrightarrow \mathcal{Y}$.*

The following construction uses the idea of 2.12, but it is applied back-and-forth to ensure that the obtained distance-preserving map is onto.

Proof. Choose dense sequences $a_1, a_2, \ldots \in \mathcal{U}$ and $b'_1, b'_2, \ldots \in \mathcal{U}'$. We can assume that $\mathcal{F} = \{a_1, \ldots, a_n\}$, $\mathcal{F}' = \{b'_1, \ldots, b'_n\}$ and $\iota(a_i) = b'_i$ for $i \leqslant n$.

The required isometry $\mathcal{U} \leftrightarrow \mathcal{U}'$ will be denoted by $u \leftrightarrow u'$. Set $a_i = b_i$ and $a'_i = b'_i$ if $i \leqslant n$.

Let us define recursively $a'_{n+1}, b_{n+1}, a'_{n+2}, b_{n+2}, \ldots$ — on the odd step we define the images of $a_{n+1}, a_{n+2}, \ldots$ and on the even steps we define inverse images of $b'_{n+1}, b'_{n+2}, \ldots$ The same argument as in the proof of 2.12 shows that we can construct two sequences $a'_1, a'_2, \ldots \in \mathcal{U}'$ and $b_1, b_2, \cdots \in \mathcal{U}$ such that

$$\begin{aligned} |a_i - a_j|_{\mathcal{U}} &= |a'_i - a'_j|_{\mathcal{U}'}, \\ |a_i - b_j|_{\mathcal{U}} &= |a'_i - b'_j|_{\mathcal{U}'}, \\ |b_i - b_j|_{\mathcal{U}} &= |b'_i - b'_j|_{\mathcal{U}'} \end{aligned}$$

for all i and j.

It remains to observe that the constructed distance-preserving bijection defined by $a_i \leftrightarrow a'_i$ and $b_i \leftrightarrow b'_i$ extends continuously to an isometry $\mathcal{U} \leftrightarrow \mathcal{U}'$. $\square$

Observe that 2.17 implies that the Urysohn space (as well as the d-Urysohn space) is finite-set-homogeneous; that is,

◇ any distance-preserving map from a finite subset to the whole space can be extended to an isometry.

Recall that $S(p,r)_{\mathcal{X}}$ denotes the sphere of radius r centered at p in a metric space $\mathcal{X}$; that is,

$$S(p,r)_{\mathcal{X}} = \{\, x \in \mathcal{X} \;:\; |p - x|_{\mathcal{X}} = r \,\}.$$

2.19. Exercise. *Choose $d \in [0, \infty]$. Denote by $\mathcal{U}_d$ the d-Urysohn space, so $\mathcal{U}_\infty$ is the Urysohn space.*

(a) Assume that $L = S(p,r)_{\mathcal{U}_d} \neq \varnothing$. Show that L is isometric to $\mathcal{U}_\ell$; find ℓ in terms of r and d.

(b) Let $\ell = |p - q|_{\mathcal{U}_d}$. Show that the subset $M \subset \mathcal{U}_d$ of midpoints between p and q is isometric to $\mathcal{U}_\ell$.

(c) Show that $\mathcal{U}_d$ is not countable-set-homogeneous; that is, there is a distance-preserving map from a countable subset of $\mathcal{U}_d$ to $\mathcal{U}_d$ that cannot be extended to an isometry of $\mathcal{U}_d$.

In fact, the Urysohn space is compact-set-homogeneous; more precisely the following theorem holds.

2.20. Theorem. *Let K be a compact set in a (d-)Urysohn space $\mathcal{U}$. Then any distance-preserving map $K \to \mathcal{U}$ can be extended to an isometry of $\mathcal{U}$.*

A proof can be obtained by modifying the proofs of 2.9 and 2.17 the same way as it is done in 2.14.

2.21. Exercise. *Let S be a unit sphere in the Urysohn space $\mathcal{U}$. Show that for any two distinct points $x, y \in \mathcal{U}$ there is a point $z \in S$ such that $|x - z| \neq |y - z|$.*

Conclude that two isometries of $\mathcal{U}$ coincide if they coincide on S.

2.22. Exercise. *Let B be an open unit ball in the Urysohn space $\mathcal{U}$. Show that $\mathcal{U} \setminus B$ is isometric to $\mathcal{U}$.*

Use it to construct an isometry of a unit sphere S in $\mathcal{U}$ that cannot be extended to an isometry of $\mathcal{U}$.

2.23. Exercise.

(a) *Show that there is a distance-preserving inclusion of the Urysohn space $\iota\colon \mathcal{U} \hookrightarrow \mathcal{U}$ such that $\mathcal{U}' = \iota(\mathcal{U})$ is nowhere dense in $\mathcal{U}$ and any isometry of $\mathcal{U}'$ can be extended to an isometry of the whole $\mathcal{U}$.*

(b) *Consider a nested sequence $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \dots$ of Urysohn spaces with each inclusion $\mathcal{U}_n \hookrightarrow \mathcal{U}_{n+1}$ as in (a). Show that the union $\bigcup_n \mathcal{U}_n$ is a noncomplete finite-set-homogeneous metric space that meets the extension property.*

2.24. Exercise. *Which of the following metric spaces are one-point-homogeneous, finite-set-homogeneous, compact-set-homogeneous, countable-set-homogeneous?*

(a) *Euclidean plane,*

(b) *Hilbert space ℓ^2,*

(c) *ℓ^∞,*

(d) *ℓ^1 — the space of all real absolutely converging series $\boldsymbol{a} = (a_1, a_2, \dots)$ with the norm $|\boldsymbol{a}|_{\ell^1} = \sum_i |a_i|$.*

2.25. Exercise. *Show that any separable one-point-homogeneous metric tree is isometric to the real line $\mathbb{R}$ or the one-point space.*

E Remarks

The statement in 2.3 was proved by Maurice René Fréchet [58, p. 162]; its extension 2.4 was given by Kazimierz Kuratowski [87]. Both maps $x \mapsto (\mathrm{dist}_x - \mathrm{dist}_{x_0})$ and $x \mapsto \mathrm{dist}_x$ can be called Kuratowski embedding.

Let us describe a closely related construction introduced by Mikhael Gromov [61]. Suppose $\mathcal{X}$ be a proper metric space. Denote by $C(\mathcal{X}, \mathbb{R})$ the space of continuous functions $\mathcal{X} \to \mathbb{R}$ equipped with the compact-open topology; that is, for any compact set $K \subset \mathcal{X}$ and any open set $U \subset \mathbb{R}$ the set of all continuous functions $f\colon \mathcal{X} \to \mathbb{R}$ such that $f(K) \subset U$ is declared to be open.

Choose a point $x_0 \in \mathcal{X}$. Consider the map $F_{\mathcal{X}}\colon \mathcal{X} \to C(\mathcal{X}, \mathbb{R})$ defined by

$$x \mapsto f_x := -|x - x_0| + \mathrm{dist}_x.$$

2.26. Exercise. *Show that if $\mathcal{X}$ is a proper length space, then $F_{\mathcal{X}}$ is an embedding. Construct a proper metric space $\mathcal{Y}$ such that $F_{\mathcal{Y}}$ is not an embedding.*

Denote by $\hat{\mathcal{X}}$ the closure of $F_{\mathcal{X}}(\mathcal{X})$ in $C(\mathcal{X}, \mathbb{R})$; observe that $\hat{\mathcal{X}}$ is compact. If $F_{\mathcal{X}}$ is an embedding, then $\hat{\mathcal{X}}$ is a compactification of $\mathcal{X}$, and it is called the horo-compactification. In this case, the complement

$$\partial_\infty \mathcal{X} = \hat{\mathcal{X}} \setminus F_{\mathcal{X}}(\mathcal{X})$$

is called the horo-absolute of $\mathcal{X}$. A variation of this construction for nonproper spaces was considered by Anders Karlsson [81].

The following two exercises show that ℓ^∞ is very different from ℓ^1. For more on the subject, see [55].

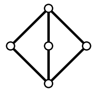
Let S be a subset of X. We say that S separates x and y if $x \in S$ and $y \notin S$ or $x \notin S$ and $y \in S$. The cut metric δ_S on X is a semimetric such that $\delta_S(x, y) = 1$ if x and y are separated by S and otherwise $\delta_S(x, y) = 0$.

2.27. Exercise. *Show that a finite metric space $\mathcal{F}$ admits a distance-preserving embedding into ℓ^1 if and only if the metric of $\mathcal{F}$ can be written as a nonnegative linear combination[1] of cut metrics on $\mathcal{F}$.*

Recall that the vertex set of any graph comes with the path metric — the distance between two vertices is the minimal number of edges in a path connecting them.

[1] that is, linear combination with nonnegative coefficients.

2.28. Exercise. *Use 2.27 to show that the metric for complete bipartite graph $K_{2,3}$ (see the diagram) does not admit a distance-preserving embedding into ℓ^1.*



The question about the existence of a separable universal space was posed by Maurice René Fréchet and answered by Pavel Urysohn [131]. Exercise 2.23 answers a question posed by Pavel Urysohn [131, §2(6)]. It was solved by Miroslav Katětov [82], but long after that, it was again mentioned as an open problem [65, p. 83].

The idea of Urysohn's construction was reused in graph theory; it produces the so-called Rado graph, also known as *Erdős–Rényi graph* or *random graph*; see [44]. In certain sense, the Urysohn space is the random metric space [136].

The (d-) Urysohn space is homeomorphic to the Hilbert space; the latter was proved by Vladimir Uspenskij [132] using the so-called Toruńczyk criterion.

The finite-set-homogeneous spaces include Euclidean spaces, hyperbolic spaces, and spheres all with standard length metrics and arbitrary finite dimensions. In fact, these are the only examples of locally compact three-point-homogeneous length spaces. The latter was proved by Herbert Busemann [40]; it also follows from the more general result of Jacques Tits about two-point-homogeneous spaces [130]. The same conclusion holds for complete all-set-homogeneous geodesic spaces with local uniqueness of geodesics; it was proved by Garrett Birkhoff [25]. The answer might be the same for complete separable all-set-homogeneous length spaces. Without the separability condition, we also get the so-called universal metric trees with finite valence [56]; no other examples seem to be known [91].

2.29. Exercise. *Show that the real projective plane $\mathbb{R}\mathrm{P}^2$ with the standard metric is two-point-homogeneous, but not three-point-homogeneous.*

2.30. Exercise. *Let Q be the set of vertices on the n-dimensional cube; assume n is large. Show that Q is three-point-homogeneous, but not four-point-homogeneous.*

2.31. Question. *Are there examples of metric spaces that are n-point-homogeneous, but not $(n+1)$-point-homogeneous for large n? See [109].*

Lecture 3

Injective spaces

Injective hull is a useful construction that provides a canonical choice of a specially nice (injective) space that includes a given metric space. This construction is similar to the convex hull in Euclidean space. The following exercise gives a bridge from the latter to the former.

3.1. Advanced exercise. *Show that $A \subset \mathbb{R}^n$ is a closed convex set if and only if for any $B \subset \mathbb{R}^n$ any short map $B \to A$ can be extended to a short map $\mathbb{R}^n \to A$.*

A Definition

3.2. Definition. *A metric space $\mathcal{Y}$ is called injective if, for any metric space $\mathcal{X}$ and any of its subspace $\mathcal{A}$, any short map $f\colon \mathcal{A} \to \mathcal{Y}$ can be extended to a short map $F\colon \mathcal{X} \to \mathcal{Y}$; that is, $f = F|_{\mathcal{A}}$.*

3.3. Exercise. *Show that any injective space is*

(a) complete, *(b) geodesic, and* *(c) contractible.*

3.4. Exercise. *Show that for any injective space $\mathcal{Y}$ there is a map $m\colon \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ (the midpoint map) such that the inequality*

$$2{\cdot}|p - m(x,y)|_{\mathcal{Y}} \leqslant |p - x|_{\mathcal{Y}} + |p - y|_{\mathcal{Y}}$$

holds for any $p, x, y \in \mathcal{Y}$.

3.5. Exercise. *Show that the following spaces are injective:*

(a) the real line;

(b) *complete metric tree;*

(c) *The space $\ell^\infty(\mathcal{S})$ for any set $\mathcal{S}$ (defined in 2.4). In particular, the coordinate plane with the metric induced by the ℓ^∞-norm.*

3.6. Exercise. *Let $\mathcal{Y}$ be an injective space.*

(a) *Show that any closed ball in $\mathcal{Y}$ is injective.*

(b) *Show that the intersection of an arbitrary collection of closed balls in $\mathcal{Y}$ is injective.*

3.7. Advanced exercise. *Let $\mathcal{Y}$ be a bounded injective space. Show that any short map $s\colon \mathcal{Y}\to\mathcal{Y}$ has a fixed point.*

B Admissible and extremal functions

Let $\mathcal{X}$ be a metric space. A function $r\colon \mathcal{X}\to(-\infty,\infty]$ is called admissible if the following inequality

➊ $$r(x)+r(y)\geqslant|x-y|_{\mathcal{X}}$$

holds for any $x,y\in\mathcal{X}$.

3.8. Observation.

(a) *Any admissible function is nonnegative.*

(b) *If $\mathcal{X}$ is a geodesic space, then a function $r\colon\mathcal{X}\to\mathbb{R}$ is admissible if and only if*

$$\overline{\mathrm{B}}[x,r(x)]\cap\overline{\mathrm{B}}[y,r(y)]\neq\varnothing$$

for any $x,y\in\mathcal{X}$.

Proof; (a). Apply ➊ for $x=y$.

(b). Apply the triangle inequality and the existence of a geodesic $[xy]$. □

A minimal admissible function will be called extremal. More precisely, an admissible function $r\colon\mathcal{X}\to\mathbb{R}$ is extremal if for any admissible function $s\colon\mathcal{X}\to\mathbb{R}$ we have

$$s\leqslant r\quad\Longrightarrow\quad s=r.$$

Applying Zorn's lemma, we get the following.

3.9. Observation. *For any admissible function s there is an extremal function r such that $r\leqslant s$.*

3.10. Lemma. *Let r be an extremal function and s an admissible function on a metric space $\mathcal{X}$. Suppose that $r \geqslant s - c$ for some constant c. Then $r \leqslant s + c$; in particular, $c \geqslant 0$.*

Proof. Note that if $c < 0$, then $r > s$. The latter is impossible since r is extremal and s is admissible.

Observe that the function $\bar{r} = \min\{\, r, s + c \,\}$ is admissible. Indeed, choose $x, y \in \mathcal{X}$. If $\bar{r}(x) = r(x)$ and $\bar{r}(y) = r(y)$, then

$$\bar{r}(x) + \bar{r}(y) = r(x) + r(y) \geqslant |x - y|.$$

Further, if $\bar{r}(x) = s(x) + c$, then

$$\begin{aligned}\bar{r}(x) + \bar{r}(y) &\geqslant [s(x) + c] + [s(y) - c] = \\ &= s(x) + s(y) \geqslant \\ &\geqslant |x - y|.\end{aligned}$$

Since r is extremal, we have $r = \bar{r}$; that is, $r \leqslant s + c$. □

3.11. Observations. *Let $\mathcal{X}$ be a metric space.*

(a) *For any point $p \in \mathcal{X}$ the distance function $r = \mathrm{dist}_p$ is extremal.*

(b) *Any extremal function r on $\mathcal{X}$ is 1-Lipschitz; that is,*

$$|r(p) - r(q)| \leqslant |p - q|$$

for any $p, q \in \mathcal{X}$. In other words, any extremal function is an extension function [see 2B].

(c) *An admissible function r on $\mathcal{X}$ is extremal if and only if for any point $p \in \mathcal{X}$ and any $\delta > 0$, there is a point $q \in \mathcal{X}$ such that*

$$r(p) + r(q) < |p - q|_{\mathcal{X}} + \delta.$$

(d) *Suppose $\mathcal{X}$ is compact. Then an admissible function r on $\mathcal{X}$ is extremal if and only if for any point $p \in \mathcal{X}$ there is a point $q \in \mathcal{X}$ such that*

$$r(p) + r(q) = |p - q|_{\mathcal{X}}.$$

Proof; (a). By the triangle inequality, ❶ holds; that is, $r = \mathrm{dist}_p$ is an admissible function.

Further, if $s \leqslant r$ is another admissible function, then $s(p) = 0$ and ❶ implies that $s(x) \geqslant |p - x|$. Whence $s = r$.

(b). By *(a)*, dist_p is admissible. Since r is admissible, we have that

$$r \geqslant \mathrm{dist}_p - r(p).$$

Since r is extremal, 3.10 implies that

$$r \leqslant \operatorname{dist}_p + r(p),$$

or, equivalently,

$$r(q) - r(p) \leqslant |p - q|$$

for any $p, q \in \mathcal{X}$. Whence the statement follows.

(c). Assume r is extremal. Arguing by contradiction, assume there is $\delta > 0$ such that

$$r(q) \geqslant \operatorname{dist}_p(q) - r(p) + \delta$$

for any q. By *(a)*, dist_p is extremal; in particular, admissible. Therefore 3.10 implies that

$$r(q) \leqslant \operatorname{dist}_p(q) + r(p) - \delta$$

for any q. Taking $q = p$, we get $r(p) \leqslant r(p) - \delta$, a contradiction.

Now suppose r is not extremal; that is, there is an admissible function $s \leqslant r$ such that $r(p) - s(p) = \delta > 0$ for some p. Then, for any q, we have

$$r(p) + r(q) \geqslant s(p) + s(q) + \delta \geqslant |p - q|_{\mathcal{X}} + \delta$$

— a contradiction.

(d). The if part follows from *(c)*.

Denote by q_n the point provided by *(c)* for $\delta = \frac{1}{n}$. Let q be a partial limit of q_n. Then

$$r(p) + r(q) \leqslant |p - q|_{\mathcal{X}}.$$

Since r is admissible, the opposite inequality holds; whence the only-if part follows. □

3.12. Exercise. *Consider the unit circle*

$$\mathbb{S}^1 = \{\, (x, y) \,:\, x^2 + y^2 = 1 \,\}$$

in the plane with induced length metric. Show that $r\colon \mathbb{S}^1 \to \mathbb{R}$ is extremal if and only if it is 1-Lipschitz and

$$r(p) + r(-p) = \pi$$

for any $p \in \mathbb{S}^1$.

3.13. Exercise. *Given a real-valued function s on a metric space $\mathcal{X}$, consider the function*

$$s^*(x) = \sup\{\, |x - y|_{\mathcal{X}} - s(y) \,:\, y \in \mathcal{X} \,\}$$

Show that the function $\frac{1}{2}\cdot(s + s^*)$ is admissible for any s.

C Equivalent conditions

3.14. Theorem. *For any metric space $\mathcal{Y}$ the following conditions are equivalent:*

(a) $\mathcal{Y}$ *is injective*

(b) If $r\colon \mathcal{Y}\to\mathbb{R}$ *is an extremal function, then there is a point* $p\in\mathcal{Y}$ *such that*
$$|p-x| = r(x)$$
for any $x\in\mathcal{Y}$.

(c) $\mathcal{Y}$ *is hyperconvex; that is, if* $\left\{\,\overline{\mathrm{B}}[x_\alpha,r_\alpha] \,:\, \alpha\in\mathcal{A}\,\right\}$ *is a family of closed balls in* $\mathcal{Y}$ *such that*
$$r_\alpha + r_\beta \geqslant |x_\alpha - x_\beta|$$
for any $\alpha,\beta\in\mathcal{A}$, *then all the balls in the family* $\{\overline{\mathrm{B}}[x_\alpha,r_\alpha]\}_{\alpha\in\mathcal{A}}$ *have a common point.*

Proof. We will prove implications *(a)*⇒*(b)*⇒*(c)*⇒*(a)*.

(a)⇒*(b)*. By 3.11*b*, r is an extension function. Applying the definition of injective space to a one-point extension of $\mathcal{Y}$, we get a point $p\in\mathcal{Y}$ such that
$$|p-x| = \mathrm{dist}_p(x) \leqslant r(x)$$
for any $x\in\mathcal{Y}$. By 3.11*a*, the distance function dist_p is extremal. Since r is extremal, we get $\mathrm{dist}_p = r$.

(b)⇒*(c)*. By 3.8*b*, part *(c)* is equivalent to the following statement:

◇ If $r\colon\mathcal{Y}\to\mathbb{R}$ is an admissible function, then there is a point $p\in\mathcal{Y}$ such that
$$\text{❶}\qquad |p-x|\leqslant r(x)$$
for any $x\in\mathcal{Y}$.

Indeed, set $r(x) := \inf\{\,r_\alpha \,:\, x_\alpha = x\,\}$. (If $x_\alpha\ne x$ for any α, then $r(x)=\infty$.) The condition in *(c)* implies that r is admissible. It remains to observe that $p\in\overline{\mathrm{B}}[x_\alpha,r_\alpha]$ for every α if and only if ❶ holds.

By 3.9, for any admissible function r there is an extremal function $\bar r\leqslant r$; hence *(b)*⇒*(c)*.

(c)⇒*(a)*. Arguing by contradiction, suppose $\mathcal{Y}$ is not injective; that is, there is a metric space $\mathcal{X}$ with a subset $\mathcal{A}$ such that a short map $f\colon\mathcal{A}\to\mathcal{Y}$ cannot be extended to a short map $F\colon\mathcal{X}\to\mathcal{Y}$. By Zorn's

lemma, we may assume that $\mathcal{A}$ is a maximal subset; that is, the domain of f cannot be enlarged by a single point.[1]

Fix a point p in the complement $\mathcal{X}\setminus\mathcal{A}$. To extend f to p, we need to choose $f(p)$ in the intersection of the balls $\overline{\mathrm{B}}[f(x),r(x)]$, where $r(x)=|p-x|$. Therefore, this intersection for all $x\in\mathcal{A}$ has to be empty.

Since f is short, we have that

$$\begin{aligned} r(x)+r(y)\geqslant |x-y|_{\mathcal{X}}\geqslant \\ \geqslant |f(x)-f(y)|_{\mathcal{Y}}. \end{aligned}$$

By *(c)* the balls $\overline{\mathrm{B}}[f(x),r(x)]$ have a common point — a contradiction. □

3.15. Exercise. *Suppose a length space $\mathcal{W}$ has two subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X}\cup\mathcal{Y}=\mathcal{W}$ and $\mathcal{X}\cap\mathcal{Y}$ is a one-point set. Assume $\mathcal{X}$ and $\mathcal{Y}$ are injective. Show that $\mathcal{W}$ is injective*

3.16. Exercise. *Show that an m-dimensional normed space is injective if and only if it is isometric to $\mathbb{R}^m$ with ℓ^∞-norm; that is,*

$$|(x_1,\dots,x_m)|=\max_i\{\,|x_i|\,\}.$$

A metric space $\mathcal{Y}$ is called finitely hyperconvex or countably hyperconvex if the condition in 3.14*c* holds only for any finite or respectively countable family of balls.

3.17. Exercise. *Show that any proper finitely hyperconvex metric space is hyperconvex.*

3.18. Exercise. *Show that the d-Urysohn space is finitely hyperconvex, but not countably hyperconvex. Conclude that the d-Urysohn space is not injective.*

Try to do the same for the Urysohn space.

3.19. Exercise. *Let $\mathcal{Y}$ be a complete metric space. Suppose $\mathcal{Y}$ is almost hyperconvex, that is, for any $\varepsilon>0$ any family of closed balls $\{\,\overline{\mathrm{B}}[x_\alpha,r_\alpha+\varepsilon]\;:\;\alpha\in\mathcal{A}\,\}$ has a common point if $r_\alpha+r_\beta\geqslant|x_\alpha-x_\beta|$ for all $\alpha,\beta\in\mathcal{A}$. Show that $\mathcal{Y}$ is hyperconvex.*

[1] In this case, $\mathcal{A}$ must be closed, but we will not use it.

D Space of extremal functions

Let $\mathcal{X}$ be a metric space. Consider the space $\operatorname{Ext}\mathcal{X}$ of extremal functions on $\mathcal{X}$ equipped with sup-norm; that is,

$$|f-g|_{\operatorname{Ext}\mathcal{X}} := \sup\{\,|f(x)-g(x)| \,:\, x\in\mathcal{X}\,\}.$$

Recall that by 3.11a, any distance function is extremal. It follows that the map $x\mapsto \operatorname{dist}_x$ produces a distance-preserving embedding $\mathcal{X}\hookrightarrow\operatorname{Ext}\mathcal{X}$. So we can (and will) treat $\mathcal{X}$ as a subspace of $\operatorname{Ext}\mathcal{X}$, or, equivalently, $\operatorname{Ext}\mathcal{X}$ as an extension of $\mathcal{X}$. In particular, from now on, a point $x\in\mathcal{X}$ can refer to the function $\operatorname{dist}_x\colon\mathcal{X}\to\mathbb{R}$ and the other way around.

Since any extremal function is 1-Lipschitz, for any $f\in\operatorname{Ext}\mathcal{X}$ and $p\in\mathcal{X}$, we have that $f(x)\leqslant f(p)+\operatorname{dist}_p(x)$. By 3.10, we also get $f(x)\geqslant -f(p)+\operatorname{dist}_p(x)$. Therefore

❶
$$\begin{aligned}|f-p|_{\operatorname{Ext}\mathcal{X}} &= \sup\{\,|f(x)-\operatorname{dist}_p(x)| \,:\, x\in\mathcal{X}\,\} = \\ &= f(p).\end{aligned}$$

In particular, the statement in 3.11c can be written as

$$|f-p|_{\operatorname{Ext}\mathcal{X}} + |f-q|_{\operatorname{Ext}\mathcal{X}} < |p-q|_{\operatorname{Ext}\mathcal{X}} + \delta.$$

3.20. Exercise. *Show that* $\operatorname{Ext}\mathcal{X}$ *is compact if and only if so is* $\mathcal{X}$.

3.21. Exercise. *Describe the set of all extremal functions on a metric space* $\mathcal{X}$ *and the metric space* $\operatorname{Ext}\mathcal{X}$ *in each of the following cases:*

(a) $\mathcal{X}$ *is a metric space with exactly two points* v,w *on distance 1 from each other.*

(b) $\mathcal{X}$ *is a metric space with exactly three points* a,b,c *such that*

$$|a-b|_{\mathcal{X}} = |b-c|_{\mathcal{X}} = |c-a|_{\mathcal{X}} = 1.$$

(c) $\mathcal{X}$ *is a metric space with exactly four points* p,q,x,y *such that*

$$|p-x|_{\mathcal{X}} = |p-y|_{\mathcal{X}} = |q-x|_{\mathcal{X}} = |q-y|_{\mathcal{X}} = 1$$

and

$$|p-q|_{\mathcal{X}} = |x-y|_{\mathcal{X}} = 2.$$

3.22. Exercise. *Assume* $\mathcal{X}$ *is a compact metric space. Recall that the map* $x\mapsto\operatorname{dist}_x$ *gives an isometric embedding* $\mathcal{X}\hookrightarrow\ell^\infty(\mathcal{X})$*; so we can think that* $\mathcal{X}$ *is a subset of* $\ell^\infty(\mathcal{X})$.

Given two points $x, y \in \mathcal{X}$, denote by $G_{x,y}$ the union of all geodesics from x to y in $\ell^\infty(\mathcal{X})$. Show that $\operatorname{Ext}\mathcal{X}$ *is isometric to*

$$G = \bigcap_{x\in\mathcal{X}}\left(\bigcup_{y\in\mathcal{X}} G_{x,y}\right).$$

3.23. Proposition. $\operatorname{Ext}\mathcal{X}$ *is injective for any metric space $\mathcal{X}$.*

3.24. Lemma. *Let $\mathcal{X}$ be a metric space. Then*

$$\sigma \in \operatorname{Ext}(\operatorname{Ext}\mathcal{X}) \quad\Longrightarrow\quad \sigma|_{\mathcal{X}} \in \operatorname{Ext}\mathcal{X}.$$

In other words, if σ is an extremal function on $\operatorname{Ext}\mathcal{X}$, then the restriction of σ to $\mathcal{X}$ is an extremal function on $\mathcal{X}$.

Proof. Arguing by contradiction, suppose that there is an admissible function $s\colon \mathcal{X}\to\mathbb{R}$ such that $s(x)\leqslant\sigma(x)$ for any $x\in\mathcal{X}$ and $s(p) < \sigma(p)$ for some point $p\in\mathcal{X}$. Consider another function $\bar\sigma\colon \operatorname{Ext}\mathcal{X}\to\mathbb{R}$ such that $\bar\sigma(f) := \sigma(f)$ if $f\ne p$ and $\bar\sigma(p) := s(p)$.

Let us show that $\bar\sigma$ is admissible; that is,

❷ $$|f-g|_{\operatorname{Ext}\mathcal{X}} \leqslant \bar\sigma(f)+\bar\sigma(g)$$

for any $f, g\in\operatorname{Ext}\mathcal{X}$.

Since σ is admissible and $\bar\sigma=\sigma$ on $(\operatorname{Ext}\mathcal{X})\setminus\{p\}$, it is sufficient to prove ❷ assuming $f\ne g=p$. By ❶, we have $|f-p|_{\operatorname{Ext}\mathcal{X}} = f(p)$. Therefore, ❷ boils down to the following inequality

❸ $$\sigma(f)+s(p)\geqslant f(p).$$

for any $f\in\operatorname{Ext}\mathcal{X}$.

Fix small $\delta>0$. Let $q\in\mathcal{X}$ be the point provided by 3.11*c*. Then

$$\sigma(f)+s(p)\geqslant[\sigma(f)-\sigma(q)]+[\sigma(q)+s(p)]\geqslant$$

since σ is 1-Lipschitz, and $\sigma(q)\geqslant s(q)$, we can continue

$$\geqslant -|q-f|_{\operatorname{Ext}\mathcal{X}}+[s(q)+s(p)]\geqslant$$

by ❶ and since s is admissible

$$\geqslant -f(q)+|p-q|>$$

and by 3.11c

$$> f(p) - \delta.$$

Since $\delta > 0$ is arbitrary, ❸ and ❷ follow.

Summarizing: the function $\bar\sigma$ is admissible, $\bar\sigma \leqslant \sigma$ and $\bar\sigma(p) < \sigma(p)$; that is, σ is not extremal — a contradiction. $\square$

Proof of 3.23. Choose a function $\sigma \in \operatorname{Ext}(\operatorname{Ext}\mathcal{X})$. By 3.24, $s := \sigma|_{\mathcal{X}} \in \operatorname{Ext}\mathcal{X}$; that is, s is extremal. By 3.14b, it is sufficient to show that

❹ $$\sigma(f) \geqslant |s - f|_{\operatorname{Ext}\mathcal{X}}$$

for any $f \in \operatorname{Ext}\mathcal{X}$.

Since σ is 1-Lipschitz (3.11b) we have that

$$s(x) - f(x) = \sigma(x) - |f - x|_{\operatorname{Ext}\mathcal{X}} \leqslant \sigma(f).$$

for any $x \in \mathcal{X}$. By 3.10, $s(x) - f(x) \geqslant -\sigma(f)$ for any $x \in \mathcal{X}$. Whence ❹ follows. $\square$

3.25. Exercise. *Let $\mathcal{X}$ be a compact metric space. Show that for any two points $f, g \in \operatorname{Ext}\mathcal{X}$ lie on a geodesic $[pq]$ with $p, q \in \mathcal{X}$.*

A metric space $\mathcal{X}$ is called δ-hyperbolic if

$$|p - q| + |x - y| \leqslant \max\{\, |p - x| + |q - y|,\ |p - y| + |q - x| \,\} + 2\cdot\delta$$

for any $p, q, x, y \in \mathcal{X}$.

3.26. Advanced exercise. *Show that $\operatorname{Ext}\mathcal{X}$ is δ-hyperbolic if and only if so is $\mathcal{X}$.*

E Injective envelope

An extension $\mathcal{E}$ of a metric space $\mathcal{X}$ will be called its injective envelope if $\mathcal{E}$ is an injective space, and there is no proper injective subspace of $\mathcal{E}$ that contains $\mathcal{X}$.

Two injective envelopes $e\colon \mathcal{X} \hookrightarrow \mathcal{E}$ and $f\colon \mathcal{X} \hookrightarrow \mathcal{F}$ are called equivalent if there is an isometry $\iota\colon \mathcal{E} \to \mathcal{F}$ such that $f = \iota \circ e$.

3.27. Theorem. *For any metric space $\mathcal{X}$, its extension $\operatorname{Ext}\mathcal{X}$ is an injective envelope.*

Moreover, any other injective envelope of $\mathcal{X}$ is equivalent to $\operatorname{Ext}\mathcal{X}$.

Proof. Suppose $S \subset \operatorname{Ext}\mathcal{X}$ is an injective subspace containing $\mathcal{X}$. Since S is injective, there is a short map $w\colon \operatorname{Ext}\mathcal{X} \to S$ that fixes all points in $\mathcal{X}$.

Suppose that $w\colon f \mapsto f'$; observe that $f(x) \geqslant f'(x)$ for any $x \in \mathcal{X}$. Since f is extremal, $f = f'$; that is, w is the identity map, and therefore $S = \operatorname{Ext}\mathcal{X}$.

Assume we have another injective envelope $e\colon \mathcal{X} \hookrightarrow \mathcal{E}$. Then there are short maps $v\colon \mathcal{E} \to \operatorname{Ext}\mathcal{X}$ and $w\colon \operatorname{Ext}\mathcal{X} \to \mathcal{E}$ such that $x = v \circ e(x)$ and $e(x) = w(x)$ for any $x \in \mathcal{X}$. From above, the composition $v \circ w$ is the identity on $\operatorname{Ext}\mathcal{X}$. In particular, w is distance-preserving.

The composition $w \circ v\colon \mathcal{E} \to \mathcal{E}$ is a short map that fixes points in $e(\mathcal{X})$. Since $e\colon \mathcal{X} \hookrightarrow \mathcal{E}$ is an injective envelope, the composition $w \circ v$ and, therefore, w are onto. Whence w is an isometry. $\square$

3.28. Exercise. *Suppose $e\colon \mathcal{X} \hookrightarrow \mathcal{E}$ and $f\colon \mathcal{X} \hookrightarrow \mathcal{F}$ are two injective envelopes of $\mathcal{X}$. Show that there is a unique isometry $\iota\colon \mathcal{E} \to \mathcal{F}$ such that $\iota \circ e = f$.*

3.29. Exercise. *Suppose $\mathcal{X}$ is a subspace of a metric space $\mathcal{U}$. Show that the inclusion $\mathcal{X} \hookrightarrow \mathcal{U}$ can be extended to a distance-preserving inclusion $\operatorname{Ext}\mathcal{X} \hookrightarrow \operatorname{Ext}\mathcal{U}$.*

3.30. Exercise. *Consider the hemisphere*

$$\mathbb{S}^2_+ = \left\{\, (x,y,z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 = 1, \quad z \geqslant 0 \,\right\}$$

and its boundary

$$\mathbb{S}^1 = \left\{\, (x,y,z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 = 1, \quad z = 0 \,\right\};$$

both with induced length metrics.

Show that there is unique isometric embedding $\iota\colon \mathbb{S}^2_+ \hookrightarrow \operatorname{Ext}\mathbb{S}^1$ such that $\iota(u) = u$ for any $u \in \mathbb{S}^1$.

F Remarks

Injective spaces were introduced by Nachman Aronszajn and Prom Panitchpakdi [12]. The injective envelope was introduced by John Isbell [75]; it is also known as tight span and hyperconvex hull.

It was observed by John Isbell [76] that *if $\mathcal{X}$ is a Banach space, then its injective hull* $\operatorname{Ext}\mathcal{X}$ *has a natural structure of Banach space* (which is unique by the Mazur–Ulam theorem). Moreover, $\mathcal{X}$ is a linear subspace of $\operatorname{Ext}\mathcal{X}$.

Let us mention that a metric space $\mathcal{X}$ is called convex if for any pair of points $x_1, x_2 \in \mathcal{X}$ and any $r_1, r_2 \geqslant 0$ we have

$$r_1 + r_2 \geqslant |x_1 - x_2|_{\mathcal{X}} \qquad \Longrightarrow \qquad \overline{\mathrm{B}}[x, r_1]_{\mathcal{X}} \cap \overline{\mathrm{B}}[y, r_2]_{\mathcal{X}} \neq \varnothing;$$

in other words, a pair of balls intersect if the triangle inequality does not forbid it. Clearly, hyperconvexity (3.14c) is stronger than convexity. Note that *any geodesic space is convex.* The converse does not hold in general, but by Menger's lemma (1.27b) *any complete convex space is geodesic.*

More generally, a metric space $\mathcal{X}$ is called n-hyperconvex if the condition in 3.14c holds only for families with at most n balls; so *convex means* 2*-hyperconvex.*

The following striking result was proved by Benjamin Miesch and Maël Pavón [100].

3.31. Theorem. *Any complete* 4*-hyperconvex space is finitely hyperconvex.*

So, by 3.17, it follows that *any proper* 4*-hyperconvex space is hyperconvex.*

3.32. Exercise. *Show that* ℓ^1 *is* 3*- but not* 4*-hyperconvex.*

Recall that if the following inequality

$$|x - z|_{\mathcal{X}} \leqslant \max\{\, |x - y|_{\mathcal{X}}, |y - z|_{\mathcal{X}} \,\}$$

holds for any three points x, y, z in a metric space $\mathcal{X}$, then $\mathcal{X}$ is called an ultrametric space. In some sense, ultrametric spaces are dual to injective spaces.

3.33. Exercise. *Suppose that a metric space* $\mathcal{X}$ *satisfies the following property: For any subspace* $\mathcal{A}$ *in* $\mathcal{X}$ *and any other metric space* $\mathcal{Y}$*, any short map* $f\colon \mathcal{A} \to \mathcal{Y}$ *can be extended to a short map* $F\colon \mathcal{X} \to \mathcal{Y}$*.*

Show that $\mathcal{X}$ *is an ultrametric space.*

A subspace $\mathcal{S}$ of a metric space $\mathcal{X}$ is called its short retract if there is a short map $\mathcal{X} \to \mathcal{S}$ that is the identity on $\mathcal{S}$.

3.34. Exercise. *Show that any compact subspace* $\mathcal{K}$ *of an ultrametric space* $\mathcal{X}$ *is its short retract.*

Construct an example of a complete ultrametric space $\mathcal{X}$ *with a closed subspace* $\mathcal{Q}$ *that is not its short retract.*

The following exercise gives a sufficient condition for the existence of a short extension.

3.35. Exercise. *Let $f\colon A\to\mathcal{K}$ be a short map from a subset A in a metric space $\mathcal{X}$ to compact metric space $\mathcal{K}$. Assume that for any finite set $F\subset\mathcal{X}$ there is a short map $F\to\mathcal{K}$ that agrees with f on $F\cap A$. Show that there is a short map $\mathcal{X}\to\mathcal{K}$ that agrees with f on A.*

Lecture 4

Space of subsets

In this lecture we define and study Hausdorff metric on subsets of a given metric space.

A Hausdorff distance

Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$, consider the distance function to A

$$\operatorname{dist}_A : \mathcal{X} \to [0, \infty)$$

defined as

$$\operatorname{dist}_A(x) := \inf \{ |a - x|_{\mathcal{X}} \ : \ a \in A \} .$$

Further, we define the so-called Hausdorff metric on all nonempty compact subsets of a given metric space $\mathcal{X}$. The obtained metric space will be denoted as $\operatorname{Haus}\mathcal{X}$.

4.1. Definition. *Let A and B be two nonempty compact subsets of a metric space $\mathcal{X}$. Then the Hausdorff distance between A and B is defined as*

$$|A - B|_{\operatorname{Haus}\mathcal{X}} := \sup_{x \in \mathcal{X}} \{ |\operatorname{dist}_A(x) - \operatorname{dist}_B(x)| \}.$$

The following observation gives a useful reformulation of the definition:

4.2. Observation. *Suppose A and B be two compact subsets of a metric space $\mathcal{X}$. Then $|A - B|_{\operatorname{Haus}\mathcal{X}} < R$ if and only if and only if B lies in an R-neighborhood of A, and A lies in an R-neighborhood of B.*

4.3. Exercise. *Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$, define its diameter as*

$$\operatorname{diam} A := \sup_{a,b\in A} |a - b|.$$

Show that

$$\operatorname{diam}\colon \operatorname{Haus}\mathcal{X} \to \mathbb{R}$$

is a 2-Lipschitz function; that is,

$$|\operatorname{diam} A - \operatorname{diam} B| \leqslant 2\cdot|A - B|_{\operatorname{Haus}\mathcal{X}}$$

for any two compact nonempty sets $A, B \subset \mathcal{X}$.

4.4. Exercise. *Let A and B be two compact subsets in the Euclidean plane $\mathbb{R}^2$. Assume $|A - B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$.*

(a) Show that $|\operatorname{Conv} A - \operatorname{Conv} B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$, where $\operatorname{Conv} A$ denoted the convex hull of A.

(b) Is it true that $|\partial A - \partial B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$, where ∂A denotes the boundary of A.

Does the converse hold? That is, assume A and B be two compact subsets in $\mathbb{R}^2$ and $|\partial A - \partial B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$; is it true that $|A - B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$?

Note that part *(a)* implies that $A \mapsto \operatorname{Conv} A$ defines a short map $\operatorname{Haus}\mathbb{R}^2 \to \operatorname{Haus}\mathbb{R}^2$.

4.5. Exercise. *Let A and B be compact subsets in metric space $\mathcal{X}$. Show that*

$$|A - B|_{\operatorname{Haus}\mathcal{X}} = \sup_f \{ \max_{a\in A}\{f(a)\} - \max_{b\in B}\{f(b)\} \},$$

where the least upper bound is taken for all 1-Lipschitz functions f.

Given a subset $A \subset \mathbb{R}^n$, the *support function* $h_A\colon \mathbb{R}^n \to \mathbb{R}$ of a nonempty closed set $A \subset \mathbb{R}^n$ is defined as

$$h_A(x) := \sup\{\langle x, a\rangle \,:\, a \in A\}.$$

4.6. Exercise. *Show that*

$$|A - B|_{\operatorname{Haus}\mathbb{R}^n} \geqslant \sup_{|u|=1} \{|h_A(u) - h_B(u)|\}$$

for any nonempty compact subsets $A, B \subset \mathbb{R}^n$.

Moreover, equality holds if both A and B are convex.

4.7. Advanced exercise. *Suppose $C_t \subset \mathcal{X}$, $t \in [0,1]$ is a family of subsets. A path $c\colon [0,1] \to \mathcal{X}$ such that $c(t) \in C_t$ for all t will be called a section of C_t.*

(a) *Construct a family of nonempty compact sets $C_t \subset \mathbb{S}^1$, $t \in [0,1]$ that is continuous in the Hausdorff topology, but does not admit a section.*

(b) *Show that any family of nonempty compact sets $C_t \subset \mathbb{R}$, $t \in [0,1]$ that is continuous in the Hausdorff topology, admits a section.*

B Hausdorff convergence

4.8. Blaschke selection theorem. *A metric space $\mathcal{X}$ is compact if and only if so is* $\operatorname{Haus}\mathcal{X}$.

The Hausdorff metric can be used to define convergence. Namely, suppose $K_1, K_2, \dots$, and K_∞ are compact sets in a metric space $\mathcal{X}$. If $|K_\infty - K_n|_{\operatorname{Haus}\mathcal{X}} \to 0$ as $n \to \infty$, then we say that the sequence K_n converges to K_∞ in the sense of Hausdorff; equivalently, K_∞ is the Hausdorff limit of the sequence K_n.

Note that the theorem implies that from any sequence of nonempty compact sets in $\mathcal{X}$ one can select a convergent subsequence; for that reason, it is called a *selection* theorem.

Proof; if part. Consider the map ι that sends each point $x \in \mathcal{X}$ to the one-point subset $\{x\}$ of $\mathcal{X}$. Note that $\iota\colon \mathcal{X} \to \operatorname{Haus}\mathcal{X}$ is distance-preserving.

Suppose that $A \subset \mathcal{X}$. Note that $\operatorname{diam} A = 0$ if and only if A is a one-point set. By 4.3, $\iota(\mathcal{X})$ is a closed subset of the compact space $\operatorname{Haus}\mathcal{X}$. It follows that $\iota(\mathcal{X})$ and, therefore, $\mathcal{X}$, are compact. □

Since the map ι above is distance-preserving, we can and will consider $\mathcal{X}$ as a subspace of $\operatorname{Haus}\mathcal{X}$.

4.9. Exercise. *Let $\mathcal{X}$ be a compact length space. Suppose that there is a short retraction* $\operatorname{Haus}\mathcal{X} \to \mathcal{X}$. *Show that $\mathcal{X}$ is contractible.*

To prove the only-if part we will need the following two lemmas.

4.10. Monotone convergence. *Let $K_1 \supset K_2 \supset \dots$ be a nested sequence of nonempty compact sets in a metric space $\mathcal{X}$. Then $K_\infty = \bigcap_n K_n$ is the Hausdorff limit of K_n; that is, $|K_\infty - K_n|_{\operatorname{Haus}\mathcal{X}} \to 0$ as $n \to \infty$.*

Proof. By finite intersection property, K_∞ is a nonempty compact set.

Arguing by contradiction, assume that there is $\varepsilon > 0$ such that for each n one can choose $x_n \in K_n$ such that $\operatorname{dist}_{K_\infty}(x_n) \geqslant \varepsilon$. Note that $x_n \in K_1$ for each n. Since K_1 is compact, there is a partial limit x_∞ of x_n; that is, a limit of a subsequence. Clearly, $\operatorname{dist}_{K_\infty}(x_\infty) \geqslant \varepsilon$.

On the other hand, since K_n is closed and $x_m \in K_n$ for $m \geqslant n$, we get $x_\infty \in K_n$ for each n. It follows that $x_\infty \in K_\infty$ and therefore $\operatorname{dist}_{K_\infty}(x_\infty) = 0$ — a contradiction. $\square$

4.11. Lemma. *If $\mathcal{X}$ is a compact metric space, then* $\operatorname{Haus}\mathcal{X}$ *is complete.*

Proof. Let $Q_1, Q_2, \dots$ be a Cauchy sequence in $\operatorname{Haus}\mathcal{X}$. Passing to a subsequence, we may assume that

$$❶ \qquad |Q_n - Q_{n+1}|_{\operatorname{Haus}\mathcal{X}} \leqslant \tfrac{1}{10^n}$$

for each n.

Denote by K_n the closed $\frac{2}{10^n}$-neighborhood of Q_n; that is,

$$K_n = \left\{\, x \in \mathcal{X} \;:\; \operatorname{dist}_{Q_n}(x) \leqslant \tfrac{2}{10^n} \,\right\}$$

Since $\mathcal{X}$ is compact so is each K_n.

From ❶, we get $K_n \supset K_{n+1}$ for each n. Set

$$K_\infty = \bigcap_{n=1}^{\infty} K_n.$$

By the monotone convergence (4.10), $|K_n - K_\infty|_{\operatorname{Haus}\mathcal{X}} \to 0$ as $n \to \infty$.

By 4.2, $|Q_n - K_n|_{\operatorname{Haus}\mathcal{X}} \leqslant \frac{2}{10^n}$. Therefore, $|Q_n - K_\infty|_{\operatorname{Haus}\mathcal{X}} \to 0$ as $n \to \infty$ — hence the lemma. $\square$

4.12. Exercise. *Let $\mathcal{X}$ be a complete metric space and $K_1, K_2, \dots$ be a sequence of compact sets that converges in the sense of Hausdorff. Show that the union $K_1 \cup K_2 \cup \dots$ has compact closure.*

Use this statement to show that in Lemma 4.11 compactness of $\mathcal{X}$ can be exchanged to completeness.

Proof of only-if part in 4.8. According to Lemma 4.11, $\operatorname{Haus}\mathcal{X}$ is complete. It remains to show that $\operatorname{Haus}\mathcal{X}$ is totally bounded (1.11c); that is, given $\varepsilon > 0$ there is a finite ε-net in $\operatorname{Haus}\mathcal{X}$.

Choose a finite ε-net A in $\mathcal{X}$. Denote by B the set of all nonempty subsets of A. Note that B is a finite set in $\operatorname{Haus}\mathcal{X}$. For each compact set $K \subset \mathcal{X}$, consider the subset K' of all points $a \in A$ such that $\operatorname{dist}_K(a) \leqslant \varepsilon$. Observe that $K' \in B$ and $|K - K'|_{\operatorname{Haus}\mathcal{X}} \leqslant \varepsilon$. In other words, B is a finite ε-net in $\operatorname{Haus}\mathcal{X}$. $\square$

4.13. Exercise. *Let $\mathcal{X}$ be a complete metric space. Show that $\mathcal{X}$ is a length space if and only if so is* $\operatorname{Haus}\mathcal{X}$.

4.14. Exercise.

(a) Show that the set of all connected compact subsets of $\mathbb{R}^2$ *is closed in* $\operatorname{Haus}\mathbb{R}^2$.

(b) Show that any connected compact subset of $\mathbb{R}^2$ *is a Hausdorff limit of a sequence of closed simple curves.*

C An application

In this section, we will sketch a proof of the isoperimetric inequality in the plane that uses the Hausdorff convergence.

It is based on the following exercise.

4.15. Exercise. *Let* $\mathcal{C}$ *be the set of all nonempty compact convex subsets in* $\mathbb{R}^2$. *Show that* $\mathcal{C}$ *is a closed subset of* $\operatorname{Haus}\mathbb{R}^2$ *and perimeter and area are continuous on* $\mathcal{C}$. *(If the set degenerates to a line segment of length* ℓ*, then its perimeter is defined as* $2{\cdot}\ell$.*)*

More precisely, if a sequence of convex compact plane sets X_n *converges to* X_∞ *in the sense of Hausdorff, then* X_∞ *is convex,*

$$\operatorname{perim} X_n \to \operatorname{perim} X_\infty, \quad \textit{and} \quad \operatorname{area} X_n \to \operatorname{area} X_\infty$$

as $n \to \infty$.

4.16. Isoperimetric inequality. *Among the plane figures bounded by closed curves of length at most* ℓ*, the round disk has the maximal area.*

Sketch. It is sufficient to consider only convex figures of the given perimeter; if a figure is not convex, pass to its convex hull and observe that it has a larger area and smaller perimeter.

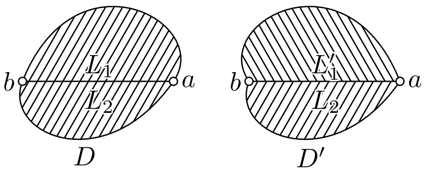
Note that the selection theorem (4.8) together with the exercise implies the existence of figure D with perimeter ℓ and maximal area.

It remains to show that D is a round disk; it will be done by means of elementary geometry.

Let us cut D along a chord $[ab]$ into two lenses, L_1 and L_2. Denote by L_1' the reflection of L_1 across the perpendicular bisector of $[ab]$. Note that D and $D' = L_1' \cup L_2$ have the same perimeter and area. That is, D' has perimeter ℓ and maximal possible area; in particular, D' is convex.

The following exercise will finish the proof. ☐

4.17. Exercise. *Suppose D is a convex figure such that for any chord $[ab]$ of D the above construction produces a convex figure D'. Show that D is a round disk.*



Another popular way to prove that D is a round disk is given by the so-called *Steiner's 4-joint method* [27].

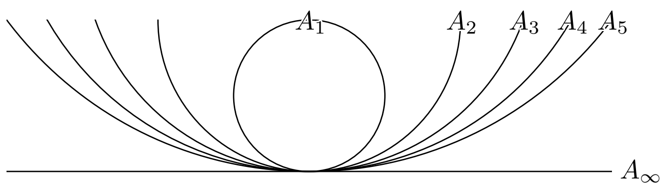
D Remarks

It seems that Hausdorff convergence was first introduced by Felix Hausdorff [71]. A couple of years later an equivalent definition was given by Wilhelm Blaschke [27].

The following refinement was introduced by Zdeněk Frolík [59] and rediscovered by Robert Wijsman [138]. This refinement is also called Hausdorff convergence; in fact, it takes an intermediate place between the original Hausdorff convergence and the so-called *closed convergence*, also introduced by Hausdorff in [71].

4.18. Definition. *Let $A_1, A_2, \ldots$ be a sequence of closed sets in a metric space $\mathcal{X}$. We say that the sequence A_n converges to a closed set A_∞ in the sense of Hausdorff if, for any $x \in \mathcal{X}$, we have $\operatorname{dist}_{A_n}(x) \to \to \operatorname{dist}_{A_\infty}(x)$ as $n \to \infty$.*

For example, suppose $\mathcal{X}$ is the Euclidean plane and A_n is the circle with radius n and center at the point $(0, n)$. If we use the standard definition (4.1), then the sequence $A_1, A_2, \ldots$ diverges, but it converges to the x-axis in the sense of Definition 4.18.



Further, consider the sequence of one-point sets $B_n = \{(n, 0)\}$ in the Euclidean plane. It diverges in the sense of the standard definition, but, in the sense of 4.18, it converges to the empty set; indeed, for any point x we have $\operatorname{dist}_{B_n}(x) \to \infty$ as $n \to \infty$ and $\operatorname{dist}_{\varnothing}(x) = \infty$ for any x.

The following exercise is analogous to the Blaschke selection theorem (4.8) for the modified Hausdorff convergence.

4.19. Exercise. *Let $\mathcal{X}$ be a proper metric space and $A_1, A_2, \dots$ be a sequence of closed sets in $\mathcal{X}$. Show that the sequence $A_1, A_2, \dots$ has a convergent subsequence in the sense of Definition 4.18.*

Lecture 5

Space of spaces

In this lecture we define and study the so-called Gromov–Hausdorff metric on the isometry classes of compact metric spaces.

A Gromov–Hausdorff metric

In this section we cook up a metric space out of all compact metric spaces. More precisely, we want to define the so-called Gromov–Hausdorff metric on the set of *isometry classes* of compact metric spaces. (Being isometric is an equivalence relation, and an isometry class is an equivalence class with respect to this relation.)

The obtained metric space will be denoted by GH. Given two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, denote by $[\mathcal{X}]$ and $[\mathcal{Y}]$ their isometry classes; that is, $\mathcal{X}' \in [\mathcal{X}]$ if and only if $\mathcal{X}' \stackrel{iso}{=\!=} \mathcal{X}$. Pedantically, the Gromov–Hausdorff distance from $[\mathcal{X}]$ to $[\mathcal{Y}]$ should be denoted as $|[\mathcal{X}]-[\mathcal{Y}]|_{\mathrm{GH}}$; but we will write it as $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}$ and say (not quite correctly) that $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}$ *is the Gromov–Hausdorff distance from* $\mathcal{X}$ *to* $\mathcal{Y}$. In other words, from now on the term *metric space* might also stand for its *isometry class.*

The metric on GH is defined as the maximal metric such that *the distance between subspaces in a metric space is not greater than the Hausdorff distance between them.* Here is a formal definition:

5.1. Definition. *The Gromov–Hausdorff distance $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}$ between compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is defined by the following relation.*

Given $r>0$, we have that $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}<r$ if and only if there exists a metric space $\mathcal{W}$ and subspaces $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{W}$ that are isometric to $\mathcal{X}$ and $\mathcal{Y}$, respectively, such that $|\mathcal{X}'-\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}}<r$. (Here $|\mathcal{X}'-$

$\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}}$ *denotes the Hausdorff distance between sets* $\mathcal{X}'$ *and* $\mathcal{Y}'$ *in* $\mathcal{W}$.*)*

5.2. Theorem. *The set of isometry classes of compact metric spaces equipped with Gromov–Hausdorff metric forms a metric space (which is denoted by* GH*).*

In other words, for arbitrary compact metric spaces $\mathcal{X}$, $\mathcal{Y}$, *and* $\mathcal{Z}$, *the following conditions hold*

(a) $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}\geqslant 0$;
(b) $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}=0$ *if and only if* $\mathcal{X}$ *is isometric to* $\mathcal{Y}$;
(c) $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}=|\mathcal{Y}-\mathcal{X}|_{\mathrm{GH}}$;
(d) $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}+|\mathcal{Y}-\mathcal{Z}|_{\mathrm{GH}}\geqslant|\mathcal{X}-\mathcal{Z}|_{\mathrm{GH}}$.

Note that *(a)*, *(c)*, and the if part of *(b)* follow directly from 5.1. Part *(d)* will be proved in Section 5B. The only-if part of *(b)* will be proved in Section 5C.

Recall that $a\cdot\mathcal{X}$ denotes $\mathcal{X}$ rescaled by a factor $a>0$; that is, $a\cdot\mathcal{X}$ is a metric space with the underlying set of $\mathcal{X}$ and the metric defined by

$$|x-y|_{a\cdot\mathcal{X}}:=a\cdot|x-y|_{\mathcal{X}}.$$

5.3. Exercise. *Let* $\mathcal{X}$ *be a compact metric space,* $\mathcal{O}$ *be the one-point metric space. Prove the following.*

(a) $|\mathcal{X}-\mathcal{O}|_{\mathrm{GH}}=\tfrac12\cdot\operatorname{diam}\mathcal{X}$.
(b) $|a\cdot\mathcal{X}-b\cdot\mathcal{X}|_{\mathrm{GH}}=\tfrac12\cdot|a-b|\cdot\operatorname{diam}\mathcal{X}$.
(c) $\iota[\mathcal{O}]=[\mathcal{O}]$ *for any isometry* $\iota\colon \mathrm{GH}\to\mathrm{GH}$.

5.4. Exercise. *Find two subsets* $A,B\subset\mathbb{R}^2$ *such that*

$$|A-B|_{\mathrm{GH}}>|A-\iota(B)|_{\mathrm{Haus}\,\mathbb{R}^2}$$

for any isometry ι *of* $\mathbb{R}^2$.

5.5. Exercise. *Let* $\mathcal{A}_r$ *be a rectangle* 1 *by* r *in the Euclidean plane and* $\mathcal{B}_r$ *be a closed line interval of length* r. *Show that*

$$|\mathcal{A}_r-\mathcal{B}_r|_{\mathrm{GH}}>\tfrac{1}{10}$$

for all large r.

5.6. Advanced exercise. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be compact metric spaces; denote by* $\hat{\mathcal{X}}$ *and* $\hat{\mathcal{Y}}$ *their injective envelopes (see 3D). Show that*

$$|\hat{\mathcal{X}}-\hat{\mathcal{Y}}|_{\mathrm{GH}}\leqslant 2\cdot|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}.$$

In other words, $\mathcal{X}\mapsto\hat{\mathcal{X}}$ *defines a 2-Lipschitz map* $\mathrm{GH}\to\mathrm{GH}$.

B Approximations and almost isometries

5.7. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces. A relation $\approx$ between points in $\mathcal{X}$ and $\mathcal{Y}$ is called ε-approximation if the following conditions hold:*

- *For any $x \in \mathcal{X}$ there is $y \in \mathcal{Y}$ such that $x \approx y$.*
- *For any $y \in \mathcal{Y}$ there is $x \in \mathcal{X}$ such that $x \approx y$.*
- *If $x \approx y$ and $x' \approx y'$ for some $x, x' \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$, then*

$$|x - x'|_{\mathcal{X}} \lesssim \big|y - y'|_{\mathcal{Y}}\big| \pm 2{\cdot}\varepsilon.$$

5.8. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two compact metric spaces. Show that*

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} < \varepsilon$$

if and only if there is an ε-approximation between $\mathcal{X}$ and $\mathcal{Y}$.

In other words, $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}}$ is the greatest lower bound of values $\varepsilon > 0$ such that there is an ε-approximation between $\mathcal{X}$ and $\mathcal{Y}$.

Proof of 5.2d. Suppose that

- $\approx_1$ is a relation between points in $\mathcal{X}$ and $\mathcal{Y}$,
- $\approx_2$ is a relation between points in $\mathcal{Y}$ and $\mathcal{Z}$.

Consider the relation $\approx_3$ between points in $\mathcal{X}$ and $\mathcal{Z}$ such that $x \approx_3 z$ if and only if there is $y \in \mathcal{Y}$ such that $x \approx_1 y$ and $y \approx_2 z$.

It is straightforward to check that if $\approx_1$ is an ε_1-approximation and $\approx_2$ is an ε_2-approximation, then $\approx_3$ is an $(\varepsilon_1 + \varepsilon_2)$-approximation.

Applying 5.8, we get that if

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} < \varepsilon_1 \quad \text{and} \quad |\mathcal{Y} - \mathcal{Z}|_{\mathrm{GH}} < \varepsilon_2,$$

then

$$|\mathcal{X} - \mathcal{Z}|_{\mathrm{GH}} < \varepsilon_1 + \varepsilon_2.$$

Hence 5.2*d* follows. □

The following weakened version of isometry is closely related to ε-approximations.

5.9. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces and $\varepsilon > 0$. A map*[1] *$f\colon \mathcal{X} \to \mathcal{Y}$ is called an ε-isometry if $f(\mathcal{X})$ is an ε-net in $\mathcal{Y}$ and*

$$|x - x'|_{\mathcal{X}} \lesssim \big|f(x) - f(x')|_{\mathcal{Y}}\big| \pm \varepsilon$$

for any $x, x' \in \mathcal{X}$.

5.10. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact metric spaces.*

(a) If $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} < \varepsilon$, then there is a $2{\cdot}\varepsilon$-isometry $f\colon \mathcal{X} \to \mathcal{Y}$.
(b) If there is an ε-isometry $f\colon \mathcal{X} \to \mathcal{Y}$, then $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} < \varepsilon$.

[1] possibly noncontinuous

C Optimal realization

Note that

$$|\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}} \geqslant |\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}},$$

where $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}'$, $\mathcal{Y}'$, and $\mathcal{W}$ are as in 5.1. The following proposition states that equality holds for some choice of $\mathcal{X}'$, $\mathcal{Y}'$, and $\mathcal{W}$.

5.11. Proposition. *For any two compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$ there is a metric space $\mathcal{W}$ with subsets $\mathcal{X}'$ and $\mathcal{Y}'$ such that $\mathcal{X}' \overset{iso}{=\!=} \mathcal{X}$, $\mathcal{Y}' \overset{iso}{=\!=} \mathcal{Y}$, and*

$$|\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}} = |\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}}.$$

Let us introduce the so-called *appropriate functions* and use them in a reinterpretation of the Gromov–Hausdorff distance.

Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}'$, $\mathcal{Y}'$, and $\mathcal{W}$ are as in 5.1. By passing to the subspace $\mathcal{X}' \cup \mathcal{Y}'$ in $\mathcal{W}$, we can assume that $\mathcal{W} = \mathcal{X}' \cup \mathcal{Y}'$. Note that in this case the metric on $\mathcal{W}$ is completely determined by the function $f\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ defined by

$$f(x,y) := |x - y|_{\mathcal{W}};$$

a function f that can appear this way will be called appropriate.

Note that a function $f\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is appropriate if and only if $x \mapsto f(x,y)$ and $y \mapsto f(x,y)$ are extension functions [see 2B]; that is, if

$$\begin{aligned} f(x,y) + f(x,y') \geqslant |y - y'|_{\mathcal{Y}} \geqslant |f(x,y) - f(x,y')|, \quad &\text{and} \\ f(x,y) + f(x',y) \geqslant |x - x'|_{\mathcal{X}} \geqslant |f(x,y) - f(x',y)| \end{aligned} \tag{❶}$$

for any $x, x', \in \mathcal{X}$ and $y, y' \in \mathcal{X}$. In other words, the following defines a semimetric on $\mathcal{X} \sqcup \mathcal{Y}$

$$|x - y|_{\mathcal{X}\sqcup\mathcal{Y}} := \begin{cases} |x-y|_{\mathcal{X}} & \text{if } x, y \in \mathcal{X}, \\ |x-y|_{\mathcal{Y}} & \text{if } x, y \in \mathcal{Y}, \\ f(x,y) & \text{if } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}, \end{cases}$$

and the corresponding metric space $\mathcal{W}$ contains isometric copies of $\mathcal{X}$ and $\mathcal{Y}$.

5.12. Observation. *Let $\mathcal{X}$, $\mathcal{Y}$ be metric spaces. Given an appropriate function $f\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, set*

$$\begin{aligned} a_f &= \max_{x\in\mathcal{X}}\{\min_{y\in\mathcal{Y}}\{f(x,y)\}\}, \\ b_f &= \max_{y\in\mathcal{Y}}\{\min_{x\in\mathcal{X}}\{f(x,y)\}\}, \\ c_f &= \max\{\, a_f, b_f \,\}. \end{aligned}$$

Then

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} = \inf\{\, c_f \,\},$$

where the greatest lower bound is taken for all appropriate functions $f\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$.

Proof of 5.11. Equip the product $\mathcal{X} \times \mathcal{Y}$ with ℓ_1-metric; that is,

$$|(x, y) - (x', y')|_{\mathcal{X}\times\mathcal{Y}} := |x - x'|_{\mathcal{X}} + |y - y'|_{\mathcal{Y}}$$

Note that any appropriate functions $f\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is 1-Lipschitz.

Let us equip the space of appropriate functions $\mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ with supnorm. Observe that the functional $f \mapsto c_f$ is continuous. By the Arzelà–Ascoli theorem, we can choose an appropriate function f with minimal possible value c_f. It remains to apply 5.12. □

5.13. Exercise. *Construct three compact metric spaces* $\mathcal{X}$, $\mathcal{Y}$, *and* $\mathcal{Z}$ *such that for any metric space* $\mathcal{W}$ *with subsets* $\mathcal{X}'$, $\mathcal{Y}'$, *and* $\mathcal{Z}'$ *such that* $\mathcal{X}' \overset{iso}{=\!=} \mathcal{X}$, $\mathcal{Y}' \overset{iso}{=\!=} \mathcal{Y}$, *and* $\mathcal{Z}' \overset{iso}{=\!=} \mathcal{Z}$ *at least one of the following three inequalities is strict*

$$\begin{aligned}
|\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}} &\geqslant |\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}},\\
|\mathcal{Y}' - \mathcal{Z}'|_{\mathrm{Haus}\,\mathcal{W}} &\geqslant |\mathcal{Y} - \mathcal{Z}|_{\mathrm{GH}},\\
|\mathcal{Z}' - \mathcal{X}'|_{\mathrm{Haus}\,\mathcal{W}} &\geqslant |\mathcal{Z} - \mathcal{X}|_{\mathrm{GH}}.
\end{aligned}$$

D Convergence

The Gromov–Hausdorff metric defines Gromov–Hausdorff convergence. Namely, a sequence of compact metric spaces $\mathcal{X}_n$ converges to compact metric spaces $\mathcal{X}_\infty$ in the sense of Gromov–Hausdorff if

$$|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \to 0 \quad \text{as} \quad n \to \infty.$$

This convergence is more important than the metric — in all applications, we use only the topology on GH, and we do not care about the particular value of the Gromov–Hausdorff distance between spaces. The following observation follows from 5.10:

5.14. Observation. *A sequence of compact metric spaces* $(\mathcal{X}_n)$ *converges to* $\mathcal{X}_\infty$ *in the sense of Gromov–Hausdorff if and only if there is a sequence* $\varepsilon_n \to 0+$ *and an* ε_n*-isometry* $f_n\colon \mathcal{X}_n \to \mathcal{X}_\infty$ *for each* n.

5.15. Exercise.

(a) *Show that a circle is not a Gromov–Hausdorff limit of compact simply-connected length spaces.*

(b) *Construct a compact non-simply-connected metric space that is a Gromov–Hausdorff limit of compact simply-connected length spaces.*

5.16. Exercise.

(a) *Show that a sequence of length metrics on the 2-sphere cannot converge to the unit disk in the sense of Gromov–Hausdorff.*

(b) *Construct a sequence of length metrics on the 3-sphere that converges to the unit 3-ball in the sense of Gromov–Hausdorff.*

E Uniformly totally bonded families

5.17. Definition. *A family $\boldsymbol{Q}$ of (isometry classes) of compact metric spaces is called* uniformly totally bonded *if it meets the following two conditions:*

(a) *Spaces in $\boldsymbol{Q}$ have uniformly bounded diameters; that is, there is $D \in \mathbb{R}$ such that*

$$\operatorname{diam} \mathcal{X} \leqslant D$$

for any space $\mathcal{X}$ in $\boldsymbol{Q}$.

(b) *For any $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that any space $\mathcal{X}$ in $\boldsymbol{Q}$ admits an ε-net with at most n points.*

5.18. Exercise. *Let $\boldsymbol{Q}$ be a family of compact spaces with uniformly bounded diameters. Show that $\boldsymbol{Q}$ is uniformly totally bonded if for any $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that*

$$\operatorname{pack}_\varepsilon \mathcal{X} \leqslant n$$

for any space $\mathcal{X}$ in $\boldsymbol{Q}$.

Fix a real constant C. A Borel measure μ on a metric space $\mathcal{X}$ is called *C-doubling* if

$$\mu[\mathrm{B}(p, 2{\cdot}r)] < C{\cdot}\mu[\mathrm{B}(p, r)]$$

for any point $p \in \mathcal{X}$ and any $r > 0$. A Borel measure is called *doubling* if it is *C-doubling* for some real constant C.

5.19. Exercise. *Let $\boldsymbol{Q}(C, D)$ be the set of all the compact metric spaces with diameter at most D that admit a C-doubling measure. Show that $\boldsymbol{Q}(C, D)$ is uniformly totally bounded.*

Fix an integer constant $M \geqslant 0$. A metric space $\mathcal{X}$ is called *M-doubling* if any $2{\cdot}r$-ball in $\mathcal{X}$ can be covered by M r-balls. A space $\mathcal{X}$ is called *doubling* if it is M-doubling for some M.

Observe that *a space is doubling if it admits a doubling measure.*

5.20. Exercise. *Given a metric space $\mathcal{X}$ consider family $\boldsymbol{B}_{\mathcal{X}}$ of all rescaled balls $\frac{1}{r}{\cdot}\overline{\mathrm{B}}[x,r]_{\mathcal{X}}$ for all $r > 0$ and $x \in \mathcal{X}$. Show that $\mathcal{X}$ is doubling if and only if $\boldsymbol{B}_{\mathcal{X}}$ is uniformly totally bounded.*

Given two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, we will write $\mathcal{X} \leqslant \mathcal{Y}$ if there is a distance-noncontracting map $f\colon \mathcal{X} \to \mathcal{Y}$; that is, if

$$|x - x'|_{\mathcal{X}} \leqslant |f(x) - f(x')|_{\mathcal{Y}}$$

for any $x, x' \in \mathcal{X}$.

5.21. Exercise.

(a) *Let $\mathcal{Y}$ be a compact metric space. Show that the set of all spaces $\mathcal{X}$ such that $\mathcal{X} \leqslant \mathcal{Y}$ is uniformly totally bounded.*

(b) *Show that for any uniformly totally bounded set $\boldsymbol{Q} \subset \mathrm{GH}$ there is a compact space $\mathcal{Y}$ such that $\mathcal{X} \leqslant \mathcal{Y}$ for any $\mathcal{X}$ in $\boldsymbol{Q}$.*

F Gromov selection theorem

The following theorem is analogous to Blaschke selection theorems (4.8).

5.22. Gromov selection theorem. *Let $\boldsymbol{Q}$ be a closed subset of* GH. *Then $\boldsymbol{Q}$ is compact if and only if the spaces in $\boldsymbol{Q}$ are uniformly totally bounded.*

5.23. Lemma. *The space* GH *is complete.*

Suppose $\mathcal{U}$ and $\mathcal{V}$ are metric spaces with isometric closed sets $A \subset \mathcal{U}$ and $A' \subset \mathcal{V}$; let $\iota\colon A \to A'$ be an isometry. Consider the gluing $\mathcal{W} = \mathcal{U} \sqcup_{\iota} \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ along ι [see 1D].

Let us identify points of $\mathcal{U}$ and $\mathcal{V}$ with their images in $\mathcal{W}$. It is straightforward to check that the metric on $\mathcal{W}$ is defined by

$$\begin{aligned}
|u - u'|_{\mathcal{W}} &:= |u - u'|_{\mathcal{U}},\\
|v - v'|_{\mathcal{W}} &:= |v - v'|_{\mathcal{V}},\\
|u - v|_{\mathcal{W}} &:= \min\{\, |u - a|_{\mathcal{U}} + |v - \iota(a)|_{\mathcal{V}} \;:\; a \in A \,\},
\end{aligned}$$

where $u, u' \in \mathcal{U}$ and $v, v' \in \mathcal{V}$.

If one applies this construction to two copies of one space $\mathcal{U}$ with a set $A \subset \mathcal{U}$ and the identity map $\iota\colon A \to A$, then the obtained space is called the doubling of $\mathcal{U}$ along A; this space can be denoted by $\sqcup_A^2 \mathcal{U}$.

Note that the inclusions $\mathcal{U} \hookrightarrow \mathcal{W}$ and $\mathcal{V} \hookrightarrow \mathcal{W}$ are distance-preserving. Therefore we can and will consider $\mathcal{U}$ and $\mathcal{V}$ as the subspaces of $\mathcal{W}$; this way the subsets A and A' will be identified and denoted further by A. Note that $A = \mathcal{U} \cap \mathcal{V} \subset \mathcal{W}$.

Proof. Let $\mathcal{X}_1, \mathcal{X}_2, \dots$ be a Cauchy sequence in GH. Passing to a subsequence if necessary, we can assume that $|\mathcal{X}_n - \mathcal{X}_{n+1}|_{\mathrm{GH}} < \frac{1}{2^n}$ for each n. In particular, for each n there is a metric space $\mathcal{V}_n$ with distance-preserving inclusions $\mathcal{X}_n \hookrightarrow \mathcal{V}_n$ and $\mathcal{X}_{n+1} \hookrightarrow \mathcal{V}_n$ such that

$$|\mathcal{X}_n - \mathcal{X}_{n+1}|_{\mathrm{Haus}\,\mathcal{V}_n} < \tfrac{1}{2^n}$$

for each n. Moreover, we may assume that $\mathcal{V}_n = \mathcal{X}_n \cup \mathcal{X}_{n+1}$.

Let us glue $\mathcal{V}_1$ to $\mathcal{V}_2$ along $\mathcal{X}_2$; to the obtained space glue $\mathcal{V}_3$ along $\mathcal{X}_3$, and so on. The obtained metric space $\mathcal{W}$ has an underlying set formed by the disjoint union of all $\mathcal{X}_n$ such that each inclusion $\mathcal{X}_n \hookrightarrow \hookrightarrow \mathcal{W}$ is distance-preserving and

$$|\mathcal{X}_n - \mathcal{X}_{n+1}|_{\mathrm{Haus}\,\mathcal{W}} < \tfrac{1}{2^n}$$

for each n. In particular,

❶ $$|\mathcal{X}_m - \mathcal{X}_n|_{\mathrm{Haus}\,\mathcal{W}} < \tfrac{1}{2^{n-1}}$$

if $m > n$.

Denote by $\bar{\mathcal{W}}$ the completion of $\mathcal{W}$. Observe that the union $\mathcal{X}_1 \cup \cup \mathcal{X}_2 \cup \dots \cup \mathcal{X}_n$ is compact and ❶ implies that it forms a $\frac{1}{2^{n-1}}$-net in $\bar{\mathcal{W}}$. Whence $\bar{\mathcal{W}}$ is compact; see 1.11*c* and 1.13.

Applying the Blaschke selection theorem (4.8), we can pass to a subsequence of $\mathcal{X}_n$ that converges in Haus $\bar{\mathcal{W}}$; denote its limit by $\mathcal{X}_\infty$. It remains to observe that $\mathcal{X}_\infty$ is the Gromov–Hausdorff limit of $\mathcal{X}_n$. □

Proof of 5.22; only-if part. Suppose that there is no sequence $\varepsilon_n \to 0$ as described in 5.17. Observe that in this case there is a sequence of spaces $\mathcal{X}_n \in \boldsymbol{Q}$ such that

$$\operatorname{pack}_\delta \mathcal{X}_n \to \infty \quad \text{as} \quad n \to \infty$$

for some fixed $\delta > 0$.

Since $\boldsymbol{Q}$ is compact, this sequence has a partial limit, say $\mathcal{X}_\infty \in \boldsymbol{Q}$. Observe that $\operatorname{pack}_\delta \mathcal{X}_\infty = \infty$. Therefore, $\mathcal{X}_\infty$ is not compact — a contradiction.

If part. Given a positive integer n consider the set of all nonempty metric spaces $\mathcal{W}_n$ with the number of points at most n and diameter $\leqslant D$. Note that $\mathcal{W}_n$ is a compact set in GH for each n.

Let D and $n = n(\varepsilon)$ be as in the definition of uniformly totally bonded families (5.17).

Note that an ε-net of any $\mathcal{X} \in \boldsymbol{Q}$ belongs to $\mathcal{W}_{n(\varepsilon)}$. Therefore, $\mathcal{W}_{n(\varepsilon)}$ is a compact ε-net of $\boldsymbol{Q}$ for any $\varepsilon > 0$. Since $\boldsymbol{Q}$ is closed in a complete space GH, it implies that $\boldsymbol{Q}$ is compact. □

5.24. Exercise. *Show that most of the compact metric spaces are homeomorphic to the Cantor set.*

More precisely, suppose $\boldsymbol{Q}$ denotes all metric spaces homeomorphic to the Cantor set. Show that $\boldsymbol{Q}$ is a dense G-delta set in GH.

5.25. Exercise. *Show that the space* GH *is*

(a) separable, *(b) length, and* *(c) geodesic.*

5.26. Exercise. *For two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $\mathcal{X} \leqslant \mathcal{Y} + \varepsilon$ if there is a map $f\colon \mathcal{X} \to \mathcal{Y}$ such that*

$$|x - x'|_{\mathcal{X}} \leqslant |f(x) - f(x')|_{\mathcal{Y}} + \varepsilon$$

for any $x, x' \in \mathcal{X}$.

(a) Show that

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}'} := \inf \{ \varepsilon > 0 \, : \, \mathcal{X} \leqslant \mathcal{Y} + \varepsilon \quad \textit{and} \quad \mathcal{Y} \leqslant \mathcal{X} + \varepsilon \}$$

defines a metric on the space of (isometry classes of) compact metric spaces.

(b) Moreover, $|{*} - {*}|_{\mathrm{GH}'}$ is equivalent to the Gromov–Hausdorff metric; that is,

$$|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \to 0 \qquad \Longleftrightarrow \qquad |\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}'} \to 0$$

as $n \to \infty$.

G Universal ambient space

Recall that a metric space is called universal if it contains an isometric copy of any separable metric space (in particular, any compact metric space). Examples of universal spaces include $\mathcal{U}_\infty$ — the Urysohn space and ℓ^∞ — the space of bounded infinite sequences with the metric defined by sup-norm; see 2.12 and 2.3.

The following proposition says that the space $\mathcal{W}$ in Definition 5.1 can be exchanged to a fixed universal space.

5.27. Proposition. *Let $\mathcal{U}$ be a universal space. Then for any compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$ we have*

$$|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}} = \inf\{|\mathcal{X}'-\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{U}}\},$$

where the greatest lower bound is taken over all pairs of sets $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{U}$ that are isometric to $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Proof of 5.27. By the definition (5.1), we have that

$$|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}} \leqslant \inf\{|\mathcal{X}'-\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{U}}\};$$

it remains to prove the opposite inequality.

Suppse $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}} < \varepsilon$; let $\mathcal{X}'$, $\mathcal{Y}'$ and $\mathcal{W}$ be as in 5.1. We can assume that $\mathcal{W} = \mathcal{X}'\cup\mathcal{Y}'$; otherwise, pass to the subspace $\mathcal{X}'\cup\mathcal{Y}'$ of $\mathcal{W}$. In this case, $\mathcal{W}$ is compact; in particular, it is separable.

Since $\mathcal{U}$ is universal, there is a distance-preserving embedding of $\mathcal{W}$ in $\mathcal{U}$; let us keep the same notation for $\mathcal{X}'$, $\mathcal{Y}'$, and their images. It follows that

$$|\mathcal{X}'-\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{U}} < \varepsilon,$$

— hence the result. □

5.28. Exercise. *Let $\mathcal{U}_\infty$ be the Urysohn space. Given two compact sets A and B in $\mathcal{U}_\infty$, define*

$$\|A-B\| := \inf\{|A-\iota(B)|_{\mathrm{Haus}\,\mathcal{U}_\infty}\},$$

where the greatest lower bound is taken for all isometrics ι of $\mathcal{U}_\infty$. Show that $\|*-*\|$ defines a semimetric on nonempty compact subsets of $\mathcal{U}_\infty$ and its corresponding metric space is isometric to GH.

The value $\|A-B\|$ is called Hausdorff distance up to isometry from A to B in $\mathcal{U}_\infty$.

H Remarks

Suppose $\mathcal{X}_n \xrightarrow{\mathrm{GH}} \mathcal{X}_\infty$, then there is a metric on the disjoint union

$$\boldsymbol{X} = \bigsqcup_{n\in\mathbb{N}\cup\{\infty\}} \mathcal{X}_n$$

that satisfies the following property:

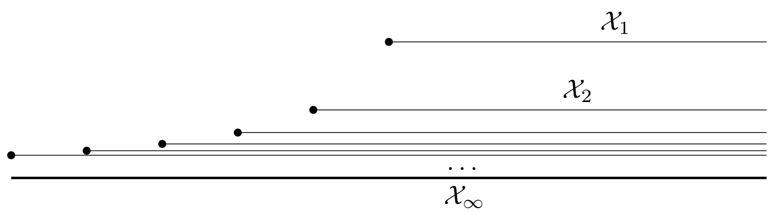
5.29. Property. *The restriction of metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric, and $\mathcal{X}_n \xrightarrow{\mathrm{H}} \mathcal{X}_\infty$ as subsets in $\boldsymbol{X}$.*

Indeed, since $\mathcal{X}_n \xrightarrow{\mathrm{GH}} \mathcal{X}_\infty$, there is a metric on $\mathcal{V}_n = \mathcal{X}_n \sqcup \mathcal{X}_\infty$ such that the restriction of metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric, and $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{Haus}\,\mathcal{V}_n} < \varepsilon_n$ for some sequence $\varepsilon_n \to 0$. Gluing all $\mathcal{V}_n$ along $\mathcal{X}_\infty$, we get the required space $\boldsymbol{X}$.

In other words, the metric on $\boldsymbol{X}$ *defines* the convergence $\mathcal{X}_n \xrightarrow{\mathrm{GH}}$ $\xrightarrow{\mathrm{GH}} \mathcal{X}_\infty$. This metric makes it possible to talk about limits of sequences $x_n \in \mathcal{X}_n$ as $n \to \infty$, as well as weak limits of a sequence of Borel measures μ_n on $\mathcal{X}_n$ and so on.

For that reason, it is useful to define convergence by specifying the metric on $\boldsymbol{X}$ that satisfies the property for the variation of Hausdorff convergence described in Section 4D.

This approach is more flexible; in particular, it can be used to define the Gromov–Hausdorff convergence of arbitrary metric spaces (not necessarily compact). A limit space for this generalized convergence is not uniquely defined. For example, if each space $\mathcal{X}_n$ in the sequence is isometric to the half-line, then its limit might be isometric to the half-line or the whole line. The first convergence is evident and the second could be guessed from the diagram.



Often the isometry class of the limit can be fixed by marking a point p_n in each space $\mathcal{X}_n$, it is called pointed Gromov–Hausdorff convergence — we say that $(\mathcal{X}_n, p_n)$ converges to $(\mathcal{X}_\infty, p_\infty)$ if there is a metric on $\boldsymbol{X}$ as in 5.29 such that $\mathcal{X}_n \xrightarrow{\mathrm{H}} \mathcal{X}_\infty$ and $p_n \to p_\infty$. For example, the sequence $(\mathcal{X}_n, p_n) = (\mathbb{R}_+, 0)$ converges to $(\mathbb{R}_+, 0)$, while $(\mathcal{X}_n, p_n) = (\mathbb{R}_+, n)$ converges to $(\mathbb{R}, 0)$.

The pointed convergence works nicely for proper metric spaces; the following theorem is an analog of Gromov's selection theorem for this convergence.

5.30. Theorem. *Let $\boldsymbol{Q}$ be a set of isometry classes of pointed proper metric spaces. Assume that for any $R > 0$, the R-balls in the spaces centered at the marked points form a uniformly totally bounded family* *of spaces. Then $\boldsymbol{Q}$ is precompact with respect to the pointed Gromov–Hausdorff convergence.*

Let us mention a characterization of doubling spaces discovered by Patrice Assouad [13], [72, 12.2].

Given a metric space $\mathcal{X}$, denote by $\mathcal{X}^\theta$ a space with the same underlying set and the metric defined by

$$|x-y|_{\mathcal{X}^\theta} := |x-y|^\theta_{\mathcal{X}};$$

here we assume that $0<\theta<1$. The space $\mathcal{X}^\theta$ is called snowflake of $\mathcal{X}$.

5.31. Theorem. *Suppose $0<\theta<1$. A metric space $\mathcal{X}$ is doubling if and only if its snowflake $\mathcal{X}^\theta$ admits a bi-Lipschitz embedding in some Euclidean space.*

Lecture 6

Ultralimits

Ultralimits provide a very general way to pass to a limit. This procedure works for *any* sequence of metric spaces, its result reminds limit in the sense of Gromov–Hausdorff, but has some strange features; for example, the limit of a constant sequence of spaces $\mathcal{X}_n = \mathcal{X}$ is *not* $\mathcal{X}$ in general (see 6.13b).

In geometry, ultralimits are used mostly as a canonical way to pass to a convergent subsequence. It is very useful in the proofs where one needs to repeat "pass to convergent subsequence" too many times.

This lecture is based on the introductory part of the paper by Bruce Kleiner and Bernhard Leeb [85].

A Faces of ultrafilters

Measure-theoretic definition. Recall that $\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers.

6.1. Definition. *A finitely additive measure ω on $\mathbb{N}$ is called an ultrafilter if it meets the following condition:*

(a) $\omega(\mathbb{N}) = 1$ and $\omega(S) = 0$ or 1 for any subset $S \subset \mathbb{N}$.

An ultrafilter ω is called nonprincipal if in addition

(b) $\omega(F) = 0$ for any finite subset $F \subset \mathbb{N}$.

If $\omega(S) = 0$ for some subset $S \subset \mathbb{N}$, we say that S is ω-small. If $\omega(S) = 1$, we say that S contains ω-almost all elements of $\mathbb{N}$.

6.2. Advanced exercise. *Let ω be an ultrafilter on $\mathbb{N}$ and $f\colon \mathbb{N} \to \mathbb{N}$. Suppose that $\omega(S) \leqslant \omega(f^{-1}(S))$ for any set $S \subset \mathbb{N}$. Show that $f(n) = n$ for ω-almost all $n \in \mathbb{N}$.*

Classical definition. More commonly, a nonprincipal ultrafilter is defined as a collection, say $\mathfrak{F}$, of subsets in $\mathbb{N}$ such that

1. if $P \in \mathfrak{F}$ and $Q \supset P$, then $Q \in \mathfrak{F}$,
2. if $P, Q \in \mathfrak{F}$, then $P \cap Q \in \mathfrak{F}$,
3. for any subset $P \subset \mathbb{N}$, either P or its complement is an element of $\mathfrak{F}$.
4. if $F \subset \mathbb{N}$ is finite, then $F \notin \mathfrak{F}$.

Setting $P \in \mathfrak{F} \Leftrightarrow \omega(P) = 1$ makes these two definitions equivalent.

A nonempty collection of sets $\mathfrak{F}$ that does not include the empty set and satisfies only conditions 1 and 2 is called a filter; if, in addition, $\mathfrak{F}$ satisfies condition 3, it is called an ultrafilter. From Zorn's lemma, it follows that every filter contains an ultrafilter. Thus there is an ultrafilter $\mathfrak{F}$ contained in the filter of all complements of finite sets. Clearly, this ultrafilter $\mathfrak{F}$ is nonprincipal.

Stone–Čech compactification. Given a set $S \subset \mathbb{N}$, consider subset Ω_S of all ultrafilters ω such that $\omega(S) = 1$. It is straightforward to check that the sets Ω_S for all subsets $S \subset \mathbb{N}$ form a topology on the set of ultrafilters on $\mathbb{N}$. The obtained space was first considered by Andrey Tikhonov and called Stone–Čech compactification of $\mathbb{N}$; it is usually denoted as $\beta\mathbb{N}$.

Let ω_n be a principal ultrafilter such that $\omega_n(\{n\}) = 1$; that is, $\omega_n(S) = 1$ if and only if $n \in S$. Note that $n \mapsto \omega_n$ defines an embedding $\mathbb{N} \hookrightarrow \beta\mathbb{N}$; so, we can (and will) consider $\mathbb{N}$ as a subset of $\beta\mathbb{N}$.

The space $\beta\mathbb{N}$ is the maximal compact Hausdorff space that contains $\mathbb{N}$ as an everywhere dense subset. More precisely, the inclusion $\mathbb{N} \hookrightarrow \beta\mathbb{N}$ has the following universal property: *for any compact Hausdorff space $\mathcal{X}$ and a map $f\colon \mathbb{N} \to \mathcal{X}$ there is a unique continuous map $\bar{f}\colon \beta\mathbb{N} \to X$ such that the restriction $\bar{f}|_{\mathbb{N}}$ coincides with f.*

B Ultralimits of points

Let us fix a nonprincipal ultrafilter ω once and for all.

Assume x_n is a sequence of points in a metric space $\mathcal{X}$. Let us define the ω-limit of a sequence $x_1, x_2, \ldots$ as the point $x_\omega \in \mathcal{X}$ such that for any $\varepsilon > 0$, point x_n lies in $\mathrm{B}(x_\omega, \varepsilon)$ for ω-almost all n; that is, if

$$S_\varepsilon = \{\, n \in \mathbb{N} \;:\; |x_\omega - x_n| < \varepsilon \,\},$$

then $\omega(S_\varepsilon) = 1$ for any $\varepsilon > 0$. In this case, we will write

$$x_\omega = \lim_{n \to \omega} x_n \quad \text{or} \quad x_n \to x_\omega \text{ as } n \to \omega.$$

For example, if ω_n is the *principal* ultrafilter defined by $\omega_n\{n\} = 1$ for some $n \in \mathbb{N}$, then $x_{\omega_n} = x_n$.

The sequence x_n can be regarded as a map $\mathbb{N} \to \mathcal{X}$ defined by $n \mapsto x_n$. If $\mathcal{X}$ is compact, then the map $\mathbb{N} \to \mathcal{X}$ can be extended to a continuous map $\beta\mathbb{N} \to \mathcal{X}$ from the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Then the ω-limit x_ω is the image of ω.

Note that the ω-limits of a sequence and its subsequence may differ. For example, sequence $y_n = -(-1)^n$ is a subsequence of $x_n = (-1)^n$, but for any ultrafilter ω, we have

$$\lim_{n\to\omega} x_n \neq \lim_{n\to\omega} y_n.$$

6.3. Proposition. *Let x_n be a sequence of points in a metric space $\mathcal{X}$. Assume $x_n \to x_\omega$ as $n \to \omega$. Then x_ω is a* partial limit *of x_n; that is, there is a subsequence of x_n that converges to x_ω in the usual sense.*

Proof. Given $\varepsilon > 0$, let $S_\varepsilon = \{\, n \in \mathbb{N} : |x_n - x_\omega| < \varepsilon \,\}$. Recall that $\omega(S_\varepsilon) = 1$ for any $\varepsilon > 0$.

Since ω is nonprincipal, the set S_ε is infinite for any $\varepsilon > 0$. Therefore, we can choose an increasing sequence n_k such that $n_k \in S_{\frac{1}{k}}$ for each $k \in \mathbb{N}$. Clearly, $x_{n_k} \to x_\omega$ as $k \to \infty$. $\square$

6.4. Proposition. *Any sequence x_n of points in a compact metric space $\mathcal{X}$ has a unique ω-limit x_ω.*

In particular, a bounded sequence of real numbers has a unique ω-limit.

The proposition is analogous to the Bolzano–Weierstrass theorem, and it can be proved the same way. The following lemma is an ultralimit analog of the Cauchy convergence test.

6.5. Lemma. *A sequence of points in a metric space converges if and only if all its subsequences have the same ω-limit.*

Proof. The only-if part is evident; it remains to prove the if part. Suppose z is a ω-limit of all subsequences of $x_1, x_2, \ldots$ By 6.3, z is a partial limit of x_n. If $x_1, x_2, \ldots$ is Cauchy, then $x_n \to z$, and the lemma is proved.

Assume $x_1, x_2, \ldots$ is not Cauchy. Then for some $\varepsilon > 0$, there is a subsequence y_n of x_n such that $|x_n - y_n| \geqslant \varepsilon$ for all n. Therefore $|x_\omega - y_\omega| \geqslant \varepsilon$ — a contradiction. $\square$

Recall that ℓ^∞ denotes the space of bounded sequences of real numbers equipped with the sup-norm.

6.6. Exercise. *Construct a linear functional $L\colon \ell^\infty \to \mathbb{R}$ such that for any sequence $\boldsymbol{s} = (s_1, s_2, \dots) \in \ell^\infty$ the image $L(\boldsymbol{s})$ is a partial limit of $s_1, s_2, \dots$*

6.7. Exercise. *Suppose that $f\colon \mathbb{N} \to \mathbb{N}$ is a map such that*

$$\lim_{n\to\omega} x_n = \lim_{n\to\omega} x_{f(n)}$$

for any bounded sequence x_n of real numbers. Show that $f(n) = n$ for ω-almost all $n \in \mathbb{N}$.

C An illustration

In this section, we illustrate the power of ultralimits by proving the following simple claim.

6.8. Claim. *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact spaces. Suppose that for every $n \in \mathbb{N}$ there is a $\tfrac{1}{n}$-isometry $f_n\colon \mathcal{X} \to \mathcal{Y}$. Then there is an isometry $\mathcal{X} \to \mathcal{Y}$.*

Proof. Consider the ω-limit f_ω of f_n; according to 6.4, f_ω is defined. Since

$$|f_n(x) - f_n(x')| \lessgtr |x - x'| \pm \tfrac{1}{n}$$

we get that

$$|f_\omega(x) - f_\omega(x')| = |x - x'|$$

for any $x, x' \in \mathcal{X}$; that is, f_ω is distance-preserving.

Further, since f_n is a $\tfrac{1}{n}$-isometry, for any $y \in \mathcal{Y}$ there is a sequence $x_1, x_2, \dots \in \mathcal{X}$ such that $|f_n(x_n) - y| \leqslant \tfrac{1}{n}$ for any n. Therefore,

$$f_\omega(x_\omega) = y,$$

where x_ω is the ω-limit of x_n; that is, f_ω is onto.

It follows that $f_\omega\colon \mathcal{X} \to \mathcal{Y}$ is an isometry. $\square$

D Ultralimits of spaces

Recall that ω is a fixed nonprincipal ultrafilter on $\mathbb{N}$.

Let $\mathcal{X}_n$ be a sequence of metric spaces. Consider all sequences of points $x_n \in \mathcal{X}_n$. On the set of all such sequences, define an ∞-semimetric by

❶ $$|(x_n) - (y_n)| := \lim_{n\to\omega} |x_n - y_n|_{\mathcal{X}_n}.$$

Note that the ω-limit on the right-hand side is always defined and takes a value in $[0,\infty]$. (The ω-convergence to ∞ is defined analogously to the usual convergence to ∞; that is, $\lim x_n = \infty \iff \lim \frac{1}{x_n} = 0$).

Let $\mathcal{X}_\omega$ be the corresponding metric space; that is, the underlying set of $\mathcal{X}_\omega$ is formed by classes of equivalence of sequences of points $x_n \in \mathcal{X}_n$ defined by

$$(x_n) \sim (y_n) \;\Leftrightarrow\; \lim_{n\to\omega} |x_n - y_n| = 0$$

and the distance is defined by ❶.

The space $\mathcal{X}_\omega$ is called the *ω-limit* of $\mathcal{X}_n$. (It is also called *ω-product*; this term is motivated by the fact that $\mathcal{X}_\omega$ is a quotient of the product $\prod \mathcal{X}_n$) Typically $\mathcal{X}_\omega$ will denote the ω-limit of sequence $\mathcal{X}_n$; we may also write

$$\mathcal{X}_n \to \mathcal{X}_\omega \text{ as } n \to \omega, \quad \text{or} \quad \mathcal{X}_\omega = \lim_{n\to\omega} \mathcal{X}_n.$$

Given a sequence $x_n \in \mathcal{X}_n$, we will denote by x_ω its equivalence class which is a point in $\mathcal{X}_\omega$; it can be written as

$$x_n \to x_\omega \text{ as } n \to \omega, \quad \text{or} \quad x_\omega = \lim_{n\to\omega} x_n.$$

6.9. Observation. *The ω-limit of any sequence of metric spaces is complete.*

We will repeat the proof of 1.9 using a slightly different language.

Proof. Let $\mathcal{X}_n$ be a sequence of metric spaces and $\mathcal{X}_n \to \mathcal{X}_\omega$ as $n \to \omega$. Choose a Cauchy sequence $x_1, x_2, \ldots \in \mathcal{X}_\omega$. Passing to a subsequence, we can assume that $|x_k - x_m|_{\mathcal{X}_\omega} < \frac{1}{k}$ if $k < m$.

Choose a double sequence $x_{n,m} \in \mathcal{X}_n$ such that for any fixed m we have $x_{n,m} \to x_m$ as $n \to \omega$. Note that for any $k < m$ the inequality $|x_{n,k} - x_{n,m}| < \frac{1}{k}$ holds for ω-almost all n.

Given $m \in \mathbb{N}$, consider the subset $S_m \subset \mathbb{N}$ defined by

$$S_m = \left\{\, n \geqslant m \,:\, |x_{n,k} - x_{n,l}| < \tfrac{1}{k} \quad \text{for all} \quad k < l \leqslant m \,\right\}.$$

Note that

- $\mathbb{N} = S_1 \supset S_2 \supset \ldots$
- $\omega(S_m) = 1$ for each m, and
- $\min S_m \geqslant m$.

Consider the sequence $y_n = x_{n,m(n)}$, where $m(n)$ is the largest value such that $n \in S_{m(n)}$; from above, $m(n) \leqslant n$. Denote by $y_\omega \in \mathcal{X}_\omega$ the ω-limit of y_n.

Observe that $|y_m - x_{n,m}| < \frac{1}{m}$ for ω-almost all n. It follows that $|x_m - y_\omega| \leqslant \frac{1}{m}$ for any m. Therefore, $x_n \to y_\omega$ as $n \to \infty$. That is, any Cauchy sequence in $\mathcal{X}_\omega$ converges. $\square$

6.10. Observation. *The ω-limit of any sequence of length spaces is geodesic.*

Proof. If $\mathcal{X}_n$ is a sequence of length spaces, then for any sequence of pairs $x_n, y_n \in X_n$ there is a sequence of $\frac{1}{n}$-midpoints z_n.

Let $x_n \to x_\omega$, $y_n \to y_\omega$ and $z_n \to z_\omega$ as $n \to \omega$. Note that z_ω is a midpoint of x_ω and y_ω in $\mathcal{X}_\omega$.

By 6.9, $\mathcal{X}_\omega$ is complete. Applying Menger's lemma (1.27), we get the statement. $\square$

6.11. Exercise. *Show that an ultralimit of metric trees is a metric tree.*

6.12. Exercise. *Suppose that $\mathcal{X}_\infty$ and $\mathcal{X}_1, \mathcal{X}_2, \dots$ are compact metric spaces. Assume $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$. Show that $\mathcal{X}_\omega \overset{iso}{=\!=} \mathcal{X}_\infty$.*

Pointed limit. If $\operatorname{diam} \mathcal{X}_n \to \infty$ as $n \to \omega$, then the metric on $\mathcal{X}_\omega$ takes value ∞; so $\mathcal{X}_\omega$ has at least two metric components.

To specify a metric component in $\mathcal{X}_\omega$, we may choose a sequence of marked points p_n in $\mathcal{X}_n$, and pass to the metric component of its ω-limit p_ω in $\mathcal{X}_\omega$. The obtained metric component $\mathcal{Z} = (\mathcal{X}_\omega)_{p_\omega}$ with marked p_ω is called the pointed ω-limit of $(\mathcal{X}_n, p_n)$. Note that $\mathcal{Z}$ is a genuine metric space.

If, in the definition of ultralimit, we consider only sequences $x_n \in \in \mathcal{X}_n$ with such that $|p_n - x_n|$ is bounded, then arrive at $\mathcal{Z}$.

For proper metric spaces, there is a relation between the pointed ultralimit and the pointed Gromov–Hausdorff limit introduced in 5H. Namely, *if $(\mathcal{X}_\infty, p_\infty)$ and $(\mathcal{X}_1, p_1), (\mathcal{X}_2, p_2), \dots$ are proper pointed metric spaces such that $(\mathcal{X}_n, p_n) \xrightarrow{\text{GH}} (\mathcal{X}_\infty, p_\infty)$ as defined in 5H, then $(\mathcal{X}_\infty, p_\infty)$ is isometric to the pointed ω-limit of $(\mathcal{X}_n, p_n)$.* The proof is the same as for 6.12.

E Ultrapower

If all the metric spaces in the sequence are identical $\mathcal{X}_n = \mathcal{X}$, its ω-limit $\lim_{n\to\omega} \mathcal{X}_n$ is denoted by $\mathcal{X}^\omega$ and called ω-power of $\mathcal{X}$ (also known as ω-completion).

6.13. Exercise. *For any point $x \in \mathcal{X}$, consider the constant sequence $x_n = x$ and set $\iota(x) = \lim_{n\to\omega} x_n \in \mathcal{X}^\omega$.*

(a) *Show that* $\iota\colon \mathcal{X} \to \mathcal{X}^\omega$ *is distance-preserving embedding. (So we can and will consider* $\mathcal{X}$ *as a subset of* $\mathcal{X}^\omega$*.)*
(b) *Show that* ι *is onto if and only if* $\mathcal{X}$ *is compact.*
(c) *Show that if* $\mathcal{X}$ *is proper, then* $\iota(\mathcal{X})$ *forms a metric component of* $\mathcal{X}^\omega$*; that is, a subset of* $\mathcal{X}^\omega$ *that lies at a finite distance from a given point.*

If $\mathcal{X}$ is a genuine metric space, then the metric component of $\mathcal{X}$ in $\mathcal{X}^\omega$ is also called the ultrapower of $\mathcal{X}$; if needed, we may call it the small ultrapower, and the whole space $\mathcal{X}^\omega$ could be called the big ultrapower of $\mathcal{X}$. Note that the small ultrapower of genuine metric space is a genuine metric space. Further, according to *(c)*, *proper metric space is isometric to its small ultrapower.*

Note that *(b)* implies that the inclusion $\mathcal{X} \hookrightarrow \mathcal{X}^\omega$ is not onto if the space $\mathcal{X}$ is not compact. However, the spaces $\mathcal{X}$ and $\mathcal{X}^\omega$ might be isometric; here is an example:

6.14. Exercise. *Let* $\mathcal{X}$ *be an infinite countable set with discrete metric; that is* $|x-y|_{\mathcal{X}} = 1$ *if* $x \neq y$*. Show that*
(a) $\mathcal{X}^\omega$ *is not isometric to* $\mathcal{X}$*, but*
(b) $\mathcal{X}^\omega$ *is isometric to* $(\mathcal{X}^\omega)^\omega$*.*

6.15. Exercise. *Given a nonprincipal ultrafilter* ω*, construct an ultrafilter* ω_1 *such that*

$$\mathcal{X}^{\omega_1} \overset{iso}{=\!=} (\mathcal{X}^\omega)^\omega$$

for any metric space $\mathcal{X}$*.*

6.16. Observation. *Let* $\mathcal{X}$ *be a complete metric space. Then* $\mathcal{X}^\omega$ *is geodesic space if and only if* $\mathcal{X}$ *is a length space.*

Proof. The if part follows from 6.10; it remains to prove the only-if part

Assume $\mathcal{X}^\omega$ is geodesic. Then any pair of points $x, y \in \mathcal{X}$ has a midpoint $z_\omega \in \mathcal{X}^\omega$. Fix a sequence of points $z_n \in \mathcal{X}$ such that $z_n \to z_\omega$ as $n \to \omega$.

Note that $|x - z_n|_{\mathcal{X}} \to \frac{1}{2}\cdot|x-y|_{\mathcal{X}}$ and $|y - z_n|_{\mathcal{X}} \to \frac{1}{2}\cdot|x-y|_{\mathcal{X}}$ as $n \to \omega$. In particular, for any $\varepsilon > 0$, the point z_n is an ε-midpoint of x and y for ω-almost all n. It remains to apply Menger's lemma (1.27). $\square$

6.17. Exercise. *Assume* $\mathcal{X}$ *is a complete length space and* $p, q \in \mathcal{X}$ *cannot be joined by a geodesic in* $\mathcal{X}$*. Show that there is at least a continuum of distinct geodesics between* p *and* q *in the ultrapower* $\mathcal{X}^\omega$*.*

6.18. Exercise. *Construct a proper metric space* $\mathcal{X}$ *such that its big ultrapower* $\mathcal{X}^\omega$ *is not locally compact.*

F Tangent and asymptotic spaces

Choose a space $\mathcal{X}$ and a sequence λ_n of positive numbers. Consider the sequence of rescalings $\mathcal{X}_n = \lambda_n\cdot\mathcal{X} = (\mathcal{X}, \lambda_n\cdot|{*}-{*}|_{\mathcal{X}})$.

Choose a point $p \in \mathcal{X}$ and denote by p_n the corresponding point in $\mathcal{X}_n$. Consider the ω-limit $\mathcal{X}_\omega$ of $\mathcal{X}_n$ (one may denote it by $\lambda_\omega\cdot\mathcal{X}$); set p_ω to be the ω-limit of p_n.

If $\lambda_n \to \infty$ as $n \to \omega$, then the metric component of p_ω in $\mathcal{X}_\omega$ is called λ_ω-tangent space at p and denoted by $\mathrm{T}_p^{\lambda_\omega}\mathcal{X}$ (or $\mathrm{T}_p^{\omega}\mathcal{X}$ if $\lambda_n = n$).

If $\lambda_n \to 0$ as $n \to \omega$, then the metric component of p_ω is called λ_ω-asymptotic space[1] and denoted by $\operatorname{Asym}\mathcal{X}$ or $\operatorname{Asym}^{\lambda_\omega}\mathcal{X}$. Note that the space $\operatorname{Asym}\mathcal{X}$ and its point p_ω do not depend on the choice of $p \in \mathcal{X}$.

The following exercise states that the constructions above depend on the sequence λ_n and a nonprincipal ultrafilter ω.

6.19. Exercise. *Construct a metric space $\mathcal{X}$ with a point p such that the tangent space $\mathrm{T}_p^{\lambda_\omega}\mathcal{X}$ (or its asymptotic cone $\operatorname{Asym}^{\lambda_\omega}\mathcal{X}$) depends on the sequence λ_n and/or ultrafilter ω.*

For nice spaces, different choices of the sequence of coefficients and ultrafilter may give the same space; some examples are given in the following exercise.

6.20. Exercise. *Let $\mathcal{T} = \operatorname{Asym}\mathcal{L}$, where $\mathcal{L}$ is one of the following spaces*

(i) Lobachevsky plane,

(ii) Lobachevsky space, or

(iii) 3-regular metric tree; that is, the degree of any vertex. Assume that each edge has unit length.

(a) Show that $\mathcal{T}$ is a complete metric tree.

(b) Show that $\mathcal{T}$ is one-point-homogeneous; that is, given two points $s, t \in \mathcal{T}$ there is an isometry of $\mathcal{T}$ that maps s to t.

(c) Show that $\mathcal{T}$ has continuum degree at any point; that is, for any point $t \in \mathcal{T}$ the set of connected components of the complement $\mathcal{T}\setminus\{t\}$ has cardinality continuum.

6.21. Exercise. *Consider the cylinder $\mathbb{S}^1\times[0,1]$ with the standard metric. Let $\mathcal{X}$ be the quotient space $\mathbb{S}^1\times[0,1]/\mathbb{S}^1\times\{0\}$; that is,*

$$|(u_1,t_1)-(u_2,t_2)|_{\mathcal{X}} := \min\{\,|(u_1,t_1)-(u_2,t_2)|_{\mathbb{S}^1\times[0,1]}, t_1+t_2\,\}.$$

Describe the ultratangent space $\mathrm{T}_o^{\omega}\mathcal{X}$, where $o \in \mathcal{X}$ is the point that corresponds to $\mathbb{S}^1\times\{0\}$.

[1] Often it is called an *asymptotic cone* despite that it is not a cone in general; this name is used since in good cases it has a cone structure.

G Remarks

A nonprincipal ultrafilter ω is called selective if for any partition of $\mathbb{N}$ into sets $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ such that $\omega(C_\alpha) = 0$ for each α, there is a set $S \subset \mathbb{N}$ such that $\omega(S) = 1$ and $S \cap C_\alpha$ is a one-point set for each $\alpha \in \mathcal{A}$.

The existence of a selective ultrafilter follows from the continuum hypothesis [120].

If needed, we may assume that the chosen ultrafilter ω is selective. In this case, *the subsequence* $(x_n)_{n\in S}$ *in 6.3 can be chosen so that* $\omega(S) = 1$.

Part II

Alexandrov geometry

Lecture 7

Introduction

A Manifesto

Alexandrov geometry can use "back to Euclid" as a slogan. Alexandrov spaces are defined via axioms similar to those given by Euclid, but certain equalities are changed to inequalities. Depending on the sign of the inequalities, we get Alexandrov spaces with curvature bounded above or curvature bounded below. The definitions of the two classes of spaces are similar, but their properties and known applications are quite different.

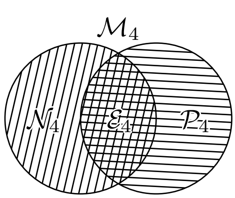
Consider the space $\mathcal{M}_4$ of all isometry classes of 4-point metric spaces. Each element in $\mathcal{M}_4$ can be described by 6 numbers — the distances between all 6 pairs of its points, say $\ell_{i,j}$ for $1 \leqslant i < j \leqslant 4$ modulo permutations of the index set $(1, 2, 3, 4)$. These 6 numbers are subject to 12 triangle inequalities; that is,

$$\ell_{i,j} + \ell_{j,k} \geqslant \ell_{i,k}$$

holds for all i, j and k, where we assume that $\ell_{j,i} = \ell_{i,j}$ and $\ell_{i,i} = 0$.

The space $\mathcal{M}_4$ comes with topology. It can defined as a quotient of the cone in $\mathbb{R}^6$ by permutations of the 4 points of the space. And, the same topology is induced on $\mathcal{M}_4$ by the Gromov–Hausdorff metric.

Consider the subset $\mathcal{E}_4 \subset \mathcal{M}_4$ of all isometry classes of 4-point metric spaces that admit isometric embeddings into Euclidean space.

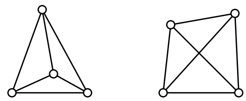


7.1. Claim. *The complement $\mathcal{M}_4 \setminus \mathcal{E}_4$ has two connected components.*

A proof of the claim can be extracted from 7.7.

The definition of Alexandrov spaces is based on this claim. Let us denote one of the components by $\mathcal{P}_4$ and the other by $\mathcal{N}_4$. Here $\mathcal{P}$ and $\mathcal{N}$ stand for positive and negative curvature because spheres have no quadruples of type $\mathcal{N}_4$ and hyperbolic space has no quadruples of type $\mathcal{P}_4$.

A metric space, with length metric, that has no quadruples of points of type $\mathcal{P}_4$ or $\mathcal{N}_4$ respectively is called an Alexandrov space with non-positive (CAT(0)) or non-negative curvature (CBB(0)).



Let us describe the subdivision into $\mathcal{P}_4$, $\mathcal{E}_4$, and $\mathcal{N}_4$ intuitively. Imagine that you move out of $\mathcal{E}_4$ — your path is a one-parameter family of 4-point metric spaces. The last thing you see in $\mathcal{E}_4$ is one of the two plane configurations shown on the diagram. If you see the left configuration then you move into $\mathcal{N}_4$; if it is the one on the right, then you move into $\mathcal{P}_4$. More degenerate pictures can be avoided; for example, a triangle with a point on a side. From such a configuration one may move in $\mathcal{N}_4$ and $\mathcal{P}_4$ (as well as come back to $\mathcal{E}_4$).

Here is an exercise, solving which would force you to rebuild a considerable part of Alexandrov geometry. It might be helpful to spend some time thinking about this exercise before proceeding.

7.2. Advanced exercise. *Assume $\mathcal{X}$ is a complete metric space with length metric, containing only quadruples of type $\mathcal{E}_4$. Show that $\mathcal{X}$ is isometric to a convex set in a Hilbert space.*

In the definition above, instead of Euclidean space one can take hyperbolic space of curvature -1. In this case, one obtains the definition of spaces with curvature bounded above or below by -1 (CAT(-1) or CBB(-1)).

To define spaces with curvature bounded above or below by 1 (CAT(1) or CBB(1)), one has to take the unit 3-sphere and specify that only the quadruples of points such that each of the four triangles has perimeter less than $2{\cdot}\pi$ are checked. The latter condition could be considered as a part of the spherical triangle inequality.

B Triangles, hinges and angles

Let $\mathcal{X}$ be a metric space.

Triangles. For a triple of points $p, q, r \in \mathcal{X}$, a choice of a triple of geodesics $([qr], [rp], [pq])$ will be called a triangle; we will use the short notation $[pqr] = [pqr]_{\mathcal{X}} = ([qr], [rp], [pq])$.

Given a triple $p, q, r \in \mathcal{X}$ there may be no triangle $[pqr]$ simply because one of the pairs of these points cannot be joined by a geodesic. Also, many different triangles with these vertices may exist, any of which can be denoted by $[pqr]$. However, if we write $[pqr]$, it means that we have made a choice of such a triangle; that is, we have fixed a choice of the geodesics $[qr]$, $[rp]$, and $[pq]$.

The value

$$|p-q| + |q-r| + |r-p|$$

will be called the *perimeter of the triangle* $[pqr]$.

Model triangles. Given three points p, q, r in a metric space $\mathcal{X}$, let us define the *model triangle* $[\tilde p\tilde q\tilde r]$ (briefly, $[\tilde p\tilde q\tilde r] = \tilde\triangle(pqr)_{\mathbb{E}^2}$) to be a triangle in the Euclidean plane $\mathbb{E}^2$ such that

$$\begin{aligned}
|\tilde p - \tilde q|_{\mathbb{E}^2} &= |p-q|_{\mathcal{X}},\\
|\tilde q - \tilde r|_{\mathbb{E}^2} &= |q-r|_{\mathcal{X}},\\
|\tilde r - \tilde p|_{\mathbb{E}^2} &= |r-p|_{\mathcal{X}}.
\end{aligned}$$

In the same way we can define the *hyperbolic* and the *spherical model triangles* $\tilde\triangle(pqr)_{\mathbb{H}^2}$, $\tilde\triangle(pqr)_{\mathbb{S}^2}$ in the hyperbolic plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2$. In the latter case the model triangle is said to be defined if in addition

$$|p-q| + |q-r| + |r-p| < 2\cdot\pi.$$

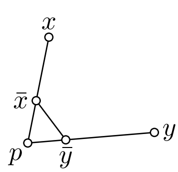
In this case, the model triangle again exists and is unique up to an isometry of $\mathbb{S}^2$.

Model angles. If $[\tilde p\tilde q\tilde r] = \tilde\triangle(pqr)_{\mathbb{E}^2}$ and $|p-q|, |p-r| > 0$, the angle measure of $[\tilde p\tilde q\tilde r]$ at $\tilde p$ will be called the *model angle* of the triple p, q, r and will be denoted by $\tilde\measuredangle(p\,{}^{q}_{r})_{\mathbb{E}^2}$. In the same way we define $\tilde\measuredangle(p\,{}^{q}_{r})_{\mathbb{H}^2}$ and $\tilde\measuredangle(p\,{}^{q}_{r})_{\mathbb{S}^2}$; in the latter case we assume in addition that the model triangle $\tilde\triangle(pqr)_{\mathbb{S}^2}$ is defined.

We may use the notation $\tilde\measuredangle(p\,{}^{q}_{r})$ if it is evident which of the model spaces $\mathbb{H}^2$, $\mathbb{E}^2$ or $\mathbb{S}^2$ is meant.

Hinges. Let $p, x, y \in \mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair of geodesics $([px],[py])$ will be called a *hinge* and will be denoted by $[p\,{}^{x}_{y}] = ([px],[py])$.

Angles. Given a hinge $[p\,{}^{x}_{y}]$, we define its *angle* as the limit



❶ $$\measuredangle[p\,{}^{x}_{y}] := \lim_{\bar x,\bar y\to p} \tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x \in \,]px]$ and $\bar y \in \,]py]$. The angle $\measuredangle[p\,{}^{x}_{y}]$ is defined if the limit exists.

It is straightforward to check that in ❶, one can use $\tilde\measuredangle(p\,{}^{\bar x}_{\bar y})_{\mathbb{S}^2}$ or $\tilde\measuredangle(p\,{}^{\bar x}_{\bar y})_{\mathbb{H}^2}$ or $\tilde\measuredangle(p\,{}^{\bar x}_{\bar y})_{\mathbb{E}^2}$, the result will be the same.

7.3. Exercise. *Give an example of a hinge $[p\,{}^{x}_{y}]$ in a metric space with undefined angle $\measuredangle[p\,{}^{x}_{y}]$.*

7.4. Exercise. *Suppose that for three geodesics $[px]$, $[py]$, and $[pz]$ in a metric space, the angles $\alpha = \measuredangle[p\,{}^{x}_{y}]$, $\beta = \measuredangle[p\,{}^{y}_{z}]$, and $\gamma = \measuredangle[p\,{}^{z}_{x}]$ are defined. Show that α, β and γ satisfy all triangle inequalities:*

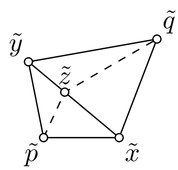
$$\alpha \leqslant \beta + \gamma, \qquad \beta \leqslant \gamma + \alpha, \qquad \gamma \leqslant \alpha + \beta,$$

C Definitions

Curvature bounded above. Given a quadruple of points p, q, x, y in a metric space $\mathcal{X}$, consider two model triangles $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{E}^2}$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)_{\mathbb{E}^2}$ with common side $[\tilde x\tilde y]$.

If the inequality

$$|p - q|_{\mathcal{X}} \leqslant |\tilde p - \tilde z|_{\mathbb{E}^2} + |\tilde z - \tilde q|_{\mathbb{E}^2}$$



holds for any point $\tilde z \in [\tilde x\tilde y]$, then we say that the quadruple p, q, x, y satisfies CAT(0) comparison.

If we do the same for spherical model triangles $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{S}^2}$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)_{\mathbb{S}^2}$, then we arrive at the definition of CAT(1) comparison. If one of the spherical model triangles is undefined,[1] then it is assumed that CAT(1) comparison automatically holds for this quadruple.

We can do the same for the model plane of curvature κ; that is, a sphere if $\kappa > 0$, Euclidean plane if $\kappa = 0$ and Lobachevsky plane if $\kappa < 0$. In this case, we arrive at the definition of CAT(κ) comparison. However we will mostly consider CAT(0) comparison and occasionally CAT(1) comparison; so, if you see CAT(κ), you can assume that κ is 0 or 1.

[1]That is, if

$$|p - x| + |p - y| + |x - y| \geqslant 2\cdot\pi \quad \text{or} \quad |q - x| + |q - y| + |x - y| \geqslant 2\cdot\pi.$$

If all quadruples in a metric space $\mathcal{X}$ satisfy CAT(κ) comparison, then we say that the space $\mathcal{X}$ is CAT(κ) (we use CAT(κ) as an adjective).

Here CAT is an acronym for Cartan, Alexandrov, and Toponogov, but usually pronounced as "cat" in the sense of "miauw". The term was coined by Mikhael Gromov in 1987. Originally, Alexandrov called these spaces $\mathfrak{R}_\kappa$ domain; this term is still in use.

7.5. Exercise. *Show that a metric space $\mathcal{X}$ is* CAT(0) *if and only if for any quadruple of points p, q, x, y in $\mathcal{X}$ there is a quadruple $\tilde p, \tilde q, \tilde x, \tilde y$ in $\mathbb{E}^2$ such that*

$$\begin{aligned} |\tilde p - \tilde q| &= |p-q|, & |\tilde x - \tilde y| &= |x-y|,\\ |\tilde p - \tilde x| &\leqslant |p-x|, & |\tilde p - \tilde y| &\leqslant |p-y|,\\ |\tilde q - \tilde x| &\leqslant |q-x|, & |\tilde q - \tilde y| &\leqslant |q-y|. \end{aligned}$$

Curvature bounded below. If the inequality

$$\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^y_z)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^z_x)_{\mathbb{E}^2} \leqslant 2\cdot\pi$$

holds for points p, x, y, z in a metric space $\mathcal{X}$, then we say that the quadruple p, x, y, z satisfies CBB(0) comparison.

If we do the same for spherical or hyperbolic model angles, then we arrive at the definition of CBB(1) or CBB(-1) comparison. Here CBB(κ) is an abbreviation of curvature bounded below by κ. If one of one of the model angles is undefined, then we assume that CBB(1) comparison automatically holds for this quadruple.

We can do the same for the model plane of curvature κ. In this case we arrive at the definition of CAT(κ) comparison. But we will mostly consider CBB(0) comparison and occasionally CBB(1) comparison; so, if you see CBB(κ), you can assume that κ is 0 or 1.

If all quadruples in a metric space $\mathcal{X}$ satisfy CBB(κ) comparison, then we say that the space $\mathcal{X}$ is CBB(κ). (Again — CBB(κ) is an adjective.)

7.6. Exercise. *Show that a metric space $\mathcal{X}$ is* CBB(0) *if and only if for any quadruple of points $p, x, y, z \in \mathcal{X}$, there is a quadruple of points $\tilde p, \tilde x, \tilde y, \tilde z \in \mathbb{E}^2$ such that*

$$\begin{aligned} |p-x|_{\mathcal{X}} &= |\tilde p - \tilde x|_{\mathbb{E}^2}, & |p-y|_{\mathcal{X}} &= |\tilde p - \tilde y|_{\mathbb{E}^2}, & |p-z|_{\mathcal{X}} &= |\tilde p - \tilde z|_{\mathbb{E}^2},\\ |x-y|_{\mathcal{X}} &\leqslant |\tilde x - \tilde y|_{\mathbb{E}^2}, & |y-z|_{\mathcal{X}} &\leqslant |\tilde y - \tilde z|_{\mathbb{E}^2}, & |z-x|_{\mathcal{X}} &\leqslant |\tilde z - \tilde x|_{\mathbb{E}^2} \end{aligned}$$

for all i and j.

7.7. Exercise. *Suppose that a quadruple of points satisfies* CAT(0) *and* CBB(0) *for all labeling. Show that the quadruple is isometric to a subset of Euclidean space.*

Observe that in order to check CAT(κ) or CBB(κ) comparison, it is sufficient to know the 6 distances between all pairs of points in the quadruple. This observation implies the following.

7.8. Proposition. *Any Gromov–Hausdorff limit (as well as ultra limit) of a sequence of* CAT(κ) *or* CBB(κ) *spaces is* CAT(κ) *or* CBB(κ) *respectively.*

In the proposition above, it does not matter which definition of convergence for metric spaces you use, as long as any quadruple of points in the limit space can be arbitrarily well approximated by quadruples in the sequence of metric spaces.

D Products and cones

Given two metric spaces $\mathcal{U}$ and $\mathcal{V}$, the *product space* $\mathcal{U}\times\mathcal{V}$ is defined as the set of all pairs (u,v) where $u\in\mathcal{U}$ and $v\in\mathcal{V}$ with the metric defined by formula

$$|(u_1,v_1)-(u_2,v_2)|_{\mathcal{U}\times\mathcal{V}}=\sqrt{|u_1-u_2|^2_{\mathcal{U}}+|v_1-v_2|^2_{\mathcal{V}}}.$$

7.9. Proposition. *Let $\mathcal{U}$ and $\mathcal{V}$ be* CAT(0) *spaces. Then the product space $\mathcal{U}\times\mathcal{V}$ is* CAT(0).

Proof. Fix a quadruple in $\mathcal{U}\times\mathcal{V}$:

$$p=(p_1,p_2),\qquad q=(q_1,q_2),\qquad x=(x_1,x_2),\qquad y=(y_1,y_2).$$

For the quadruple p_1,q_1,x_1,y_1 in $\mathcal{U}$, construct two model triangles $[\tilde p_1\tilde x_1\tilde y_1]=\tilde\triangle(p_1x_1y_1)_{\mathbb{E}^2}$ and $[\tilde q_1\tilde x_1\tilde y_1]=\tilde\triangle(q_1x_1y_1)_{\mathbb{E}^2}$. Similarly, for the quadruple p_2,q_2,x_2,y_2 in $\mathcal{V}$ construct two model triangles $[\tilde p_2\tilde x_2\tilde y_2]$ and $[\tilde q_2\tilde x_2\tilde y_2]$.

Consider four points in $\mathbb{E}^4=\mathbb{E}^2\times\mathbb{E}^2$

$$\tilde p=(\tilde p_1,\tilde p_2),\qquad \tilde q=(\tilde q_1,\tilde q_2),\qquad \tilde x=(\tilde x_1,\tilde x_2),\qquad \tilde y=(\tilde y_1,\tilde y_2).$$

Note that the triangles $[\tilde p\tilde x\tilde y]$ and $[\tilde q\tilde x\tilde y]$ in $\mathbb{E}^4$ are isometric to the model triangles $\tilde\triangle(pxy)_{\mathbb{E}^2}$ and $\tilde\triangle(qxy)_{\mathbb{E}^2}$.

If $\tilde z=(\tilde z_1,\tilde z_2)\in[\tilde x\tilde y]$, then $\tilde z_1\in[\tilde x_1\tilde y_1]$ and $\tilde z_2\in[\tilde x_2\tilde y_2]$ and

$$\begin{aligned}
|\tilde z-\tilde p|^2_{\mathbb{E}^4}&=|\tilde z_1-\tilde p_1|^2_{\mathbb{E}^2}+|\tilde z_2-\tilde p_2|^2_{\mathbb{E}^2},\\
|\tilde z-\tilde q|^2_{\mathbb{E}^4}&=|\tilde z_1-\tilde q_1|^2_{\mathbb{E}^2}+|\tilde z_2-\tilde q_2|^2_{\mathbb{E}^2},\\
|p-q|^2_{\mathcal{U}\times\mathcal{V}}&=|p_1-q_1|^2_{\mathcal{U}}+|p_2-q_2|^2_{\mathcal{V}}.
\end{aligned}$$

Therefore CAT(0) comparison for the quadruples p_1,q_1,x_1,y_1 in $\mathcal{U}$ and p_2,q_2,x_2,y_2 in $\mathcal{V}$ implies CAT(0) comparison for the quadruples p,q,x,y in $\mathcal{U}\times\mathcal{V}$. □

7.10. Exercise. *Assume $\mathcal{U}$ and $\mathcal{V}$ are* CBB(0) *spaces. Show that the product space $\mathcal{U}\times\mathcal{V}$ is* CBB(0)*.*

The cone $\mathcal{V}=\operatorname{Cone}\mathcal{U}$ over a metric space $\mathcal{U}$ is defined as the metric space whose underlying set consists of equivalence classes in $[0,\infty)\times\mathcal{U}$ with the equivalence relation "$\sim$" given by $(0,p)\sim(0,q)$ for any points $p,q\in\mathcal{U}$, and whose metric is given by the cosine rule

$$|(p,s)-(q,t)|_{\mathcal{V}}=\sqrt{s^2+t^2-2{\cdot}s{\cdot}t{\cdot}\cos\alpha},$$

where $\alpha=\min\{\pi,|p-q|_{\mathcal{U}}\}$.

The point in the cone $\mathcal{V}$ formed by the equivalence class of $0\times\mathcal{U}$ is called the tip of the cone and is denoted by 0 or $0_{\mathcal{V}}$. The distance $|0-v|_{\mathcal{V}}$ is called the norm of v and is denoted by $|v|$ or $|v|_{\mathcal{V}}$. The space $\mathcal{U}$ can be identified with the subset $x\in\mathcal{V}$ such that $|x|=1$.

The points in the cone $\mathcal{V}$ can be multiplied by a real number $\lambda\geqslant0$; namely, if $x=(x',r)$, then $\lambda{\cdot}x:=(x',\lambda{\cdot}r)$.

7.11. Proposition. *Let $\mathcal{U}$ be a metric space. Then* $\operatorname{Cone}\mathcal{U}$ *is* CAT(0) *if and only if $\mathcal{U}$ is* CAT(1)*.*

Proof; if part. Given a point $x\in\operatorname{Cone}\mathcal{U}$, denote by x' its projection to $\mathcal{U}$ and by $|x|$ the distance from x to the tip of the cone; if x is the tip, then $|x|=0$ and we can take any point of $\mathcal{U}$ as x'.

Let p, q, x, y be a quadruple in $\operatorname{Cone}\mathcal{U}$. Assume that the spherical model triangles $[\tilde p'\tilde x'\tilde y']_{\mathbb{S}^2}=\tilde\triangle(p'x'y')_{\mathbb{S}^2}$ and $[\tilde q'\tilde x'\tilde y']_{\mathbb{S}^2}=\tilde\triangle(q'x'y')_{\mathbb{S}^2}$ are defined. Consider the following points in $\mathbb{E}^3=\operatorname{Cone}\mathbb{S}^2$:

$$\tilde p=|p|{\cdot}\tilde p',\qquad \tilde q=|q|{\cdot}\tilde q',\qquad \tilde x=|x|{\cdot}\tilde x',\qquad \tilde y=|y|{\cdot}\tilde y'.$$

Note that $[\tilde p\tilde x\tilde y]_{\mathbb{E}^3}\overset{iso}{=\!=}\tilde\triangle(pxy)_{\mathbb{E}^2}$ and $[\tilde q\tilde x\tilde y]_{\mathbb{E}^3}\overset{iso}{=\!=}\tilde\triangle(qxy)_{\mathbb{E}^2}$. Further note that if $\tilde z\in[\tilde x\tilde y]_{\mathbb{E}^3}$, then $\tilde z'=\tilde z/|\tilde z|$ lies on the geodesic $[\tilde x'\tilde y']_{\mathbb{S}^2}$. Therefore the CAT(1) comparison for $|p'-q'|$ with $\tilde z'\in[\tilde x'\tilde y']_{\mathbb{S}^2}$ implies the CAT(0) comparison for $|p-q|$ with $\tilde z\in[\tilde x\tilde y]_{\mathbb{E}^3}$.

If at least one of the model triangles $\tilde{\triangle}(p'x'y')_{\mathbb{S}^2}$ and $\tilde{\triangle}(q'x'y')_{\mathbb{S}^2}$ is undefined, then the statement follows from the triangle inequalities

$$|p'-x'|_{\mathcal{U}}+|q'-x'|_{\mathcal{U}}\geqslant|p'-q'|_{\mathcal{U}}$$
$$|p'-y'|_{\mathcal{U}}+|q'-y'|_{\mathcal{U}}\geqslant|p'-q'|_{\mathcal{U}}$$

This case is left as an exercise.

Only-if part. Suppose that $\tilde{p}'$, $\tilde{q}'$, $\tilde{x}'$, $\tilde{y}'$ are defined as above. Assume all these points lie in a half-space of $\mathbb{E}^3=\operatorname{Cone}\mathbb{S}^2$ with origin at its boundary. Then we can choose positive values a, b, c, and d such that the points $a{\cdot}\tilde{p}'$, $b{\cdot}\tilde{q}'$, $c{\cdot}\tilde{x}'$, $d{\cdot}\tilde{y}'$ lie in one plane. Consider the corresponding points $a{\cdot}p'$, $b{\cdot}q'$, $c{\cdot}x'$, $d{\cdot}y'$ in $\operatorname{Cone}\mathcal{U}$. Applying the CAT(0) comparison for these points leads to CAT(1) comparison for the quadruple p', q', x', y' in $\mathcal{U}$.

It remains to consider the case when $\tilde{p}'$, $\tilde{q}'$, $\tilde{x}'$, $\tilde{y}'$ do not in a half-space. Fix $\tilde{z}'\in[\tilde{x}'\tilde{y}']_{\mathbb{S}^2}$. Observe that

$$|\tilde{p}'-\tilde{x}'|_{\mathbb{S}^2}+|\tilde{q}'-\tilde{x}'|_{\mathbb{S}^2}\leqslant|\tilde{p}'-\tilde{z}'|_{\mathbb{S}^2}+|\tilde{q}'-\tilde{z}'|_{\mathbb{S}^2}$$

or

$$|\tilde{p}'-\tilde{y}'|_{\mathbb{S}^2}+|\tilde{q}'-\tilde{y}'|_{\mathbb{S}^2}\leqslant|\tilde{p}'-\tilde{z}'|_{\mathbb{S}^2}+|\tilde{q}'-\tilde{z}'|_{\mathbb{S}^2}.$$

That is, in this case, the CAT(1) comparison follows from the triangle inequality. ☐

E Geodesics

7.12. Proposition. *Let $\mathcal{X}$ be a complete length* CAT(0) *space. Then any two points in $\mathcal{X}$ are joint by a unique geodesic.*

Proof. Fix two points $x,y\in\mathcal{X}$. Choose a sequence of approximate midpoints p_n for x and y; that is,

❶ $$|x-p_n|\to\tfrac12{\cdot}|x-y|\quad\text{and}\quad|y-p_n|\to\tfrac12{\cdot}|x-y|$$

as $n\to\infty$.

Consider model triangles $[\tilde{p}_n\tilde{x}\tilde{y}]=\tilde{\triangle}(p_nxy)$. Let $\tilde{z}$ be the midpoint of $\tilde{x}$ and $\tilde{y}$. By ❶, we have that

$$|\tilde{p}_n-\tilde{z}|\to0$$

as $n\to\infty$.

By CAT(0) comparison,

$$|p_n - p_m|_{\mathcal{X}} \leqslant |\tilde{p}_n - \tilde{z}| + |\tilde{p}_m - \tilde{z}|.$$

Therefore $|p_n - p_m| \to 0$ as $m, n \to \infty$; that is, (p_n) is Cauchy. Clearly the limit of the sequence p_n is a midpoint of x and y. Applying 1.27b, we get that $\mathcal{X}$ is geodesic.

It remains to prove uniqueness. Suppose there are two geodescics between x and y. Then we can choose two points $p \neq q$ on these geodesics such that $|x - p| = |x - q|$ and therefore $|y - p| = |y - q|$.

Observe that the model triangles $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$ and $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\triangle}(qxy)$ are degenerate and moreover $\tilde{p} = \tilde{q}$. Applying CAT(0) comparison with $\tilde{z} = \tilde{p} = \tilde{q}$, we get that $|p - q| = 0$, a contradiction. $\square$

The following exercise is an analogous statement for CBB spaces. In general complete length CBB(0) space might fail to be geodesic and uniqueness of geodesic usually does not hold.

7.13. Exercise. *Let $\mathcal{X}$ be a complete length* CBB(0) *space. Show that if two geodesics from x to y share yet another point z, then they coincide.*

F Alexandrov's lemma

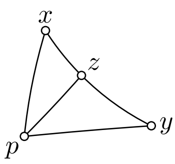
7.14. Lemma. *Let p, x, y, z be distinct points in a metric space such that $z \in \,]xy[$. Then the following expressions for the Euclidean model angles have the same sign:*

(a) $\tilde{\measuredangle}(x\,{}^p_y) - \tilde{\measuredangle}(x\,{}^p_z)$,

(b) $\tilde{\measuredangle}(z\,{}^p_x) + \tilde{\measuredangle}(z\,{}^p_y) - \pi$.

Moreover,

$$\tilde{\measuredangle}(p\,{}^x_y) \geqslant \tilde{\measuredangle}(p\,{}^x_z) + \tilde{\measuredangle}(p\,{}^z_y),$$

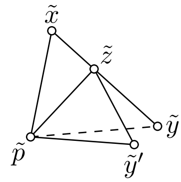


with equality if and only if the expressions in (a) and (b) vanish.

The same holds for the hyperbolic and spherical model angles, but in the latter case one has to assume in addition that

$$|p - z| + |p - y| + |x - y| < 2{\cdot}\pi.$$

Proof. Consider the model triangle $[\tilde{x}\tilde{p}\tilde{z}] = \tilde{\triangle}(xpz)$. Take a point $\tilde{y}$ on the extension of $[\tilde{x}\tilde{z}]$ beyond $\tilde{z}$ so that $|\tilde{x} - \tilde{y}| = |x - y|$ (and therefore $|\tilde{x} - \tilde{z}| = |x - z|$).



Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:

(i) $\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(x\,{}^{p}_{y})$,

(ii) $|\tilde p - \tilde y| - |p - y|$,

(iii) $\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(z\,{}^{p}_{y})$.

Since

$$\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] = \measuredangle[\tilde x\,{}^{\tilde p}_{\tilde z}] = \tilde\measuredangle(x\,{}^{p}_{z})$$

and

$$\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] = \pi - \measuredangle[\tilde z\,{}^{\tilde x}_{\tilde p}] = \pi - \tilde\measuredangle(z\,{}^{x}_{p}),$$

the first statement follows.

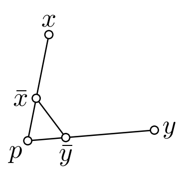
For the second statement, construct a model triangle $[\tilde p\tilde z\tilde y'] = \tilde\triangle(pzy)_{\mathbb{E}^2}$ on the opposite side of $[\tilde p\tilde z]$ from $[\tilde x\tilde p\tilde z]$. Note that

$$\begin{aligned}|\tilde x - \tilde y'| &\leqslant |\tilde x - \tilde z| + |\tilde z - \tilde y'| = \\ &= |x - z| + |z - y| = \\ &= |x - y|.\end{aligned}$$

Therefore

$$\begin{aligned}\tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{z}_{y}) &= \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde z}] + \measuredangle[\tilde p\,{}^{\tilde z}_{\tilde y'}] = \\ &= \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y'}] \leqslant \\ &\leqslant \tilde\measuredangle(p\,{}^{x}_{y}).\end{aligned}$$

Equality holds if and only if $|\tilde x - \tilde y'| = |x - y|$, as required. $\square$



7.15. Exercise. *Given $[p\,{}^{x}_{y}]$ in a metric space $\mathcal{X}$, consider the function*

$$f\colon (|p - \bar x|, |p - \bar y|) \mapsto \tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x \in\ ]px]$ and $\bar y \in\ ]py]$.

(a) Suppose $\mathcal{X}$ is CAT(0)*. Show that f is nondecreasing in each argument.*

(b) Suppose $\mathcal{X}$ is CBB(0)*. Show that f is nonincreasing in each argument.*

Conclude that any hinge in a CAT(0) *or* CBB(0) *space has defined angle.*

7.16. Exercise. *Fix a point p in a be a complete length* CAT(0) *space $\mathcal{X}$. Given a point $x \in \mathcal{X}$, denote by γ_x a (necessarily unique) geodesic path from p to x.*

Show that the family of maps $h_t\colon \mathcal{X}\to\mathcal{X}$ defined by

$$h_t(x)=\gamma_x(t)$$

is a homotopy; it is called geodesic homotopy*. Conclude that $\mathcal{X}$ is contractible.*

The geodesic homotopy introduced in the previous exercise should help to solve the next one.

7.17. Exercise. *Let $\mathcal{X}$ be a complete length* CAT(0) *space. Assume $\mathcal{X}$ is a topological manifold. Show that any geodesic in $\mathcal{X}$ can be extended as a two-side infinite geodesic.*

G Thin and fat triangles

Recall that a triangle $[xyz]$ in a space $\mathcal{X}$ is a triple of minimizing geodesics $[xy]$, $[yz]$ and $[zx]$. Consider the model triangle $[\tilde x\tilde y\tilde z]=\tilde\triangle(xyz)_{\mathbb{E}^2}$ in the Euclidean plane. The natural map $[\tilde x\tilde y\tilde z]\to[xyz]$ sends a point $\tilde p\in[\tilde x\tilde y]\cup[\tilde y\tilde z]\cup[\tilde z\tilde x]$ to the corresponding point $p\in[xy]\cup[yz]\cup[zx]$; that is, if $\tilde p$ lies on $[\tilde y\tilde z]$, then $p\in[yz]$ and $|\tilde y-\tilde p|=|y-p|$ (and therefore $|\tilde z-\tilde p|=|z-p|$).

In the same way, the natural map can be defined for the spherical model triangle $\tilde\triangle(xyz)_{\mathbb{S}^2}$.

7.18. Definition. *A triangle $[xyz]$ in the metric space $\mathcal{X}$ is called* thin *(or* fat*) if the natural map $\tilde\triangle(xyz)_{\mathbb{E}^2}\to[xyz]$ is distance nonincreasing (or respectively distance nondecreasing).*

Analogously, a triangle $[xyz]$ is called spherically thin *or* spherically fat *if the natural map from the spherical model triangle $\tilde\triangle(xyz)_{\mathbb{S}^2}$ to $[xyz]$ is distance nonincreasing or nondecreasing.*

7.19. Proposition. *A geodesic space is* CAT(0) *(*CAT(1)*) if and only if all its triangles are thin (respectively, all its triangles of perimeter $<2\cdot\pi$ are spherically thin).*

Proof; if part. Apply the triangle inequality and thinness of triangles $[pxy]$ and $[qxy]$, where p, q, x, and y are as in the definition of CAT(κ) comparison (7C).

Only-if part. Applying CAT(0) comparison to a quadruple p,q,x,y with $q\in[xy]$ shows that any triangle satisfies point-side comparison, that is, the distance from a vertex to a point on the opposite side is no greater than the corresponding distance in the Euclidean model triangle.

Now consider a triangle $[xyz]$ and let $p \in [xy]$ and $q \in [xz]$. Let $\tilde p$, $\tilde q$ be the corresponding points on the sides of the model triangle $\tilde\triangle(xyz)_{\mathbb{E}^2}$. Applying 7.15$a$, we get that

$$\tilde\measuredangle(x\,{}^y_z)_{\mathbb{E}^2} \geqslant \tilde\measuredangle(x\,{}^p_q)_{\mathbb{E}^2}.$$

Therefore $|\tilde p - \tilde q|_{\mathbb{E}^2} \geqslant |p - q|$.

The CAT(1) argument is the same. □

7.20. Exercise. *Show that any triangle is a* CBB(0) *space is fat.*

7.21. Exercise. *Suppose $\gamma_1, \gamma_2 \colon [0,1] \to \mathcal{U}$ be two geodesic paths in a complete length* CAT(0) *space $\mathcal{U}$. Show that*

$$t \mapsto |\gamma_1(t) - \gamma_2(t)|_{\mathcal{U}}$$

is a convex function.

7.22. Exercise. *Let A be a convex closed set in a proper length* CAT(0) *space $\mathcal{U}$; that is, if $x, y \in A$, then $[xy] \subset A$. Show that for any $r > 0$ the closed r-neighborhood of A is convex; that is, the set*

$$A_r = \{\, x \in \mathcal{U} \;:\; \operatorname{dist}_A x \leqslant r \,\}$$

is convex.

7.23. Exercise. *Let $\mathcal{U}$ be a proper length* CAT(0) *space and $K \subset \mathcal{U}$ be a closed convex set. Show that:*

(a) For each point $p \in \mathcal{U}$ there is a unique point $p^* \in K$ that minimizes the distance $|p - p^*|$.

(b) The closest-point projection $p \mapsto p^*$ defined by (a) is short.

Recall that a set A in a metric space $\mathcal{U}$ is called locally convex if for any point $p \in A$ there is an open neighborhood $\mathcal{U} \ni p$ such that any geodesic in $\mathcal{U}$ with ends in A lies in A.

7.24. Exercise. *Let $\mathcal{U}$ be a proper length* CAT(0) *space. Show that any closed, connected, locally convex set in $\mathcal{U}$ is convex.*

H Other descriptions

In this section, we will list few ways to describe CAT(0) and CBB(0) spaces. We do not give proofs of these statements, altho they are not hard [see 5, and the references therein].

These conditions will not be used in the sequel, but they might help to build right intuition.

Convexity of function. The following condition might help to adapt intuition from real analysis.

Let $\mathcal{X}$ be a metric space and $\lambda \in \mathbb{R}$. A function $f\colon \mathcal{X} \to \mathbb{R}$ is called *λ-convex* (*λ-concave*) if the real-to-real function

$$t \mapsto f \circ \gamma(\gamma) - \tfrac{\lambda}{2}\cdot t^2$$

is convex (respectively concave) for any geodesic $\gamma\colon \mathbb{I} \to \mathbb{R}$.

The λ-convex and λ-concave functions can be thought as functions satisfying inequalities $f'' \geqslant \lambda$ and respectively $f'' \leqslant \lambda$ in a generalized sense. Note that a smooth real-to-real function f is λ-convex (λ-concave) if it satisfies inequality $f'' \geqslant \lambda$ (respectively $f'' \leqslant \lambda$).

7.25. Proposition. *Let $\mathcal{X}$ be a geodesic space. Then $\mathcal{X}$ is* $\mathrm{CAT}(0)$ *(respectively* $\mathrm{CBB}(0)$*) if and only if for any point $p \in \mathcal{X}$ the function*

$$f(x) = \tfrac{1}{2}\cdot|p - x|^2_{\mathcal{X}}$$

is 1-convex (respectively 1-concave).

Angle comparison. The following condition might help to adapt intuition from Euclidean geometry.

Recall that in $\mathrm{CAT}(0)$ and $\mathrm{CBB}(0)$ spaces any hinge has defined angle; see 7.15.

7.26. Proposition. *Let $\mathcal{X}$ be a geodesic space such that any hinge in $\mathcal{X}$ has defined angle. Then*

(a) $\mathcal{X}$ is $\mathrm{CAT}(0)$ *if and only if*

$$\measuredangle[p\,{}^x_y] \leqslant \tilde\measuredangle(p\,{}^x_y).$$

(b) $\mathcal{X}$ is $\mathrm{CBB}(0)$ *if and only if*

$$\measuredangle[p\,{}^x_y] \geqslant \tilde\measuredangle(p\,{}^x_y)$$

and

$$\measuredangle[p\,{}^x_y] + \measuredangle[p\,{}^x_z] = \pi$$

for any adjacent hinges $[p\,{}^x_y]$ and $[p\,{}^x_z]$; that is, the union of the sides $[px]$ and $[pz]$ of the hinges form a geodesic $[xy]$.

It is unknown if the condition on adjacent hinges in *(b)* can be removed (even in the two-dimesional case).

Kirszbraun property. We include the following condition only because it is beautiful.

The following theorem was proved by Mojżesz Kirszbraun [84] and rediscovered later by Frederick Valentine [135].

7.27. Theorem. *Let $A \subset \mathbb{E}^m$. Then any short map $f\colon A \to \mathbb{E}^n$ admits a short extension $f\colon \mathbb{E}^m \to \mathbb{E}^n$.*

The conclusion of the theorem holds for some other metric spaces instead of $\mathbb{E}^m$ and $\mathbb{E}^n$. For example, instead of $\mathbb{E}^n$ one might take any injective space (3.2) and instead of $\mathbb{E}^m$ one may take any compact ultrametric space (3.33). On the other hand, the existence of extension to/from a Euclidean space is a much weaker condition than in 3.2 and 3.33. As the following theorems state, these conditions are closely related to the CBB(0) and CAT(0) conditions.

7.28. Theorem. *Let $\mathcal{X}$ be a complete length space and $n \geqslant 2$. Then $\mathcal{X}$ is* CBB(0) *if and only if for any set $A \subset \mathcal{X}$, any short map $f\colon A \to \mathbb{E}^n$ admits a short extension $F\colon \mathcal{X} \to \mathbb{E}^n$.*

7.29. Theorem. *Let $\mathcal{Y}$ be a metric space with and $m \geqslant 2$. Assume any two points in $\mathcal{Y}$ are joint by unique geodesic. Then $\mathcal{Y}$ is* CAT(0) *if and only if for any set $A \subset \mathbb{E}^m$, any short map $f\colon \mathbb{E}^m \to \mathcal{Y}$ admits a short extension $F\colon \mathbb{E}^m \to \mathcal{Y}$.*

I History

The idea that the essence of curvature lies in a condition on quadruples of points apparently originated with Abraham Wald. It is found in his publication on "coordinate-free differential geometry" [137] written under the supervision of Karl Menger; the story of this discovery can be found in [97]. In 1941, similar definitions were rediscovered independently by Alexandr Danilovich Alexandrov [8]. In Alexandrov's work the first fruitful applications of this approach were given. Mainly:

- ◇ Alexandrov's embedding theorem — *metrics of non-negative curvature on the sphere, and only they, are isometric to closed convex surfaces in Euclidean 3-space.*
- ◇ Gluing theorem, which tells when the sphere obtained by gluing of two discs along their boundaries has non-negative curvature in the sense of Alexandrov.

These two results together gave a very intuitive geometric tool for studying embeddings and bending of surfaces in Euclidean space, and changed this subject dramatically. They formed the foundation of the branch of geometry now called Alexandrov geometry.

The study of spaces with curvature bounded above started later. The first paper on the subject was written by Alexandrov [9]. It was based on work of Herbert Busemann [41], who studied spaces satisfying a weaker condition.

Lecture 8

Gluing and billiards

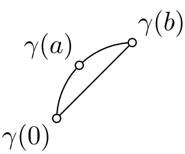
A Inheritance lemma

The inheritance lemma 8.2 proved below plays a central role in the theory of $\mathrm{CAT}(\kappa)$ spaces.

A curve $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a local geodesic if for any $t \in \mathbb{I}$ there is a neighborhood U of t in $\mathbb{I}$ such that the restriction $\gamma|_U$ is a geodesic.

8.1. Proposition. *Suppose $\mathcal{U}$ is a proper length $\mathrm{CAT}(0)$ space. Then any local geodesic in $\mathcal{U}$ is a geodesic.*

Analogously, if $\mathcal{U}$ is a proper length $\mathrm{CAT}(1)$ space, then any local geodesic in $\mathcal{U}$ which is shorter than π is a geodesic.



Proof. Suppose $\gamma\colon [0,\ell] \to \mathcal{U}$ is a local geodesic that is not a geodesic. Choose a to be the maximal value such that γ is a geodesic on $[0,a]$. Further choose $b > a$ so that γ is a geodesic on $[a,b]$.

Since the triangle $[\gamma(0)\gamma(a)\gamma(b)]$ is thin and $|\gamma(0) - \gamma(b)| < b$ we have

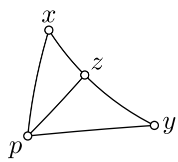
$$|\gamma(a-\varepsilon) - \gamma(a+\varepsilon)| < 2\cdot\varepsilon$$

for all small $\varepsilon > 0$. That is, γ is not length-minimizing on the interval $[a-\varepsilon, a+\varepsilon]$ for any $\varepsilon > 0$, a contradiction.

The spherical case is done in the same way. $\square$

Now let us formulate the main result of this section.

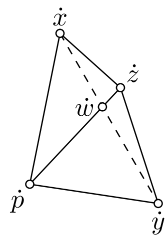
8.2. Inheritance lemma. *Assume that a triangle $[pxy]$ in a metric space is decomposed into two triangles $[pxz]$ and $[pyz]$; that is, $[pxz]$ and $[pyz]$ have a common side $[pz]$, and the sides $[xz]$ and $[zy]$ together form the side $[xy]$ of $[pxy]$.*



If both triangles $[pxz]$ and $[pyz]$ are thin, then the triangle $[pxy]$ is also thin.

Analogously, if $[pxy]$ has perimeter $< 2\cdot\pi$ and both triangles $[pxz]$ and $[pyz]$ are spherically thin, then triangle $[pxy]$ is spherically thin.

Proof. Construct the model triangles $[\dot p\dot x\dot z] = \tilde\triangle(pxz)_{\mathbb{E}^2}$ and $[\dot p\dot y\dot z] = \tilde\triangle(pyz)_{\mathbb{E}^2}$ so that $\dot x$ and $\dot y$ lie on opposite sides of $[\dot p\dot z]$.



Let us show that

$$\text{❶} \qquad \tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) \geqslant \pi.$$

If not, then for some point $\dot w \in [\dot p\dot z]$, we have

$$|\dot x - \dot w| + |\dot w - \dot y| < |\dot x - \dot z| + |\dot z - \dot y| = |x - y|.$$

Let $w \in [pz]$ correspond to $\dot w$; that is, $|z - w| = |\dot z - \dot w|$. Since $[pxz]$ and $[pyz]$ are thin, we have

$$|x - w| + |w - y| < |x - y|,$$

contradicting the triangle inequality.

Denote by $\dot D$ the union of two solid triangles $[\dot p\dot x\dot z]$ and $[\dot p\dot y\dot z]$. Further, denote by $\tilde D$ the solid triangle $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{E}^2}$. By ❶, there is a short map $F\colon \tilde D \to \dot D$ that sends

$$\tilde p \mapsto \dot p, \qquad \tilde x \mapsto \dot x, \qquad \tilde z \mapsto \dot z, \qquad \tilde y \mapsto \dot y.$$

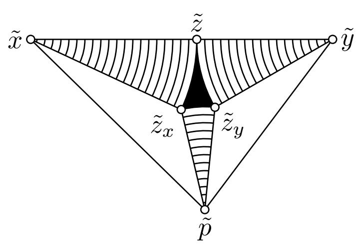
Indeed, by Alexandrov's lemma (7.14), there are nonoverlapping triangles

$$[\tilde p\tilde x\tilde z_x] \overset{iso}{=\!=} [\dot p\dot x\dot z]$$

and

$$[\tilde p\tilde y\tilde z_y] \overset{iso}{=\!=} [\dot p\dot y\dot z]$$

inside the triangle $[\tilde p\tilde x\tilde y]$.



Connect the points in each pair $(\tilde z, \tilde z_x)$, $(\tilde z_x, \tilde z_y)$ and $(\tilde z_y, \tilde z)$ with arcs of circles centered at $\tilde y$, $\tilde p$, and $\tilde x$ respectively. Define F as follows:

- ⋄ Map $\mathrm{Conv}[\tilde p\tilde x\tilde z_x]$ isometrically onto $\mathrm{Conv}[\dot p\dot x\dot z]$; similarly map $\mathrm{Conv}[\tilde p\tilde y\tilde z_y]$ onto $\mathrm{Conv}[\dot p\dot y\dot z]$.
- ⋄ If x is in one of the three circular sectors, say at distance r from its center, set $F(x)$ to be the point on the corresponding segment $[pz]$, $[xz]$ or $[yz]$ whose distance from the left-hand endpoint of the segment is r.
- ⋄ Finally, if x lies in the remaining curvilinear triangle $\tilde z\tilde z_x\tilde z_y$, set $F(x)=z$.

By construction, F satisfies the conditions.

By assumption, the natural maps $[\dot p\dot x\dot z]\to[pxz]$ and $[\dot p\dot y\dot z]\to[pyz]$ are short. By composition, the natural map from $[\tilde p\tilde x\tilde y]$ to $[pyz]$ is short, as claimed.

The spherical case is done along the same lines. □

B Reshetnyak's gluing

Suppose $\mathcal U^1$ and $\mathcal U^2$ are proper length spaces with isometric closed convex sets $A^i\subset\mathcal U^i$ and let $\iota\colon A^1\to A^2$ be an isometry. Consider the space $\mathcal W$ of all equivalence classes in $\mathcal U^1\sqcup\mathcal U^2$ with the equivalence relation given by $a\sim\iota(a)$ for any $a\in A^1$.

It is straightforward to see that $\mathcal W$ is a proper length space when equipped with the following metric

$$\begin{aligned}
|x-y|_{\mathcal W}&:=|x-y|_{\mathcal U^i}\\
&\qquad\text{if}\quad x,y\in\mathcal U^i,\quad\text{and}\\
|x-y|_{\mathcal W}&:=\min\left\{\,|x-a|_{\mathcal U^1}+|y-\iota(a)|_{\mathcal U^2}\ :\ a\in A^1\,\right\}\\
&\qquad\text{if}\quad x\in\mathcal U^1\quad\text{and}\quad y\in\mathcal U^2.
\end{aligned}$$

Abusing notation, we denote by x and y the points in $\mathcal U^1\sqcup\mathcal U^2$ and their equivalence classes in $\mathcal U^1\sqcup\mathcal U^2/{\sim}$.

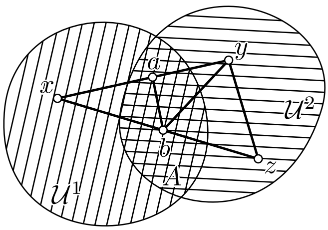
The space $\mathcal W$ is called the gluing of $\mathcal U^1$ and $\mathcal U^2$ along ι. If one applies this construction to two copies of one space $\mathcal U$ with a set $A\subset\mathcal U$ and the identity map $\iota\colon A\to A$, then the obtained space is called the double of $\mathcal U$ along A.

We can (and will) identify $\mathcal U^i$ with its image in $\mathcal W$; this way both subsets $A^i\subset\mathcal U^i$ will be identified and denoted further by A. Note that $A=\mathcal U^1\cap\mathcal U^2\subset\mathcal W$, therefore A is also a convex set in $\mathcal W$.

8.3. Reshetnyak gluing. *Suppose $\mathcal U^1$ and $\mathcal U^2$ are proper length* $\mathrm{CAT}(0)$ *spaces with isometric closed convex sets $A^i\subset\mathcal U^i$, and $\iota\colon A^1\to$ $\to A^2$ is an isometry. Then the gluing of $\mathcal U^1$ and $\mathcal U^2$ along ι is a* $\mathrm{CAT}(0)$ *proper length space.*

Proof. By construction of the gluing space, the statement can be reformulated in the following way:

8.4. Reformulation of 8.3. *Let $\mathcal{W}$ be a proper length space that has two closed convex sets $\mathcal{U}^1$,$\mathcal{U}^2 \subset \mathcal{W}$ such that $\mathcal{U}^1 \cup \mathcal{U}^2 = \mathcal{W}$ and $\mathcal{U}^1$, $\mathcal{U}^2$ are* CAT(0). *Then $\mathcal{W}$ is* CAT(0).



It suffices to show that any triangle $[xyz]$ in $\mathcal{W}$ is thin. This is obviously true if all three points x, y, z lie in one of $\mathcal{U}^i$. Thus, without loss of generality, we may assume that $x \in \mathcal{U}^1$ and $y, z \in \mathcal{U}^2$.

Choose points $a, b \in A = \mathcal{U}^1 \cap \mathcal{U}^2$ that lie respectively on the sides $[xy]$, $[xz]$. Note that

- ⋄ the triangle $[xab]$ lies in $\mathcal{U}^1$,
- ⋄ both triangles $[yab]$ and $[ybz]$ lie in $\mathcal{U}^2$.

In particular, each triangle $[xab]$, $[yab]$, and $[ybz]$ is thin.

Applying the inheritance lemma (8.2) twice, we get that $[xyb]$ and consequently $[xyz]$ is thin. □

8.5. Exercise. *Suppose $\mathcal{U}$ is a geodesic space and $A \subset \mathcal{U}$ is a closed subset. Assume that the doubling of $\mathcal{U}$ in A is* CAT(0). *Show that A is a convex set of $\mathcal{U}$.*

C Puff pastry

In this section, we introduce the notion of Reshetnyak puff pastry. This construction will be used in the next section to prove the collision theorem (8.16).

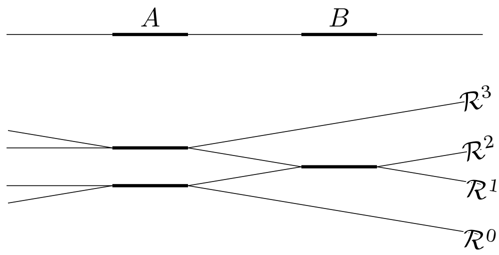
Let $\boldsymbol{A} = (A^1, \ldots, A^N)$ be an array of convex closed sets in the Euclidean space $\mathbb{E}^m$. Consider an array of $N+1$ copies of $\mathbb{E}^m$. Assume that the space $\mathcal{R}$ is obtained by gluing successive pairs of spaces along $A^1, \ldots, A^N$ respectively.

The resulting space $\mathcal{R}$ will be called the Reshetnyak puff pastry for array $\boldsymbol{A}$. The copies of $\mathbb{E}^m$ in the puff pastry $\mathcal{R}$ will be called levels; they will be denoted by $\mathcal{R}^0, \ldots, \mathcal{R}^N$. The point in the k-th level $\mathcal{R}^k$ that corresponds to $x \in \mathbb{E}^m$ will be denoted by x^k.

Given $x \in \mathbb{E}^m$, any point $x^k \in \mathcal{R}$ is called a lifting of x. The map $x \mapsto x^k$ defines an isometry $\mathbb{E}^m \to \mathcal{R}^k$; in particular, we can talk about liftings of subsets in $\mathbb{E}^m$.

Note that:

- ⋄ The intersection $A^1 \cap \cdots \cap A^N$ admits a unique lifting in $\mathcal{R}$.



Puff pastry for (A, B, A).

- Moreover, $x^i = x^j$ for some $i < j$ if and only if

$$x \in A^{i+1} \cap \dots \cap A^j.$$

- The restriction $\mathcal{R}^k \to \mathbb{E}^m$ of the natural projection $x^k \mapsto x$ is an isometry.

8.6. Observation. *Any Reshetnyak puff pastry is a proper length* CAT(0) *space.*

Proof. Apply Reshetnyak gluing theorem (8.3) recursively for the convex sets in the array. □

8.7. Proposition. *Assume $(A^1, \dots, A^N)$ and $(\check{A}^1, \dots, \check{A}^N)$ are two arrays of convex closed sets in $\mathbb{E}^m$ such that $A^k \subset \check{A}^k$ for each k. Let $\mathcal{R}$ and $\check{\mathcal{R}}$ be the corresponding Reshetnyak puff pastries. Then the map $\mathcal{R} \to \check{\mathcal{R}}$ defined by $x^k \mapsto \check{x}^k$ is short.*

Moreover, if

❶ $$|x^i - y^j|_{\mathcal{R}} = |\check{x}^i - \check{y}^j|_{\check{\mathcal{R}}}$$

for some $x, y \in \mathbb{E}^m$ and $i, j \in \{0, \dots, n\}$, then the unique geodesic $[\check{x}^i\check{y}^j]_{\check{\mathcal{R}}}$ is the image of the unique geodesic $[x^iy^j]_{\mathcal{R}}$ under the map $x^i \mapsto \check{x}^i$.

Proof. The first statement in the proposition follows from the construction of Reshetnyak puff pastries.

By Observation 8.6, $\mathcal{R}$ and $\check{\mathcal{R}}$ are proper length CAT(0) spaces; hence $[x^iy^j]_{\mathcal{R}}$ and $[\check{x}^i\check{y}^j]_{\check{\mathcal{R}}}$ are unique. By ❶, since the map $\mathcal{R} \to \check{\mathcal{R}}$ is short, the image of $[x^iy^j]_{\mathcal{R}}$ is a geodesic of $\check{\mathcal{R}}$ joining $\check{x}^i$ to $\check{y}^j$. Hence the second statement follows. □

8.8. Definition. *Consider a Reshetnyak puff pastry $\mathcal{R}$ with the levels $\mathcal{R}^0, \dots, \mathcal{R}^N$. We say that $\mathcal{R}$ is end-to-end convex if $\mathcal{R}^0 \cup \mathcal{R}^N$, the* *union of its lower and upper levels forms a convex set in $\mathcal{R}$; that is, if $x, y \in \mathcal{R}^0 \cup \mathcal{R}^N$, then $[xy]_{\mathcal{R}} \subset \mathcal{R}^0 \cup \mathcal{R}^N$.*

Note that if $\mathcal{R}$ is the Reshetnyak puff pastry for an array of convex sets $\boldsymbol{A} = (A^1, \dots, A^N)$, then $\mathcal{R}$ is end-to-end convex if and only if the union of the lower and the upper levels $\mathcal{R}^0 \cup \mathcal{R}^N$ is isometric to the double of $\mathbb{E}^m$ along the nonempty intersection $A^1 \cap \dots \cap A^N$.

8.9. Observation. *Let $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ be arrays of convex bodies in $\mathbb{E}^m$. Assume that array $\boldsymbol{A}$ is obtained by inserting in $\check{\boldsymbol{A}}$ several copies of the bodies which were already listed in $\check{\boldsymbol{A}}$.*

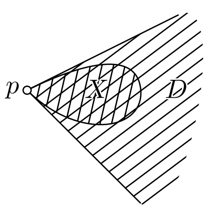
For example, if $\check{\boldsymbol{A}} = (A, C, B, C, A)$, by placing B in the second place and A in the fourth place, we obtain $\boldsymbol{A} = (A, B, C, A, B, C, A)$.

Denote by $\check{\mathcal{R}}$ and $\mathcal{R}$ the Reshetnyak puff pastries for $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ respectively.

If $\check{\mathcal{R}}$ is end-to-end convex, then so is $\mathcal{R}$.

Proof. Without loss of generality, we may assume that $\boldsymbol{A}$ is obtained by inserting one element in $\check{\boldsymbol{A}}$, say at the place number k.

Note that $\check{\mathcal{R}}$ is isometric to the puff pastry for $\boldsymbol{A}$ with A^k replaced by $\mathbb{E}^m$. It remains to apply Proposition 8.7. □



Let X be a convex set in a Euclidean space. By a d i h e d r a l a n g l e, we understand an intersection of two half-spaces; the intersection of corresponding hyperplanes is called the e d g e of the angle. We say that a dihedral angle D supports X at a point $p \in X$ if D contains X and the edge of D contains p.

8.10. Lemma. *Let A and B be two convex sets in $\mathbb{E}^m$. Assume that any dihedral angle supporting $A \cap B$ has angle measure at least α. Then the Reshetnyak puff pastry for the array*

$$(\underbrace{A, B, A, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \ \textit{times}}).$$

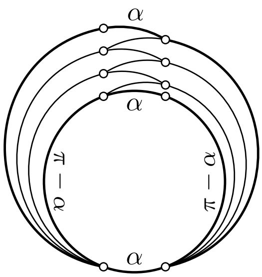
is end-to-end convex.

The proof of the lemma is based on a partial case, which we formulate as a sublemma.

8.11. Sublemma. *Let $\ddot{A}$ and $\ddot{B}$ be two half-planes in $\mathbb{E}^2$, where $\ddot{A} \cap \ddot{B}$ is an angle with measure α. Then the Reshetnyak puff pastry for the array*

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \ \textit{times}})$$

is end-to-end convex.



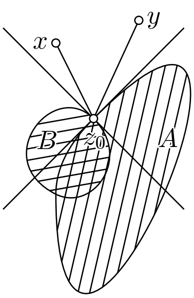
Proof. Note that the puff pastry $\ddot{\mathcal{R}}$ is isometric to the cone over the space glued from the unit circles as shown on the diagram.

All the short arcs on the diagram have length α; the long arcs have length $\pi-\alpha$, so making a circuit along any path will take $2\cdot\pi$.

The end-to-end convexity of $\ddot{\mathcal{R}}$ is equivalent to the fact that any geodesic shorter than π with the ends on the inner and the outer circles lies completely in the union of these two circles.

The latter holds if the zigzag line in the picture has length at least π. This line is formed by $\lceil\frac{\pi}{\alpha}\rceil$ arcs with length α each. Hence the sublemma. $\square$

In the proof of 8.10, we will use the following exercise in convex geometry:



8.12. Exercise. *Let A and B be two closed convex sets in $\mathbb{E}^m$ and $A\cap B\ne\varnothing$. Given two points $x,y\in\mathbb{E}^m$ let $f(z)=|x-z|+|y-z|$.*

Let $z_0\in A\cap B$ be a point of minimum of $f|_{A\cap B}$.

Show that there are half-spaces $\dot{A}$ and $\dot{B}$ such that $\dot{A}\supset A$ and $\dot{B}\supset B$ and z_0 is also a point of minimum of the restriction $f|_{\dot{A}\cap\dot{B}}$.

Proof of 8.10. Fix arbitrary $x,y\in\mathbb{E}^m$. Choose a point $z\in A\cap B$ for which the sum

$$|x-z|+|y-z|$$

is minimal. To show the end-to-end convexity of $\mathcal{R}$, it is sufficient to prove the following:

❷ *The geodesic $[x^0y^N]_{\mathcal{R}}$ contains $z^0=z^N\in\mathcal{R}$.*

Without loss of generality, we may assume that $z\in\partial A\cap\partial B$. Indeed, since the puff pastry for the 1-array (B) is end-to-end convex, Proposition 8.7 together with Observation 8.9 imply ❷ in case z lies in the interior of A. In the same way we can treat the case when z lies in the interior of B.

Note that $\mathbb{E}^m$ admits an isometric splitting $\mathbb{E}^{m-2}\times\mathbb{E}^2$ such that

$$\dot{A}=\mathbb{E}^{m-2}\times\ddot{A}$$
$$\dot{B}=\mathbb{E}^{m-2}\times\ddot{B}$$

where $\ddot{A}$ and $\ddot{B}$ are half-planes in $\mathbb{E}^2$.

Using Exercise 8.12, let us replace each A by $\dot{A}$ and each B by $\dot{B}$ in the array, to get the array

$$(\underbrace{\dot{A}, \dot{B}, \dot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil+1 \text{ times}}).$$

The corresponding puff pastry $\dot{\mathcal{R}}$ splits as a product of $\mathbb{E}^{m-2}$ and a puff pastry, call it $\ddot{\mathcal{R}}$, glued from the copies of the plane $\mathbb{E}^2$ for the array

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil+1 \text{ times}}).$$

Note that the dihedral angle $\dot{A}\cap\dot{B}$ is at least α. Therefore the angle measure of $\ddot{A}\cap\ddot{B}$ is also at least α. According to Sublemma 8.11 and Observation 8.9, $\ddot{\mathcal{R}}$ is end-to-end convex.

Since $\dot{\mathcal{R}} \overset{iso}{=\!=} \mathbb{E}^{m-2}\times\ddot{\mathcal{R}}$, the puff pastry $\dot{\mathcal{R}}$ is also end-to-end convex.

It follows that the geodesic $[\dot{x}^0\dot{y}^N]_{\dot{\mathcal{R}}}$ contains $\dot{z}^0 = \dot{z}^N \in \dot{\mathcal{R}}$. By Proposition 8.7, the image of $[\dot{x}^0\dot{y}^N]_{\dot{\mathcal{R}}}$ under the map $\dot{x}^k \mapsto x^k$ is the geodesic $[x^0y^N]_{\mathcal{R}}$. Hence Claim ❷, and the lemma follow. $\square$

D Wide corners

We say that a closed convex set $A \subset \mathbb{E}^m$ has ε-wide corners for given $\varepsilon > 0$ if together with each point p, the set A contains a small right circular cone with the tip at p and aperture ε; that is, ε is the maximum angle between two generating lines of the cone.

For example, a plane polygon has ε-wide corners if all its interior angles are at least ε.

We will consider finite collections of closed convex sets $A^1, \dots, A^n \subset \subset \mathbb{E}^m$ such that for any subset $F \subset \{1, \dots, n\}$, the intersection $\bigcap_{i\in F} A^i$ has ε-wide corners. In this case, we may say briefly *all intersections of A^i have ε-wide corners.*

8.13. Exercise. *Assume $A^1, \dots, A^n \subset \mathbb{E}^m$ are compact, convex sets with a common interior point. Show that all intersections of A^i have ε-wide corners for some positive ε.*

8.14. Exercise. *Assume $A^1, \dots, A^n \subset \mathbb{E}^m$ are convex sets with nonempty interiors that have a common center of symmetry. Show that all intersections of A^i have ε-wide corners for some positive ε.*

The proof of the following proposition is based on Lemma 8.10; this lemma is essentially the case $n = 2$ in the proposition.

8.15. Proposition. *Given $\varepsilon > 0$ and a positive integer n, there is an array of integers $\boldsymbol{j}_\varepsilon(n) = (j_1, \dots, j_N)$ such that:*

(a) For each k we have $1 \leqslant j_k \leqslant n$, and each number $1, \dots, n$ appears in $\boldsymbol{j}_\varepsilon$ at least once.

(b) If $A^1, \dots, A^n$ is a collection of closed convex sets in $\mathbb{E}^m$ with a common point and all their intersections have ε-wide corners, then the puff pastry for the array $(A^{j_1}, \dots, A^{j_N})$ is end-to-end convex.

Moreover we can assume that $N \leqslant (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^n$.

Proof. The array $\boldsymbol{j}_\varepsilon(n) = (j_1, \dots, j_N)$ is constructed recursively. For $n = 1$, we can take $\boldsymbol{j}_\varepsilon(1) = (1)$.

Assume that $\boldsymbol{j}_\varepsilon(n)$ is constructed. Let us replace each occurrence of n in $\boldsymbol{j}_\varepsilon(n)$ by the alternating string

$$\underbrace{n, n+1, n, \dots}_{\lceil \frac{\pi}{\varepsilon} \rceil + 1 \text{ times}}.$$

Denote the obtained array by $\boldsymbol{j}_\varepsilon(n+1)$.

By Lemma 8.10, end-to-end convexity of the puff pastry for $\boldsymbol{j}_\varepsilon(n+1)$ follows from end-to-end convexity of the puff pastry for the array where each string

$$\underbrace{A^n, A^{n+1}, A^n, \dots}_{\lceil \frac{\pi}{\varepsilon} \rceil + 1 \text{ times}}$$

is replaced by $Q = A^n \cap A^{n+1}$. End-to-end convexity of the latter follows by the assumption on $\boldsymbol{j}_\varepsilon(n)$, since all the intersections of $A^1, \dots, A^{n-1}, Q$ have ε-wide corners.

The upper bound on N follows directly from the construction. $\square$

E Billiards

Let $A^1, A^2, \dots A^n$ be a finite collection of closed convex sets in $\mathbb{E}^m$. Assume that for each i the boundary ∂A^i is a smooth hypersurface.

Consider the billiard table formed by the closure of the complement

$$T = \overline{\mathbb{E}^m \setminus \bigcup_i A^i}.$$

The sets A^i will be called walls of the table T and the billiards described above will be called billiards with convex walls.

A billiard trajectory on the table is a unit-speed broken line γ that follows the standard law of billiards at the breakpoints on ∂A^i

— in particular, the angle of reflection is equal to the angle of incidence. The breakpoints of the trajectory will be called collisions. We assume the trajectory meets only one wall at a time.

Recall that the definition of sets with ε-wide corners is given in 8D.

8.16. Collision theorem. *Assume $T \subset \mathbb{E}^m$ is a billiard table with n convex walls. Assume that the walls of T have a common interior point and all their intersections have ε-wide corners. Then the number of collisions of any trajectory in T is bounded by a number N which depends only on n and ε.*

As we will see from the proof, the value N can be found explicitly; $N = (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^{n^2}$ will do.

The collision theorem was proved by Dmitri Burago, Serge Ferleger, and Alexey Kononenko [35]; we present their proof with minor improvements.

Let us formulate and prove a corollary of the collision theorem; it answers a question formulated by Yakov Sinai [60].

8.17. Corollary. *Consider n homogeneous hard balls moving freely and colliding elastically in $\mathbb{R}^3$. Every ball moves along a straight line with constant speed until two balls collide, and then the new velocities of the two balls are determined by the laws of classical mechanics. We assume that only two balls can collide at the same time.*

Then the total number of collisions cannot exceed some number N that depends on the radii and masses of the balls. If the balls are identical, then N depends only on n.

8.18. Exercise. *Show that in the case of identical balls in the one-dimensional space (in $\mathbb{R}$) the total number of collisions cannot exceed $N = \frac{n\cdot(n-1)}{2}$.*

The proof below admits a straightforward generalization to all dimensions.

Proof. Denote by $a_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ the center of the i-th ball. Consider the corresponding point in $\mathbb{R}^{3\cdot N}$

$$\begin{aligned} \boldsymbol{a} &= (a_1, a_2, \dots, a_n) = \\ &= (x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_n, y_n, z_n). \end{aligned}$$

The i-th and j-th balls intersect if

$$|a_i - a_j| \leqslant R_i + R_j,$$

where R_i denotes the radius of the i-th ball. These inequalities define $\frac{n\cdot(n-1)}{2}$ cylinders

$$C_{i,j} = \left\{ (a_1, a_2, \dots, a_n) \in \mathbb{R}^{3\cdot n} \ : \ |a_i - a_j| \leqslant R_i + R_j \right\}.$$

The closure of the complement

$$T = \overline{\mathbb{R}^{3\cdot n} \setminus \bigcup_{i<j} C_{i,j}}$$

is the configuration space of our system. Its points correspond to valid positions of the system of balls.

The evolution of the system of balls is described by the motion of the point $\boldsymbol{a} \in \mathbb{R}^{3\cdot n}$. It moves along a straight line at a constant speed until it hits one of the cylinders $C_{i,j}$; this event corresponds to a collision in the system of balls.

Consider the norm of $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{R}^{3\cdot n}$ defined by

$$\|\boldsymbol{a}\| = \sqrt{M_1 \cdot |a_1|^2 + \dots + M_n \cdot |a_n|^2},$$

where $|a_i| = \sqrt{x_i^2 + y_i^2 + z_i^2}$ and M_i denotes the mass of the i-th ball. In the metric defined by $\|{*}\|$, the collisions follow the standard law of billiards.

By construction, the number of collisions of hard balls that we need to estimate is the same as the number of collisions of the corresponding billiard trajectory on the table with $C_{i,j}$ as the walls.

Note that each cylinder $C_{i,j}$ is a convex set; it has smooth boundary, and it is centrally symmetric around the origin. By Exercise 8.14, all the intersections of the walls have ε-wide corners for some $\varepsilon > 0$ that depend on the radiuses R_i and the masses M_i. It remains to apply the collision theorem (8.16). $\square$

Now we present the proof of the collision theorem (8.16) based on the results developed in the previous section.

Proof of 8.16. Let us apply induction on n.

Base: $n = 1$. The number of collisions cannot exceed 1. Indeed, by the convexity of A^1, if the trajectory is reflected once in ∂A^1, then it cannot return to A^1.

Step. Assume γ is a trajectory that meets the walls in the order $A^{i_1}, \dots, A^{i_N}$ for a large integer N.

Consider the array

$$\boldsymbol{A}_\gamma = (A^{i_1}, \dots, A^{i_N}).$$

The induction hypothesis implies:

❶ *There is a positive integer M such that any M consecutive elements of $\boldsymbol{A}_\gamma$ contain each A^i at least once.*

Let $\mathcal{R}_\gamma$ be the Reshetnyak puff pastry for $\boldsymbol{A}_\gamma$.

Consider the lift of γ to $\mathcal{R}_\gamma$, defined by $\bar\gamma(t) = \gamma^k(t) \in \mathcal{R}_\gamma$ for any moment of time t between the k-th and $(k+1)$-th collisions. Since γ follows the standard law of billiards at breakpoints, the lift $\bar\gamma$ is locally a geodesic in $\mathcal{R}_\gamma$. By Observation 8.6, the puff pastry $\mathcal{R}_\gamma$ is a proper length CAT(0) space. Therefore $\bar\gamma$ is a geodesic.

Since γ does not meet $A^1 \cap \dots \cap A^n$, the lift $\bar\gamma$ does not lie in $\mathcal{R}^0_\gamma \cup \mathcal{R}^N_\gamma$. In particular, $\mathcal{R}_\gamma$ is not end-to-end convex.

Let

$$\boldsymbol{B} = (A^{j_1}, \dots, A^{j_K})$$

be the array provided by Proposition 8.15; so $\boldsymbol{B}$ contains each A^i at least once and the puff pastry $\mathcal{R}_{\boldsymbol{B}}$ for $\boldsymbol{B}$ is end-to-end convex. If N is sufficiently large, namely $N \geqslant K\cdot M$, then ❶ implies that $\boldsymbol{A}_\gamma$ can be obtained by inserting a finite number of A^i's in $\boldsymbol{B}$.

By Observation 8.9, $\mathcal{R}_\gamma$ is end-to-end convex, a contradiction. □

F Comments

The gluing theorem (8.3) was proved by Yuri Reshetnyak [117]. It can be extended to the class of geodesic CAT(0) spaces, which by 7.12 includes all complete length CAT(0) spaces. It also admits a natural generalization to length CAT(κ) spaces; see the book of Martin Bridson and André Haefliger [31] and our book [5] for details.

Puff pastry is used to bound topological entropy of the billiard flow and to approximate the shortest billiard path that touches given lines in a given order; see the papers of Dmitri Burago with Serge Ferleger and Alexey Kononenko [36], and with Dimitri Grigoriev and Anatol Slissenko [37]. The lecture of Dmitri Burago [33] gives a short survey on the subject.

Note that the interior points of the walls play a key role in the proof despite the fact that trajectories never go inside the walls. In a similar fashion, puff pastry was used by Stephanie Alexander and Richard Bishop [2] to find the upper curvature bound for warped products.

Joel Hass [69] constructed an example of a Riemannian metric on the 3-ball with negative curvature and concave boundary. This example might decrease your appetite for generalizing the collision theorem — while locally such a 3-ball looks as good as the billiards table in the theorem, the number of collisions is obviously infinite.

It was shown by Dmitri Burago and Sergei Ivanov [38] that the number of collisions that may occur between n identical balls in $\mathbb{R}^3$ grows at least exponentially in n; the two-dimensional case is open so far.

Lecture 9

Globalization

This lecture is nearly a copy of [4, Sections 3.1–3.3]; here we introduce locally CAT(0) spaces and prove the globalization theorem that provides a sufficient condition for locally CAT(0) spaces to be globally CAT(0).

A Locally CAT spaces

We say that a space $\mathcal{U}$ is l o c a l l y CAT(0) (or l o c a l l y CAT(1)) if a small closed ball centered at any point p in $\mathcal{U}$ is CAT(0) (or CAT(1), respectively).

For example, the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is locally isometric to $\mathbb{R}$, and so $\mathbb{S}^1$ is locally CAT(0). On the other hand, $\mathbb{S}^1$ is not CAT(0), since closed local geodesics in $\mathbb{S}^1$ are not geodesics, so $\mathbb{S}^1$ does not satisfy Proposition 8.1.

If $\mathcal{U}$ is a proper length space, then it is locally CAT(0) (or locally CAT(1)) if and only if each point $p \in \mathcal{U}$ admits an open neighborhood Ω that is geodesic and such that any triangle in Ω is thin (or spherically thin, respectively).

B Space of local geodesic paths

A constant-speed parameterization of a local geodesic by the unit interval $[0, 1]$ is called a l o c a l g e o d e s i c p a t h.

In this section, we will study behavior of local geodesics in locally CAT(κ) spaces. The results will be used in the proof of the globalization theorem (9.6).

Recall that a path is a curve parametrized by $[0,1]$. The space of paths in a metric space $\mathcal{U}$ comes with the natural metric

❶ $$|\alpha - \beta| = \sup\{\, |\alpha(t) - \beta(t)|_{\mathcal{U}} \,:\, t \in [0,1]\,\}.$$

9.1. Proposition. *Let $\mathcal{U}$ be a proper length, locally* CAT(κ) *space.*

Assume $\gamma_n\colon[0,1]\to\mathcal{U}$ is a sequence of local geodesic paths converging to a path $\gamma_\infty\colon[0,1]\to\mathcal{U}$. Then γ_∞ is a local geodesic path. Moreover

$$\operatorname{length}\gamma_n \to \operatorname{length}\gamma_\infty$$

as $n\to\infty$.

Proof; CAT(0) *case.* Fix $t\in[0,1]$. Let $R>0$ be sufficiently small, so that $\overline{\mathrm{B}}[\gamma_\infty(t),R]$ forms a proper length CAT(0) space.

Assume that a local geodesic σ is shorter than $R/2$ and intersects the ball $\mathrm{B}(\gamma_\infty(t),R/2)$. Then σ cannot leave the ball $\overline{\mathrm{B}}[\gamma_\infty(t),R]$. Hence, by Proposition 8.1, σ is a geodesic. In particular, for all sufficiently large n, any arc of γ_n of length $R/2$ or less containing $\gamma_n(t)$ is a geodesic.

Since $\mathcal{B}=\overline{\mathrm{B}}[\gamma_\infty(t),R]$ is a proper length CAT(0) space, by 7.12, geodesic segments in $\mathcal{B}$ depend uniquely on their endpoint pairs. Thus there is a subinterval $\mathbb{I}$ of $[0,1]$, that contains a neighborhood of t in $[0,1]$ and such that the arc $\gamma_n|_{\mathbb{I}}$ is minimizing for all large n. It follows that $\gamma_\infty|_{\mathbb{I}}$ is a geodesic, and therefore γ_∞ is a local geodesic.

The CAT(1) case is done in the same way, but one has to assume in addition that $R<\pi$. □

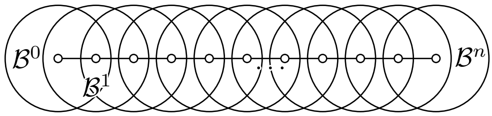
The following lemma and its proof were suggested to us by Alexander Lytchak. This lemma allows a local geodesic path to be moved continuously so that its endpoints follow given trajectories. This statement was originally proved by Stephanie Alexander and Richard Bishop [3] using a different method.

9.2. Patchwork along a geodesic. *Let $\mathcal{U}$ be a proper length, locally* CAT(0) *space, and $\gamma\colon[0,1]\to\mathcal{U}$ be a locally geodesic path.*

Then there is a proper length CAT(0) *space $\mathcal{N}$, an open set $\hat\Omega\subset\mathcal{N}$, and a geodesic path $\hat\gamma\colon[0,1]\to\hat\Omega$, such that there is an open locally distance-preserving map $\Phi\colon\hat\Omega\looparrowright\mathcal{U}$ satisfying $\Phi\circ\hat\gamma=\gamma$.*

If $\operatorname{length}\gamma<\pi$*, then the same holds in the* CAT(1) *case. Namely we assume that $\mathcal{U}$ is a proper length, locally* CAT(1) *space and construct a proper length* CAT(1) *space $\mathcal{N}$ with the same property as above.*

Proof. Fix $r>0$ so that for each $t\in[0,1]$, the closed ball $\overline{\mathrm{B}}[\gamma(t),r]$ forms a proper length CAT(0) space.



Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that

$$\mathrm{B}(\gamma(t_i), r) \supset \gamma([t_{i-1}, t_i])$$

for all $n > i > 0$. Set $\mathcal{B}_i = \overline{\mathrm{B}}[\gamma(t_i), r]$. We can assume in addition that $\mathcal{B}_{i-1} \cap \mathcal{B}_{i+1} \subset \mathcal{B}_i$ if $0 < i < n$.

Consider the disjoint union $\bigsqcup_i \mathcal{B}_i = \{ (i, x) : x \in \mathcal{B}_i \}$ with the minimal equivalence relation $\sim$ such that $(i, x) \sim (i-1, x)$ for all i. Let $\mathcal{N}$ be the space obtained by gluing the $\mathcal{B}_i$ along $\sim$.

Note that $A_i = \mathcal{B}_i \cap \mathcal{B}_{i-1}$ is convex in $\mathcal{B}_i$ and in $\mathcal{B}_{i-1}$. Applying the Reshetnyak gluing theorem (8.3) n times, we conclude that $\mathcal{N}$ is a proper length CAT(0) space.

For $t \in [t_{i-1}, t_i]$, define $\hat{\gamma}(t)$ as the equivalence class of $(i, \gamma(t))$ in $\mathcal{N}$. Let $\hat{\Omega}$ be the ε-neighborhood of $\hat{\gamma}$ in $\mathcal{N}$, where $\varepsilon > 0$ is chosen so that $\mathrm{B}(\gamma(t), \varepsilon) \subset \mathcal{B}_i$ for all $t \in [t_{i-1}, t_i]$.

Define $\Phi\colon \hat{\Omega} \to \mathcal{U}$ by sending the equivalence class of (i, x) to x. It is straightforward to check that Φ, $\hat{\gamma}$, and $\hat{\Omega} \subset \mathcal{N}$ satisfy the conclusion of the lemma.

The CAT(1) case is proved in the same way. □

The following two corollaries follow from: (1) patchwork (9.2), (2) Proposition 8.1, which states that local geodesics are geodesics in any CAT(0) space, and (3) Proposition 7.12 on uniqueness of geodesics.

9.3. Corollary. *If $\mathcal{U}$ is a proper length, locally* CAT(0) *space, then for any pair of points $p, q \in \mathcal{U}$, the space of all local geodesic paths from p to q is discrete; that is, for any local geodesic path γ connecting p to q, there is $\varepsilon > 0$ such that for any other local geodesic path δ from p to q we have $|\gamma(t) - \delta(t)|_{\mathcal{U}} > \varepsilon$ for some $t \in [0, 1]$.*

Analogously, if $\mathcal{U}$ is a proper length, locally CAT(1) *space, then for any pair of points $p, q \in \mathcal{U}$, the space of all local geodesic paths shorter than π from p to q is discrete.*

9.4. Corollary. *If $\mathcal{U}$ is a proper length, locally* CAT(0) *space, then for any path α there is a choice of local geodesic path γ_α connecting the ends of α such that the map $\alpha \mapsto \gamma_\alpha$ is continuous, and if α is a local geodesic path then $\gamma_\alpha = \alpha$.*

Analogously, if $\mathcal{U}$ is a proper length, locally CAT(1) *space, then for any path α shorter than π, there is a choice of local geodesic path γ_α*

shorter than π connecting the ends of α such that the map $\alpha \mapsto \gamma_\alpha$ is continuous, and if α is a local geodesic path then $\gamma_\alpha = \alpha$.

Proof of 9.4. We do the CAT(0) case; the CAT(1) case is analogous.

Consider the maximal interval $\mathbb{I} \subset [0,1]$ containing 0 such that there is a continuous one-parameter family of local geodesic paths γ_t for $t \in \mathbb{I}$ connecting $\alpha(0)$ to $\alpha(t)$, with $\gamma_t(0) = \gamma_0(t) = \alpha(0)$ for any t.

By Proposition 9.1, $\mathbb{I}$ is closed, so we may assume $\mathbb{I} = [0,s]$ for some $s \in [0,1]$.

Applying patchwork (9.2) to γ_s, we find that $\mathbb{I}$ is also open in $[0,1]$. Hence $\mathbb{I} = [0,1]$. Set $\gamma_\alpha = \gamma_1$.

By construction, if α is a local geodesic path, then $\gamma_\alpha = \alpha$.

Moreover, from Corollary 9.3, the construction $\alpha \mapsto \gamma_\alpha$ produces close results for sufficiently close paths in the metric defined by ➊; that is, the map $\alpha \mapsto \gamma_\alpha$ is continuous. $\square$

Given a path $\alpha\colon [0,1] \to \mathcal{U}$, we denote by $\bar{\alpha}$ the same path traveled in the opposite direction; that is,

$$\bar{\alpha}(t) = \alpha(1-t).$$

The product of two paths will be denoted with "$*$"; if two paths α and β connect the same pair of points, then the product $\bar{\alpha} * \beta$ is a closed curve.

9.5. Exercise. *Assume $\mathcal{U}$ is a proper length, locally* CAT(1) *space. Consider the construction $\alpha \mapsto \gamma_\alpha$ provided by Corollary 9.4.*

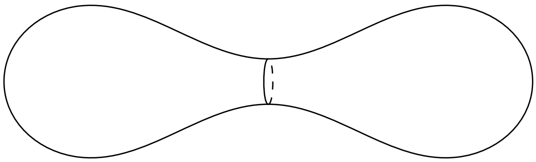
Assume that α and β are two paths connecting the same pair of points in $\mathcal{U}$, where each is shorter than π and the product $\bar{\alpha} * \beta$ is null-homotopic in the class of closed curves shorter than $2{\cdot}\pi$. Show that $\gamma_\alpha = \gamma_\beta$.

C Globalization

9.6. Globalization theorem. *If a proper length, locally* CAT(0) *space is simply connected, then it is* CAT(0).

Analogously, if $\mathcal{U}$ is a proper length, locally CAT(1) *space such that any closed curve $\gamma\colon \mathbb{S}^1 \to \mathcal{U}$ shorter than $2{\cdot}\pi$ is null-homotopic in the class of closed curves shorter than $2{\cdot}\pi$. Then $\mathcal{U}$ is* CAT(1).

The surface on the diagram is an example of a simply connected space that is locally CAT(1) but not CAT(1). To contract the marked curve one has to increase its length to $2{\cdot}\pi$ or more; in particular, the surface does not satisfy the assumption of the globalization theorem.



The proof of the globalization theorem relies on the following theorem, which is essentially [6, Satz 9].

9.7. Patchwork globalization theorem. *A proper length, locally* CAT(0) *space* $\mathcal{U}$ *is* CAT(0) *if and only if all pairs of points in* $\mathcal{U}$ *are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

Analogously, a proper length, locally CAT(1) *space* $\mathcal{U}$ *is* CAT(1) *if and only if all pairs of points in* $\mathcal{U}$ *at distance less than* π *are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

The proof uses a thin-triangle decomposition with the inheritance lemma (8.2) and the following construction:

9.8. Line-of-sight map. *Let* p *be a point and* α *be a curve of finite length in a length space* $\mathcal{X}$*. Let* $\mathring{\alpha} : [0,1] \to \mathcal{U}$ *be the constant-speed parametrization of* α*. If* $\gamma_t \colon [0,1] \to \mathcal{U}$ *is a geodesic path from* p *to* $\mathring{\alpha}(t)$*, we say*

$$[0,1] \times [0,1] \to \mathcal{U} \colon (t,s) \mapsto \gamma_t(s)$$

is a line-of-sight map from p *to* α*.*

Proof of the patchwork globalization theorem (9.7). Note that the implication "only if" follows from 7.12 and 7.21; it remains to prove the "if" part.

Fix a triangle $[pxy]$ in $\mathcal{U}$. We need to show that $[pxy]$ is thin.

By the assumptions, the line-of-sight map $(t,s) \mapsto \gamma_t(s)$ from p to $[xy]$ is uniquely defined and continuous.

Fix a partition

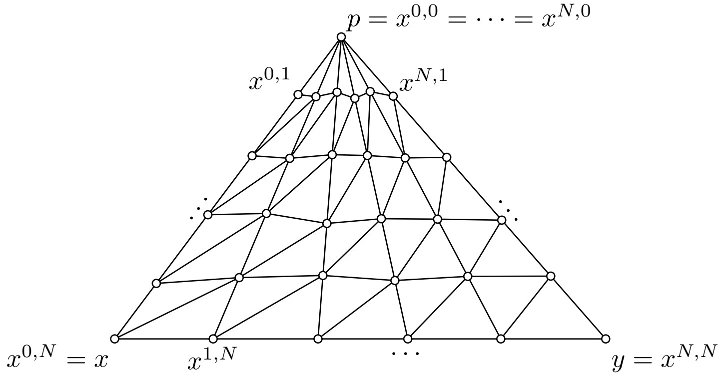
$$0 = t^0 < t^1 < \ldots < t^N = 1,$$

and set $x^{i,j} = \gamma_{t^i}(t^j)$. Since the line-of-sight map is continuous and $\mathcal{U}$ is locally CAT(0), we may assume that the triangles

$$[x^{i,j}x^{i,j+1}x^{i+1,j+1}] \quad \text{and} \quad [x^{i,j}x^{i+1,j}x^{i+1,j+1}]$$

are thin for each pair i, j.

Now we show that the thin property propagates to $[pxy]$ by repeated application of the inheritance lemma (8.2):



- ◇ For fixed i, sequentially applying the lemma shows that the triangles $[px^{i,1}x^{i+1,2}]$, $[px^{i,2}x^{i+1,2}]$, $[px^{i,2}x^{i+1,3}]$, and so on are thin.

In particular, for each i, the long triangle $[px^{i,N}x^{i+1,N}]$ is thin.

- ◇ By the same lemma the triangles $[px^{0,N}x^{2,N}]$, $[px^{0,N}x^{3,N}]$, and so on, are thin.

In particular, $[pxy] = [px^{0,N}x^{N,N}]$ is thin. □

Proof of the globalization theorem; CAT(0) *case.* Let $\mathcal{U}$ be a proper length, locally CAT(0) space that is simply connected. Given a path α in $\mathcal{U}$, denote by γ_α the local geodesic path provided by 9.4. Since the map $\alpha \mapsto \gamma_\alpha$ is continuous, by 9.3 we have $\gamma_\alpha = \gamma_\beta$ for any pair of paths α and β homotopic relative to the ends.

Since $\mathcal{U}$ is simply connected, any pair of paths with common ends are homotopic. In particular, if α and β are local geodesics from p to q, then $\alpha = \gamma_\alpha = \gamma_\beta = \beta$ by Corollary 9.4. It follows that any two points $p, q \in \mathcal{U}$ are joined by a unique local geodesic that depends continuously on (p, q).

Since $\mathcal{U}$ is geodesic, it remains to apply the patchwork globalization theorem (9.7).

CAT(1) *case.* The proof goes along the same lines, but one needs to use Exercise 9.5. □

9.9. Corollary. *Any compact length, locally* CAT(0) *space that contains no closed local geodesics is* CAT(0)*.*

Analogously, any compact length, locally CAT(1) *space that contains no closed local geodesics shorter than* $2{\cdot}\pi$ *is* CAT(1)*.*

Proof. By the globalization theorem (9.6), we need to show that the space is simply connected. Assume the contrary. Fix a nontrivial homotopy class of closed curves.

Denote by ℓ the exact lower bound for the lengths of curves in the class. Note that $\ell > 0$; otherwise, there would be a closed noncontractible curve in a CAT(0) neighborhood of some point, contradicting 7.16.

Since the space is compact, the class contains a length-minimizing curve, which must be a closed local geodesic.

The CAT(1) case is analogous, one only has to consider a homotopy class of closed curves shorter than $2\cdot\pi$. □

9.10. Exercise. *Prove that any compact length, locally* CAT(0) *space* $\mathcal{X}$ *that is not* CAT(0) *contains a geodesic circle; that is, a simple closed curve* γ *such that for any two points* $p, q \in \gamma$*, one of the arcs of* γ *with endpoints* p *and* q *is a geodesic.*

Formulate and prove the analogous statement for CAT(1) *spaces.*

9.11. Advanced exercise. *Let* $\mathcal{U}$ *be a proper length* CAT(0) *space. Assume* $\tilde{\mathcal{U}} \to \mathcal{U}$ *is a metric double cover branching along a geodesic. (For example, 3-dimensional Euclidean space admits a double cover branching along a line.)*

Show that $\tilde{\mathcal{U}}$ *is* CAT(0)*.*

D Remarks

Riemannian manifolds with nonpositive sectional curvature are locally CAT(0). The original formulation of the globalization theorem, or Hadamard–Cartan theorem, states that *if* M *is a complete Riemannian manifold with sectional curvature at most* 0*, then the exponential map at any point* $p \in M$ *is a covering*; in particular, it implies that *the universal cover of* M *is diffeomorphic to the Euclidean space of the same dimension.*

In this generality, this theorem appeared in the lectures of Elie Cartan [45]. This theorem was proved for surfaces in Euclidean 3-space by Hans von Mangoldt [94] and a few years later independently for two-dimensional Riemannian manifolds by Jacques Hadamard [68].

Formulations for metric spaces of different generality were proved by Herbert Busemann [41], Willi Rinow [118], Mikhael Gromov [63, p. 119]. A detailed proof of Gromov's statement was given by Werner Ballmann [15] when $\mathcal{U}$ is proper, and by Stephanie Alexander and Richard Bishop [3] in more generality.

For proper CAT(1) spaces, the globalization theorem was proved by Brian Bowditch [29].

The globalization theorem holds for complete length spaces (not necessarily proper spaces) [5].

The patchwork globalization (9.7) is proved by Alexandrov [6, Satz 9]. For proper spaces one can remove the continuous dependence from the formulation; it follows from uniqueness. For complete spaces, the later is not true [31, Chapter I, Exercise 3.14].

For spaces with curvature bounded below globalization requires no additional condition. Namely the following theorem holds [see 5, and the references therein].

9.12. Globalization theorem. *Any complete length locally* CBB(κ) *space is* CBB(κ).

Lecture 10

Polyhedral spaces

This lecture is nearly a copy of [4, Sections 1.7, 3.4, and 3.5]; here we give a condition for polyhedral spaces that grantees that it is $\mathrm{CAT}(0)$.

A Space of directions and tangent space

In this section, we introduce a metric analog of (unit) tangent bundle that makes sense in Alexandrov geometry.

Let $\mathcal{X}$ be a metric space with defined angles for all hinges; by 7.15 it holds for any $\mathrm{CBB}(\kappa)$ or $\mathrm{CAT}(\kappa)$ space. Fix a point $p \in \mathcal{X}$.

Consider the set $\mathfrak{S}_p$ of all nontrivial geodesics that start at p. By 7.4, the triangle inequality holds for $\measuredangle$ on $\mathfrak{S}_p$, so $(\mathfrak{S}_p, \measuredangle)$ forms a semi-metric space; that is, $\measuredangle$ satisfies all the conditions of a metric on $\mathfrak{S}_p$, except that the angle between distinct geodesics might vanish.

The metric space corresponding to $(\mathfrak{S}_p, \measuredangle)$ is called the space of geodesic directions at p, denoted by Σ'_p or $\Sigma'_p\mathcal{X}$. Elements of Σ'_p are called geodesic directions at p. Each geodesic direction is formed by an equivalence class of geodesics in $\mathfrak{S}_p$ for the equivalence relation

$$[px] \sim [py] \quad \Longleftrightarrow \quad \measuredangle[p\,{}^x_y] = 0.$$

The completion of Σ'_p is called the space of directions at p and is denoted by Σ_p or $\Sigma_p\mathcal{X}$. Elements of Σ_p are called directions at p.

The Euclidean cone $\mathrm{Cone}\,\Sigma_p$ over the space of directions Σ_p is called the tangent space at p and is denoted by T_p or $\mathrm{T}_p\mathcal{X}$.

10.1. Exercise. *Assume $\mathcal{U}$ is a proper length $\mathrm{CAT}(0)$ space with extendable geodesics; that is, any geodesic is an arc in a local geodesic $\mathbb{R} \to \mathcal{U}$.*

Show that the space of geodesic directions at any point in $\mathcal{U}$ is complete.

Does the statement remain true if $\mathcal{U}$ is complete, but not required to be proper?

The tangent space T_p could also be defined directly, without introducing the space of directions. To do so, consider the set $\mathfrak{T}_p$ of all geodesics with constant-speed parametrizations starting at p. Given $\alpha, \beta \in \mathfrak{T}_p$, set

❶ $$|\alpha - \beta|_{\mathfrak{T}_p} = \lim_{\varepsilon\to 0} \frac{|\alpha(\varepsilon) - \beta(\varepsilon)|_{\mathcal{X}}}{\varepsilon}$$

Since the angles in $\mathcal{X}$ are defined, ❶ defines a semimetric on $\mathfrak{T}_p$.

The corresponding metric space admits a natural isometric identification with the cone $\mathrm{T}'_p = \mathrm{Cone}\,\Sigma'_p$. The elements of T'_p are equivalence classes for the relation

$$\alpha \sim \beta \quad\iff\quad |\alpha(t) - \beta(t)|_{\mathcal{X}} = o(t).$$

The completion of T'_p is therefore naturally isometric to T_p.

Elements of T_p will be called *tangent vectors* at p, regardless of the fact that T_p is only a metric cone and need not be a vector space. Elements of T'_p will be called *geodesic tangent vectors* at p.

10.2. Exercise. *Let $\mathcal{X}$ be a complete length* $\mathrm{CAT}(0)$ *space. Show that for any point $p \in \mathcal{X}$ the tangent space $\mathrm{T}_p\mathcal{X}$ is isometric to a subset of the ultra-tangent space $\mathrm{T}^\omega_p\mathcal{X}$ (defined in 6F).*

Use 7.8 to conclude that $\mathrm{T}_p\mathcal{X}$ is $\mathrm{CAT}(0)$.

10.3. Exercise. *Let $\mathcal{X}$ be a complete length* $\mathrm{CAT}(0)$ *space. Show that for any point $p \in \mathcal{X}$ the tangent space $\mathrm{T}_p\mathcal{X}$ is a length space.*

B Suspension

Suspension is a spherical analog of cone construction (7D).

The *suspension* $\mathcal{V} = \mathrm{Susp}\,\mathcal{U}$ over a metric space $\mathcal{U}$ is defined as the metric space whose underlying set consists of equivalence classes in $[0, \pi] \times \mathcal{U}$ with the equivalence relation "$\sim$" given by $(0, p) \sim (0, q)$ and $(\pi, p) \sim (\pi, q)$ for any points $p, q \in \mathcal{U}$, and whose metric is given by the spherical cosine rule

$$\cos|(p, s) - (q, t)|_{\mathrm{Susp}\,\mathcal{U}} = \cos s\cdot \cos t - \sin s\cdot \sin t\cdot \cos\alpha,$$

where $\alpha = \min\{\pi, |p - q|_{\mathcal{U}}\}$.

The points in $\mathcal{V}$ formed by the equivalence classes of $0 \times \mathcal{U}$ and $\pi \times \mathcal{U}$ are called the north and the south poles of the suspension.

10.4. Exercise. *Let $\mathcal{U}$ be a metric space. Show that the spaces*

$$\mathbb{R} \times \mathrm{Cone}\,\mathcal{U} \quad \textit{and} \quad \mathrm{Cone}[\mathrm{Susp}\,\mathcal{U}]$$

are isometric.

The following statement is a direct analog of 7.11 and it can be proved along the same lines.

10.5. Proposition. *Let $\mathcal{U}$ be a metric space. Then $\mathrm{Susp}\,\mathcal{U}$ is $\mathrm{CAT}(1)$ if and only if $\mathcal{U}$ is $\mathrm{CAT}(1)$.*

C Definitions

10.6. Definition. *A length space $\mathcal{P}$ is called a (spherical) polyhedral space if it admits a finite triangulation τ such that every simplex in τ is isometric to a simplex in a Euclidean space (or respectively a unit sphere) of appropriate dimension.*

By a triangulation of a polyhedral space we will always understand a triangulation as above.

Note that according to the above definition, all polyhedral spaces are compact. However, most of the statements below admit straightforward generalizations to locally polyhedral spaces; that is, complete length spaces, any point of which admits a closed neighborhood isometric to a polyhedral space. The latter class of spaces includes in particular infinite covers of polyhedral spaces.

The dimension of a polyhedral space $\mathcal{P}$ is defined as the maximal dimension of the simplices in one (and therefore any) triangulation of $\mathcal{P}$.

Links. Let $\mathcal{P}$ be a polyhedral space and σ be a simplex in a triangulation τ of $\mathcal{P}$.

The simplices that contain σ form an abstract simplicial complex called the link of σ, denoted by Link_σ. If m is the dimension of σ, then the set of vertices of Link_σ is formed by the $(m+1)$-simplices that contain σ; the set of its edges is formed by the $(m+2)$-simplices that contain σ; and so on.

The link Link_σ can be identified with the subcomplex of τ formed by all the simplices σ' such that $\sigma \cap \sigma' = \varnothing$ but both σ and σ' are faces of a simplex of τ.

The points in Link_σ can be identified with the normal directions to σ at a point in its interior. The angle metric between directions makes

Link_σ into a spherical polyhedral space. We will always consider the link with this metric.

Tangent space and space of directions. Let $\mathcal{P}$ be a polyhedral space (Euclidean or spherical) and τ be its triangulation. If a point $p \in \mathcal{P}$ lies in the interior of a k-simplex σ of τ then the tangent space $\mathrm{T}_p = \mathrm{T}_p\mathcal{P}$ is naturally isometric to

$$\mathbb{E}^k \times (\mathrm{Cone}\,\mathrm{Link}_\sigma).$$

Equivalently, the space of directions $\Sigma_p = \Sigma_p\mathcal{P}$ can be isometrically identified with the k-times iterated suspension over Link_σ; that is,

$$\Sigma_p \stackrel{iso}{=\!=} \mathrm{Susp}^k(\mathrm{Link}_\sigma).$$

If $\mathcal{P}$ is an m-dimensional polyhedral space, then for any $p \in \mathcal{P}$ the space of directions Σ_p is a spherical polyhedral space of dimension at most $m-1$.

In particular, for any point p in σ, the isometry class of Link_σ together with $k = \dim\sigma$ determines the isometry class of Σ_p, and the other way around — Σ_p and k determines the isometry class of Link_σ.

A small neighborhood of p is isometric to a neighborhood of the tip of $\mathrm{Cone}\,\Sigma_p$. (If $\mathcal{P}$ is a spherical polyhedral space, then a small neighborhood of p is isometric to a neighborhood of the north pole in $\mathrm{Susp}\,\Sigma_p$.) In fact, if this property holds at any point of a compact length space $\mathcal{P}$, then $\mathcal{P}$ is a polyhedral space [90].

D CAT test

The following theorem provides a combinatorial description of polyhedral spaces with curvature bounded above.

10.7. Theorem. *Let $\mathcal{P}$ be a polyhedral space and τ be its triangulation. Then $\mathcal{P}$ is locally* CAT(0) *if and only if the link of each simplex in τ has no closed local geodesic shorter than $2\cdot\pi$.*

Analogously, let $\mathcal{P}$ be a spherical polyhedral space and τ be its triangulation. Then $\mathcal{P}$ is CAT(1) *if and only if neither $\mathcal{P}$ nor the link of any simplex in τ has a closed local geodesic shorter than $2\cdot\pi$.*

Proof. The "only if" part follows from 8.1, 10.5, and 7.11.

To prove the "if" part, we apply induction on $\dim\mathcal{P}$. The base case $\dim\mathcal{P} = 0$ is evident. Let us start with the CAT(1) case.

Step. Assume that the theorem is proved in the case $\dim\mathcal{P} < m$. Suppose $\dim\mathcal{P} = m$.

Fix a point $p \in \mathcal{P}$. A neighborhood of p is isometric to a neighborhood of the north pole in the suspension over the space of directions Σ_p.

Note that Σ_p is a spherical polyhedral space, and its links are isometric to links of $\mathcal{P}$. By the induction hypothesis, Σ_p is CAT(1). Thus, by the second part of Exercise 7.11, $\mathcal{P}$ is locally CAT(1).

Applying the second part of Corollary 9.9, we get the statement.

The CAT(0) case is done in exactly the same way except we need to use the first part of Exercise 7.11 and the first part of Corollary 9.9 on the last step. □

10.8. Exercise. *Let $\mathcal{P}$ be a polyhedral space such that any two points can be connected by a unique geodesic. Show that $\mathcal{P}$ is* CAT(0)*.*

10.9. Advanced exercise. *Construct a Euclidean polyhedral metric on $\mathbb{S}^3$ such that the total angle around each edge in its triangulation is at least $2{\cdot}\pi$.*

E Flag complexes

10.10. Definition. *A simplicial complex $\mathcal{S}$ is called flag if whenever $\{v^0,\dots,v^k\}$ is a set of distinct vertices of $\mathcal{S}$ that are pairwise joined by edges, then the vertices $v^0,\dots,v^k$ span a k-simplex in $\mathcal{S}$.*

If the above condition is satisfied for $k=2$, then we say that $\mathcal{S}$ satisfies the no-triangle condition.

Note that every flag complex is determined by its one-skeleton. Moreover, for any graph, its cliques (that is, complete subgraphs) define a flag complex. For that reason, flag complexes are also called clique complexes.

10.11. Exercise. *Show that the barycentric subdivision of any simplicial complex is a flag complex.*

Use the flag condition (see 10.14 below) to conclude that any finite simplicial complex is homeomorphic to a proper length CAT(1) *space.*

10.12. Proposition. *A simplicial complex $\mathcal{S}$ is flag if and only if $\mathcal{S}$ as well as all the links of all its simplices satisfy the no-triangle condition.*

From the definition of flag complex, we get the following.

10.13. Observation. *Any link of any simplex in a flag complex is flag.*

Proof of 10.12. By Observation 10.13, the no-triangle condition holds for any flag complex and the links of all its simplices.

Now assume that a complex $\mathcal{S}$ and all its links satisfy the no-triangle condition. It follows that $\mathcal{S}$ includes a 2-simplex for each triangle. Applying the same observation for each edge we get that $\mathcal{S}$ includes a 3-simplex for any complete graph with 4 vertices. Repeating this observation for triangles, 4-simplices, 5-simplices, and so on, we get that $\mathcal{S}$ is flag. □

All-right triangulation. A triangulation of a spherical polyhedral space is called an all-right triangulation if each simplex of the triangulation is isometric to a spherical simplex all of whose angles are right. Similarly, we say that a simplicial complex is equipped with an all-right spherical metric if it is a length metric and each simplex is isometric to a spherical simplex all of whose angles are right.

Spherical polyhedral CAT(1) spaces glued from right-angled simplices admit the following characterization discovered by Mikhael Gromov [63, p. 122].

10.14. Flag condition. *Assume that a spherical polyhedral space $\mathcal{P}$ admits an all-right triangulation τ. Then $\mathcal{P}$ is* CAT(1) *if and only if τ is flag.*

Proof; only-if part. Assume there are three vertices v^1, v^2 and v^3 of τ that are pairwise joined by edges but do not span a triangle. Note that in this case

$$\measuredangle[v^1\,{}^{v^2}_{v^3}] = \measuredangle[v^2\,{}^{v^3}_{v^1}] = \measuredangle[v^3\,{}^{v^1}_{v^2}] = \pi.$$

Equivalently,

❶ *The product of the geodesics $[v^1v^2]$, $[v^2v^3]$ and $[v^3v^1]$ forms a locally geodesic loop in $\mathcal{P}$ of length $\tfrac{3}{2}\cdot\pi$.*

Now assume that $\mathcal{P}$ is CAT(1). Then by 10.7, $\mathrm{Link}_\sigma \mathcal{P}$ is CAT(1) for every simplex σ in τ.

Each of these links is an all-right spherical complex and by 10.7, none of these links can contain a geodesic circle shorter than $2\cdot\pi$.

Therefore Proposition 10.12 and ❶ imply the "only if" part.

If part. By 10.13 and 10.7, it is sufficient to show that any closed local geodesic γ in a flag complex $\mathcal{S}$ with all-right metric has length at least $2\cdot\pi$.

Recall that the closed star of a vertex v (briefly $\overline{\mathrm{Star}}_v$) is formed by all the simplices containing v. Similarly, Star_v, the open star of v, is the union of all simplices containing v with faces opposite v removed.

Choose a vertex v such that Star_v contains a point $\gamma(t_0)$ of γ. Consider the maximal arc γ_v of γ that contains the point $\gamma(t_0)$ and runs in Star_v. Note that the distance $|v-\gamma_v(t)|_{\mathcal{P}}$ behaves in exactly the same way as the distance from the north pole in $\mathbb{S}^2$ to a geodesic in the north hemisphere; that is, there is a geodesic $\tilde\gamma_v$ in the north hemisphere of $\mathbb{S}^2$ such that for any t we have

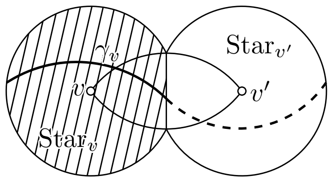
$$|v-\gamma_v(t)|_{\mathcal{P}} = |n-\tilde\gamma_v(t)|_{\mathbb{S}^2},$$

where n denotes the north pole of $\mathbb{S}^2$. In particular,

$$\operatorname{length}\gamma_v = \pi;$$

that is, γ spends time π on every visit to Star_v.

After leaving Star_v, the local geodesic γ has to enter another simplex, say σ'. Since τ is flag, the simplex σ' has a vertex v' not joined to v by an edge; that is,



$$\mathrm{Star}_v \cap \mathrm{Star}_{v'} = \varnothing$$

The same argument as above shows that γ spends time π on every visit to $\mathrm{Star}_{v'}$. Therefore the total length of γ is at least $2{\cdot}\pi$. $\square$

10.15. Exercise. *Assume that a spherical polyhedral space $\mathcal{P}$ admits a triangulation τ such that all edge lengths of all simplices are at least $\frac{\pi}{2}$. Show that $\mathcal{P}$ is* CAT(1) *if τ is flag.*

10.16. Exercise. *Let P be a convex polyhedron in $\mathbb{E}^3$ with n faces $F_1,\dots,F_n$. Suppose that each face of P has only obtuse or right angles. Let us take 2^n copies of P indexed by an n-bit array. Glue two copies of P along F_i if their arrays differ only in i-th bit. Show that the obtained space is a locally* CAT(0) *topological manifold.*

The space of trees. The following construction is given by Louis Billera, Susan Holmes, and Karen Vogtmann [22].

Let $\mathcal{T}_n$ be the set of all metric trees with n end vertices labeled by $a^1,\dots,a^n$. To describe one tree in $\mathcal{T}_n$ we may fix a topological tree t with end vertices $a^1,\dots,a^n$, and all other vertices of degree 3, and prescribe the lengths of $2{\cdot}n-3$ edges. If the length of an edge vanishes, we assume that this edge degenerates; such a tree can be also described using a different topological tree t'. The subset of $\mathcal{T}_n$ corresponding to the given topological tree t can be identified with the octant

$$\left\{\,(x_1,\dots,x_{2\cdot n-3})\in\mathbb{R}^{2\cdot n-3}\;:\;x_i\geqslant 0\,\right\}.$$

Equip each such subset with the metric induced from $\mathbb{R}^{2\cdot n-3}$ and consider the length metric on $\mathcal{T}_n$ induced by these metrics.

10.17. Exercise. *Show that $\mathcal{T}_n$ with the described metric is* CAT(0).

F Remarks

Let us formulate a test for spaces with lower curvature bound.

10.18. Theorem. *Let $\mathcal{P}$ be a polyhedral space and τ be a triangulation of $\mathcal{P}$. Then $\mathcal{P}$ is* CBB(0) *if and only if the following conditions hold.*

(a) τ is pure; that is, any simplex in τ is a face of some simplex of dimension exactly m.
(b) The link of any simplex of dimension $m-1$ is formed by single point or two points.
(c) The link of any simplex of dimension $\leqslant m-2$ is connected.
(d) Any link of any simplex of dimension $m-2$ has diameter at most π.

The proof relies on 9.12. The condition *(c)* can be reformulated in the following way:

(c)′ Any path γ in $\mathcal{P}$ is a limit of paths that cross only simplexes of dimension m and $m-1$.

Further, modulo the other conditions, the condition *(d)* is equivalent to the following:

(d)′ The link of any simplex of dimension $m-2$ is isometric to a circle of length $\leqslant 2\cdot\pi$ or a closed real interval of length $\leqslant \pi$.

Lecture 11

Exotic aspherical manifolds

This lecture is nearly a copy of [4, Sections 3.6–3.8]; here we describe a set of rules for gluing Euclidean cubes that produce a locally CAT(0) space and use these rules to construct exotic examples of aspherical manifolds.

A Cubical complexes

The definition of a cubical complex mostly repeats the definition of a simplicial complex, with simplices replaced by cubes.

Formally, a *cubical complex* is defined as a subcomplex of the unit cube in the Euclidean space $\mathbb{R}^N$ of large dimension; that is, a collection of faces of the cube such that together with each face it contains all its sub-faces. Each cube face in this collection will be called a *cube* of the cubical complex.

Note that according to this definition, any cubical complex is finite.

The union of all the cubes in a cubical complex $\mathcal{Q}$ will be called its *underlying space*. A homeomorphism from the underlying space of $\mathcal{Q}$ to a topological space $\mathcal{X}$ is called a *cubulation of* $\mathcal{X}$.

The underlying space of a cubical complex $\mathcal{Q}$ will be always considered with the length metric induced from $\mathbb{R}^N$. In particular, with this metric, each cube of $\mathcal{Q}$ is isometric to the unit cube of the corresponding dimension.

It is straightforward to construct a triangulation of the underlying space of $\mathcal{Q}$ such that each simplex is isometric to a Euclidean simplex. In particular, the underlying space of $\mathcal{Q}$ is a Euclidean polyhedral space.

The link of a cube in a cubical complex is defined similarly to the link of a simplex in a simplicial complex. It is a simplicial complex that admits a natural all-right triangulation — each simplex corresponds to an adjusted cube.

Cubical analog of a simplicial complex. Let $\mathcal{S}$ be a finite simplicial complex and $\{v_1, \dots, v_N\}$ be the set of its vertices.

Consider $\mathbb{R}^N$ with the standard basis $\{e_1, \dots, e_N\}$. Denote by $\square^N$ the standard unit cube in $\mathbb{R}^N$; that is,

$$\square^N = \left\{ (x_1, \dots, x_N) \in \mathbb{R}^N \ : \ 0 \leqslant x_i \leqslant 1 \text{ for each } i \right\}.$$

Given a k-dimensional simplex $\langle v_{i_0}, \dots, v_{i_k} \rangle$ in $\mathcal{S}$, mark the $(k+1)$-dimensional faces in $\square^N$ (there are 2^{N-k} of them) which are parallel to the coordinate $(k+1)$-plane spanned by $e_{i_0}, \dots, e_{i_k}$.

Note that the set of all marked faces of $\square^N$ forms a cubical complex; it will be called the cubical analog of $\mathcal{S}$ and will be denoted as $\square_{\mathcal{S}}$.

11.1. Proposition. *Let $\mathcal{S}$ be a finite connected simplicial complex and $\mathcal{Q} = \square_{\mathcal{S}}$ be its cubical analog. Then the underlying space of $\mathcal{Q}$ is connected and the link of any vertex of $\mathcal{Q}$ is isometric to $\mathcal{S}$ equipped with the spherical right-angled metric.*

In particular, if $\mathcal{S}$ is a flag complex, then $\mathcal{Q}$ is a locally CAT(0), *and therefore its universal cover $\tilde{\mathcal{Q}}$ is* CAT(0).

Proof. The first part of the proposition follows from the construction of $\square_{\mathcal{S}}$.

If $\mathcal{S}$ is flag, then by the flag condition (10.14) the link of any cube in $\mathcal{Q}$ is CAT(1). Therefore, by the cone construction (7.11) $\mathcal{Q}$ is locally CAT(0). It remains to apply the globalization theorem (9.6). $\square$

From Proposition 11.1, it follows that the cubical analog of any flag complex is aspherical. The following exercise states that the converse also holds; see [52, 5.4].

11.2. Exercise. *Show that a finite simplicial complex is flag if and only if its cubical analog is aspherical.*

B Construction

By 7.16, any complete length CAT(0) space is contractible. Therefore, by the globalization theorem (9.6), all proper length, locally CAT(0) spaces are aspherical; that is, they have contractible universal covers. This observation can be used to construct examples of aspherical spaces.

Let $\mathcal{X}$ be a proper topological space. Recall that $\mathcal{X}$ is called *simply connected at infinity* if for any compact set $K \subset \mathcal{X}$ there is a bigger compact set $K' \supset K$ such that $\mathcal{X} \setminus K'$ is path-connected and any loop which lies in $\mathcal{X} \setminus K'$ is null-homotopic in $\mathcal{X} \setminus K$.

Recall that path-connected spaces are not empty by definition. Therefore compact spaces are not simply connected at infinity.

The following example was constructed by Michael Davis [51].

11.3. Proposition. *For any $m \geqslant 4$ there is a closed aspherical m-dimensional manifold whose universal cover is not simply connected at infinity.*

In particular, the universal cover of this manifold is not homeomorphic to the m-dimensional Euclidean space.

The proof requires the following lemma.

11.4. Lemma. *Let $\mathcal{S}$ be a finite flag complex, $\mathcal{Q} = \square_{\mathcal{S}}$ be its cubical analog and $\tilde{\mathcal{Q}}$ be the universal cover of $\mathcal{Q}$.*

Assume $\tilde{\mathcal{Q}}$ is simply connected at infinity. Then $\mathcal{S}$ is simply connected.

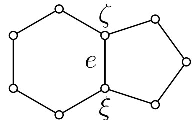
Proof. Assume $\mathcal{S}$ is not simply connected. Equip $\mathcal{S}$ with an all-right spherical metric. Choose a shortest noncontractible circle $\gamma\colon \mathbb{S}^1 \to \mathcal{S}$ formed by the edges of $\mathcal{S}$.

Note that γ forms a one-dimensional subcomplex of $\mathcal{S}$ which is a closed local geodesic. Denote by G the subcomplex of $\mathcal{Q}$ which corresponds to γ.

Fix a vertex $v \in G$; let G_v be the connected component of v in G. Let $\tilde{G}$ be a connected component of the inverse image of G_v in $\tilde{\mathcal{Q}}$ for the universal cover $\tilde{\mathcal{Q}} \to \mathcal{Q}$. Fix a point $\tilde{v} \in \tilde{G}$ in the inverse image of v.

Note that

❶ *$\tilde{G}$ is a convex set in $\tilde{\mathcal{Q}}$.*



Indeed, according to Proposition 11.1, $\tilde{\mathcal{Q}}$ is CAT(0). By Exercise 7.24, it is sufficient to show that $\tilde{G}$ is locally convex in $\tilde{\mathcal{Q}}$, or equivalently, G is locally convex in $\mathcal{Q}$.

Note that the latter can only fail if γ contains two vertices, say ξ and ζ in $\mathcal{S}$, which are joined by an edge not in γ; denote this edge by e.

Each edge of $\mathcal{S}$ has length $\frac{\pi}{2}$. Therefore each of two circles formed by e and an arc of γ from ξ to ζ is shorter than γ. Moreover, at least one of them is noncontractible since γ is noncontractible. That is, γ is not a shortest noncontractible circle, a contradiction. △

Further, note that $\tilde{G}$ is homeomorphic to the plane since $\tilde{G}$ is a two-dimensional manifold without boundary which by the above is CAT(0) and hence is contractible.

Denote by C_R the circle of radius R in $\tilde{G}$ centered at $\tilde{v}$. All C_R are homotopic to each other in $\tilde{G} \setminus \{\tilde{v}\}$ and therefore in $\tilde{\mathcal{Q}} \setminus \{\tilde{v}\}$.

Note that the map $\tilde{\mathcal{Q}} \setminus \{\tilde{v}\} \to \mathcal{S}$ which returns the direction of $[\tilde{v}x]$ for any $x \neq \tilde{v}$, maps C_R to a circle homotopic to γ. Therefore C_R is not contractible in $\tilde{\mathcal{Q}} \setminus \{\tilde{v}\}$.

If R is large, the circle C_R lies outside of any fixed compact set K' in $\tilde{\mathcal{Q}}$. From above C_R is not contractible in $\tilde{\mathcal{Q}} \setminus K$ if $K \supset \tilde{v}$. It follows that $\tilde{\mathcal{Q}}$ is not simply connected at infinity, a contradiction. $\square$

The proof of the following exercise is analogous. It will be used later in the proof of Proposition 11.6 — a more geometric version of Proposition 11.3.

11.5. Exercise. *Under the assumptions of Lemma 11.4, for any vertex v in $\mathcal{S}$ the complement $\mathcal{S} \setminus \{v\}$ is simply connected.*

Proof of 11.3. Let Σ^{m-1} be an $(m-1)$-dimensional smooth homology sphere that is not simply connected, and bounds a contractible smooth compact m-dimensional manifold $\mathcal{W}$.

For $m \geqslant 5$, the existence of such $(\mathcal{W}, \Sigma)$ is proved by Michel Kervaire [83]. For $m = 4$, it follows from construction of Barry Mazur [95].

Pick any triangulation τ of W and let $\mathcal{S}$ be the resulting subcomplex that triangulates Σ.

We can assume that $\mathcal{S}$ is flag; otherwise, pass to the barycentric subdivision of τ and apply Exercise 10.11.

Let $\mathcal{Q} = \square_{\mathcal{S}}$ be the cubical analog of $\mathcal{S}$.

By Proposition 11.1, $\mathcal{Q}$ is a homology manifold. It follows that $\mathcal{Q}$ is a piecewise linear manifold with a finite number of singularities at its vertices.

Removing a small contractible neighborhood V_v of each vertex v in $\mathcal{Q}$, we can obtain a piecewise linear manifold $\mathcal{N}$ whose boundary is formed by several copies of Σ.

Let us glue a copy of $\mathcal{W}$ along its boundary to each copy of Σ in the boundary of $\mathcal{N}$. This results in a closed piecewise linear manifold $\mathcal{M}$ which is homotopically equivalent to $\mathcal{Q}$.

Indeed, since both V_v and $\mathcal{W}$ are contractible, the identity map of their common boundary Σ can be extended to a homotopy equivalence $V_v \to \mathcal{W}$ relative to the boundary. Therefore the identity map on $\mathcal{N}$ extends to homotopy equivalences $f \colon \mathcal{Q} \to \mathcal{M}$ and $g \colon \mathcal{M} \to \mathcal{Q}$.

Finally, by Lemma 11.4, the universal cover $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ is not simply connected at infinity.

The same holds for the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$. The latter follows since the constructed homotopy equivalences $f\colon \mathcal{Q} \to \mathcal{M}$ and $g\colon \mathcal{M} \to \mathcal{Q}$ lift to proper maps $\tilde{f}\colon \tilde{\mathcal{Q}} \to \tilde{\mathcal{M}}$ and $\tilde{g}\colon \tilde{\mathcal{M}} \to \tilde{\mathcal{Q}}$; that is, for any compact sets $A \subset \tilde{\mathcal{Q}}$ and $B \subset \tilde{\mathcal{M}}$, the inverse images $\tilde{g}^{-1}(A)$ and $\tilde{f}^{-1}(B)$ are compact. □

The following proposition was proved by Fredric Ancel, Michael Davis, and Craig Guilbault [10]; it could be considered as a more geometric version of Proposition 11.3.

11.6. Proposition. *Given $m \geqslant 5$, there is a Euclidean polyhedral space $\mathcal{P}$ such that:*

(a) $\mathcal{P}$ is homeomorphic to a closed m-dimensional manifold.
(b) $\mathcal{P}$ is locally CAT(0)*.*
(c) The universal cover of $\mathcal{P}$ is not simply connected at infinity.

Dale Rolfsen [119] has shown that there are no three-dimensional examples of that type. Paul Thurston [128] conjectured that the same holds in the four-dimensional case.

Proof. Apply Exercise 11.5 to the barycentric subdivision of the simplicial complex $\mathcal{S}$ provided by Exercise 11.7. □

11.7. Exercise. *Given an integer $m \geqslant 5$, construct a finite $(m-1)$-dimensional simplicial complex $\mathcal{S}$ such that* $\operatorname{Cone}\mathcal{S}$ *is homeomorphic to $\mathbb{E}^m$ and $\pi_1(\mathcal{S} \setminus \{v\}) \neq 0$ for some vertex v in $\mathcal{S}$.*

C Remarks

As was mentioned earlier, the motivation for the notion of CAT(κ) spaces comes from the fact that a Riemannian manifold is locally CAT(κ) if and only if it has sectional curvature at most κ. This easily follows from Rauch comparison for Jacobi fields and Proposition 7.19.

In the globalization theorem (9.6), properness can be weakened to completeness [see 5, and the references therein].

The condition on polyhedral CAT(κ) spaces given in Theorem 10.7 might look easy to use, but in fact, it is hard to check even in very simple cases. For example, the description of those coverings of $\mathbb{S}^3$ branching at three great circles which are CAT(1) requires quite a bit of work [46] — try to guess the answer before reading.

Another example is the space $\mathcal{B}_n$ that is the universal cover of $\mathbb{C}^n$ infinitely branching in complex hyperplanes $z_i = z_j$ with the induced length metric. So far it is not known if $\mathcal{B}_n$ is CAT(0) for any $n \geqslant 4$ [105]. Understanding this space could be helpful for studying the braid group. This circle of questions is closely related to the generalization of the flag condition (10.14) to spherical simplices with few acute dihedral angles.

The construction used in the proof of Proposition 11.3 admits a number of interesting modifications, several of which are discussed in a survey by Michael Davis [52].

A similar argument was used by Michael Davis, Tadeusz Januszkiewicz, and Jean-François Lafont [54]. They constructed a closed smooth four-dimensional manifold M with universal cover $\tilde{M}$ diffeomorphic to $\mathbb{R}^4$, such that M admits a polyhedral metric which is locally CAT(0), but does not admit a Riemannian metric with nonpositive sectional curvature. Another example of that type was constructed by Stephan Stadler [125]. There are no lower-dimensional examples of this type — the two-dimensional case follows from the classification of surfaces, and the three-dimensional case follows from the geometrization conjecture.

It is noteworthy that any complete, simply connected Riemannian manifold with nonpositive curvature is homeomorphic to the Euclidean space of the same dimension. In fact, by the globalization theorem (9.6), the exponential map at a point of such a manifold is a homeomorphism. In particular, there is no Riemannian analog of Proposition 11.6.

Recall that a triangulation of an m-dimensional manifold defines a piecewise linear structure if the link of every simplex Δ is homeomorphic to the sphere of dimension $m - 1 - \dim\Delta$. According to Stone's theorem [53, 126], the triangulation of $\mathcal{P}$ in Proposition 11.6 cannot be made piecewise linear — despite the fact that $\mathcal{P}$ is a manifold, its triangulation does not induce a piecewise linear structure.

The flag condition also leads to the so-called hyperbolization procedure, a flexible tool for constructing aspherical spaces; a good survey on the subject is given by Ruth Charney and Michael Davis [47].

The CAT(0) property of a cube complex admits interesting (and useful) geometric descriptions if one exchanged the ℓ^2-metric to a natural ℓ^1 or ℓ^∞ on each cube.

11.8. Theorem. *The following three conditions are equivalent.*

(a) A cube complex Q equipped with ℓ^2-metric is CAT(0).

(b) A cube complex Q equipped with ℓ^∞-metric is injective.

(c) A cube complex Q equipped with ℓ^1-metric is median. The latter means that for any three points x, y, z there is a unique point m (it is called the median of x, y, and z) that lies on some geodesics $[xy]$, $[xz]$ and $[yz]$.

A very readable paper on the subject was written by Brian Bowditch [30]; two easy parts of the theorem are included in the following exercise.

11.9. Exercise. *Prove the implication (b)$\Rightarrow$(a) and/or (c)$\Rightarrow$(a) in the theorem.*

All the topics discussed in this lecture link Alexandrov geometry with the fundamental group. The theory of hyperbolic groups, a branch of geometric group theory, introduced by Mikhael Gromov [63], could be considered as a further step in this direction.

A striking result that connects this topic with injective envelopes obtained by Urs Lang [88]. In particular, he proved that injective envelope of word hyperbolic groups is finite dimensional.

Lecture 12

Subsets

This lecture is nearly a copy of [4, Chapter 4]; here we give a partial answer to the following question: *Which subsets of Euclidean space, equipped with their induced length-metrics, are* CAT(0)*?*

A Motivating examples

Consider three subgraphs of different quadric surfaces:

$$\begin{aligned} A &= \{\, (x,y,z) \in \mathbb{E}^3 \;:\; z \leqslant x^2 + y^2 \,\}, \\ B &= \{\, (x,y,z) \in \mathbb{E}^3 \;:\; z \leqslant -x^2 - y^2 \,\}, \\ C &= \{\, (x,y,z) \in \mathbb{E}^3 \;:\; z \leqslant x^2 - y^2 \,\}. \end{aligned}$$

12.1. Question. *Which of the sets A, B and C, if equipped with the induced length metric, are* CAT(0) *and why?*

The answers are given below, but it is instructive to think about these questions before reading further.

A. No, A is not CAT(0).

The boundary ∂A is the paraboloid described by $z = x^2 + y^2$; in particular it bounds an open convex set in $\mathbb{E}^3$ whose complement is A. The closest point projection of $A \to \partial A$ is short (Exercise 7.23). It follows that ∂A is a convex set in A equipped with its induced length metric.

Therefore if A is CAT(0), then so is ∂A. The latter is not true: ∂A is a smooth convex surface, and has strictly positive curvature by the Gauss formula.

B. Yes, B is CAT(0).

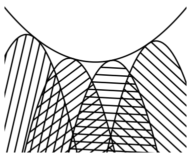
Evidently B is a convex closed set in $\mathbb{E}^3$. Therefore the length metric on B coincides with the Euclidean metric and CAT(0) comparison holds.

C. Yes, C is CAT(0), but the proof is not as easy as before. We give a sketch here; a complete proof of a more general statement is given in Section 12C.

Set $f_t(x,y) = x^2 - y^2 - 2{\cdot}(x-t)^2$. Consider the one-parameter family of sets

$$V_t = \left\{ (x,y,z) \in \mathbb{E}^3 \ :\ z \leqslant f_t(x,y) \right\}.$$

Each set V_t is a solid paraboloid tangent to ∂C along the parabola $y \mapsto (t, y, t^2 - y^2)$. The set V_t is closed and convex for any t, and



$$C = \bigcup_t V_t.$$

Further note that the function $t \mapsto f_t(x,y)$ is concave for any fixed x,y. Therefore

❶ *if $a < b < c$, then $V_b \supset V_a \cap V_c$.*

Consider the finite union

$$C' = V_{t_1} \cup \dots \cup V_{t_n}.$$

The inclusion ❶ makes it possible to apply Reshetnyak gluing theorem 8.3 recursively and show that C' is CAT(0).

By approximation, the CAT(0) comparison holds for any 4 points in C; hence C is CAT(0). More precisely, choose $x_1, x_2, x_3, x_4 \in C$ and 6 geodesics $[x_i x_j]_C$ between them. Choose $\varepsilon > 0$, shift each $[x_i x_j]_C$ down by ε, and reconnect it to x_i and x_j by vertical ε-segments. Denote the obtained curve by $\gamma_{i,j}$; note that

$$\operatorname{length} \gamma_{i,j} = |x_i - x_j|_C + 2{\cdot}\varepsilon.$$

We may assume that C' contains all $\gamma_{i,j}$. It follows that

$$|x_i - x_j|_C \leqslant |x_i - x_j|_{C'} \leqslant |x_i - x_j|_C + 2{\cdot}\varepsilon$$

Since and C' is CAT(0), $\varepsilon > 0$ is arbitrary, so is C.

Remark. The set C is not convex, but it is *two-convex* as defined in the next section. As you will see, two-convexity is closely related to the inheritance of an upper curvature bound by a subset.

B Two-convexity

12.2. Definition. *We say that a subset $K \subset \mathbb{E}^m$ is* two-convex *if the following condition holds for any plane $W \subset \mathbb{E}^m$: If γ is a simple closed curve in $W \cap K$ that is null-homotopic in K, then it is null-homotopic in $W \cap K$, and in particular the disc in W bounded by γ lies in K.*

Note that two-convex sets do not have to be connected or simply connected. The following two propositions follow immediately from the definition.

12.3. Proposition. *Any subset in $\mathbb{E}^2$ is two-convex.*

12.4. Proposition. *The intersection of an arbitrary collection of two-convex sets in $\mathbb{E}^m$ is two-convex.*

12.5. Proposition. *Show that the interior of any two-convex set in $\mathbb{E}^m$ is a two-convex set.*

Proof. Fix a two-convex set $K \subset \mathbb{E}^m$ and a 2-plane W; denote by $\operatorname{Int} K$ the interior of K. Let γ be a closed simple curve in $W \cap \operatorname{Int} K$ that is contractible in the interior of K.

Since K is two-convex, the plane disc D bounded by γ lies in K. The same holds for the translations of D by small vectors. Therefore D lies in $\operatorname{Int} K$; that is, $\operatorname{Int} K$ is two-convex. □

12.6. Definition. *Given a subset $K \subset \mathbb{E}^m$, define its two-convex hull (briefly, $\operatorname{Conv}_2 K$) as the intersection of all two-convex subsets containing K.*

Note that by Proposition 12.4, the two-convex hull of any set is two-convex. Further, by 12.5, the two-convex hull of an open set is open.

The next proposition describes closed two-convex sets with smooth boundary.

12.7. Proposition. *Let $K \subset \mathbb{E}^m$ be a closed subset.*

Assume that the boundary of K is a smooth hypersurface S. Consider the unit normal vector field ν on S that points outside of K. Denote by $k_1 \leqslant \ldots \leqslant k_{m-1}$ the principal curvature functions of S with respect to ν (note that if K is convex, then $k_1 \geqslant 0$).

Then K is two-convex if and only if $k_2(p) \geqslant 0$ for any point $p \in S$. Moreover, if $k_2(p) < 0$ at some point p, then Definition 12.2 fails for some curve γ forming a triangle in an arbitrary small neighborhood of p.

The following proof was given by Mikhael Gromov [64, §$\frac{1}{2}$], but we added a few details. The proof uses a straightforward modification of the Morse theory for manifolds with boundary; the paper of Sergei Vakhrameev [134] contains all the necessary lemmas.

Proof; only-if part. If $k_2(p) < 0$ for some $p \in S$, consider the plane W containing p and spanned by the first two principal directions at p. Choose a small triangle in W which surrounds p and move it slightly in the direction of $\nu(p)$. We get a triangle $[xyz]$ which is null-homotopic in K, but the solid triangle $\Delta = \mathrm{Conv}\{x, y, x\}$ bounded by $[xyz]$ does not lie in K completely. Therefore K is not two-convex. (See figure in the "only if" part of the smooth two-convexity theorem (12.10).)

If part. Recall that a smooth function $f\colon \mathbb{E}^m \to \mathbb{R}$ is called strongly convex if its Hessian is positive definite at each point.

Suppose $f\colon \mathbb{E}^m \to \mathbb{R}$ is a smooth strongly convex function such that the restriction $f|_S$ is a Morse function. Note that a generic smooth strongly convex function $f\colon \mathbb{E}^m \to \mathbb{R}$ has this property.

For a critical point p of $f|_S$, the outer normal vector $\nu(p)$ is parallel to the gradient $\nabla_p f$; we say that p is a positive critical point if $\nu(p)$ and $\nabla_p f$ point in the same direction, and negative otherwise. If f is generic, then we can assume that the sign is defined for all critical points; that is, $\nabla_p f \neq 0$ for any critical point p of $f|_S$.

Since $k_2 \geqslant 0$ and the function f is strongly convex, the negative critical points of $f|_S$ have index at most 1.

Given a real value s, set

$$K_s = \{\, x \in K \;:\; f(x) < s \,\}.$$

Assume $\varphi_0\colon \mathbb{D} \to K$ is a continuous map of the disc $\mathbb{D}$ such that $\varphi_0(\partial\mathbb{D}) \subset K_s$.

Note that by the Morse lemma, there is a homotopy $\varphi_t\colon \mathbb{D} \to K$ rel $\partial\mathbb{D}$ such that $\varphi_1(\mathbb{D}) \subset K_s$.

Indeed, we can construct a homotopy $\varphi_t\colon \mathbb{D} \to K$ that decreases the maximum of $f \circ \varphi$ on $\mathbb{D}$ until the maximum occurs at a critical point p of $f|_S$. This point cannot be negative; otherwise, its index would be at least 2. If this critical point is positive, then it is easy to decrease the maximum a little by pushing the disc from S into K in the direction of $-\nabla f_p$.

Consider a closed curve $\gamma\colon \mathbb{S}^1 \to K$ that is null-homotopic in K. Note that the distance function

$$f_0(x) = |\mathrm{Conv}\,\gamma - x|_{\mathbb{E}^m}$$

is convex. Therefore f_0 can be approximated by smooth strongly convex functions f in general position. From above, there is a disc in

K with boundary γ that lies arbitrarily close to $\operatorname{Conv}\gamma$. Since K is closed, the statement follows. $\square$

Note that the "if" part proves a somewhat stronger statement. Namely, any plane curve γ (not necessary simple) which is contractible in K is also contractible in the intersection of K with the plane of γ. The latter condition does not hold for the complement of two planes in $\mathbb{E}^4$, which is two-convex by Proposition 12.4; see also Exercise 12.18 below. The following proposition shows that there are no such examples in $\mathbb{E}^3$.

12.8. Proposition. *Let $\Omega \subset \mathbb{E}^3$ be an open two-convex subset. Then for any plane $W \subset \mathbb{E}^3$, any closed curve in $W \cap \Omega$ that is null-homotopic in Ω is also null-homotopic in $W \cap \Omega$.*

This statement is intuitively obvious, but the proof is not trivial; it use the following classical result. An alternative definition of two-convexity using homology instead of homotopy is mentioned in the last section. For this definition the proof is simpler.

12.9. Loop theorem. *Let M be a three-dimensional manifold with nonempty boundary ∂M. Assume $f\colon (\mathbb{D}, \partial\mathbb{D}) \to (M, \partial M)$ is a continuous map from the disc $\mathbb{D}$ such that the boundary curve $f|_{\partial\mathbb{D}}$ is not null-homotopic in ∂M. Then there is an embedding $h\colon (\mathbb{D}, \partial\mathbb{D}) \to (M, \partial M)$ with the same property.*

The theorem is due to Christos Papakyriakopoulos [a proof can be found in 70].

Proof of 12.8. Fix a closed plane curve γ in $W \cap \Omega$ that is null-homotopic in Ω. Suppose γ is not contractible in $W \cap \Omega$.

Let $\varphi\colon \mathbb{D} \to \Omega$ be a map of the disc with the boundary curve γ.

Since Ω is open we can first change φ slightly so that $\varphi(x) \notin W$ for $1 - \varepsilon < |x| < 1$ for some small $\varepsilon > 0$. By further changing φ slightly we can assume that it is transversal to W on $\operatorname{Int}\mathbb{D}$ and agrees with the previous map near $\partial\mathbb{D}$.

This means that $\varphi^{-1}(W) \cap \operatorname{Int}\mathbb{D}$ consists of finitely many simple closed curves which cut $\mathbb{D}$ into several components. Consider one of the "innermost" components c'; that is, c' is a boundary curve of a disc $\mathbb{D}' \subset \mathbb{D}$, $\varphi(c')$ is a closed curve in W and $\varphi(\mathbb{D}')$ completely lies in one of the two half-spaces with boundary W. Denote this half-space by H.

If $\varphi(c')$ is not contractible in $W \cap \Omega$, then applying the loop theorem to $M^3 = H \cap \Omega$ we conclude that there exists a simple closed curve $\gamma' \subset \Omega \cap W$ which is not contractible in $\Omega \cap W$ but is contractible in $\Omega \cap H$. This contradicts two-convexity of Ω.

Hence $\varphi(c')$ is contractible in $W \cap \Omega$. Therefore φ can be changed in a small neighborhood U of $\mathbb{D}'$ so that the new map $\hat{\varphi}$ maps U to one side of W. In particular, the set $\hat{\varphi}^{-1}(W)$ consists of the same curves as $\varphi^{-1}(W)$ with the exception of c'.

Repeating this process several times we reduce the problem to the case where $\varphi^{-1}(W) \cap \operatorname{Int} \mathbb{D} = \varnothing$. This means that $\varphi(\mathbb{D})$ lies entirely in one of the half-spaces bounded by W.

Again applying the loop theorem, we obtain a simple closed curve in $W \cap \Omega$ which is not contractible in $W \cap \Omega$ but is contractible in Ω. This again contradicts two-convexity of Ω. Hence γ is contractible in $W \cap \Omega$ as claimed. $\square$

C Sets with smooth boundary

In this section, we characterize the subsets with smooth boundary in $\mathbb{E}^m$ that form $\mathrm{CAT}(0)$ spaces.

12.10. Smooth two-convexity theorem. *Let K be a closed, simply connected subset in $\mathbb{E}^m$ equipped with the induced length metric. Assume K is bounded by a smooth hypersurface. Then K is* $\mathrm{CAT}(0)$ *if and only if K is two-convex.*

This theorem is a baby case of a result of Stephanie Alexander, David Berg, and Richard Bishop [1], which is briefly discussed at the end of the lecture. The proof below is based on the argument in Section 12A.

Proof. Denote by S and by Ω the boundary and the interior of K respectively. Since K is connected and S is smooth, Ω is also connected.

Denote by $k_1(p) \leqslant \ldots \leqslant k_{m-1}(p)$ the principal curvatures of S at $p \in S$ with respect to the normal vector $\nu(p)$ pointing out of K. By Proposition 12.7, K is two-convex if and only if $k_2(p) \geqslant 0$ for any $p \in S$.

Only-if part. Assume K is not two-convex. Then by Proposition 12.7, there is a triangle $[xyz]$ in K which is null-homotopic in K, but the solid triangle $\Delta = \operatorname{Conv}\{x, y, z\}$ does not lie in K completely. Evidently the triangle $[xyz]$ is not thin in K. Hence K is not $\mathrm{CAT}(0)$.

If part. Since K is simply connected, by the globalization theorem (9.6) it suffices to show that any point $p \in K$ admits a $\mathrm{CAT}(0)$ neighborhood.

If $p \in \operatorname{Int} K$, then it admits a neighborhood isometric to a $\mathrm{CAT}(0)$ subset of $\mathbb{E}^m$. Fix $p \in S$. Assume that $k_2(p) > 0$. Fix a sufficiently small $\varepsilon > 0$ and set $K' = K \cap \overline{\mathrm{B}}[p, \varepsilon]$. Let us show that

❶ *K' is* CAT(0)*.*

Consider the coordinate system with the origin at p and the principal directions and $\nu(p)$ as the coordinate directions. For small $\varepsilon > 0$, the set K' can be described as a subgraph

$$K' = \left\{ (x_1, \dots, x_m) \in \overline{\mathrm{B}}[p, \varepsilon] \,:\, x_m \leqslant f(x_1, \dots, x_{m-1} \right\}.$$

Fix $s \in [-\varepsilon, \varepsilon]$. Since ε is small and $k_2(p) > 0$, the restriction $f|_{x_1=s}$ is concave in the $(m-2)$-dimensional cube defined by the inequalities $|x_i| < 2{\cdot}\varepsilon$ for $2 \leqslant i \leqslant m-1$.

Fix a negative real value $\lambda < k_1(p)$. Given $s \in (-\varepsilon, \varepsilon)$, consider the set

$$V_s = \left\{ (x_1, \dots, x_m) \in K' \,:\, x_m \leqslant f(x_1, \dots, x_{m-1}) + \lambda{\cdot}(x_1 - s)^2 \right\}.$$

Note that the function

$$(x_1, \dots, x_{m-1}) \mapsto f(x_1, \dots, x_{m-1}) + \lambda{\cdot}(x_1 - s)^2$$

is concave near the origin. Since ε is small, we can assume that the V_s are convex subsets of $\mathbb{E}^m$.

Further note that

$$K' = \bigcup_{s \in [-\varepsilon, \varepsilon]} V_s.$$

Also, the same argument as in 12.1 shows that

❷ *If $a < b < c$, then $V_b \supset V_a \cap V_c$.*

Given an array of values $s^1 < \dots < s^k$ in $[-\varepsilon, \varepsilon]$, set $V^i = V_{s^i}$ and consider the unions

$$W^i = V^1 \cup \dots \cup V^i$$

equipped with the induced length metric.

Note that the array (s^n) can be chosen in such a way that W^k is arbitrarily close to K' in the sense of Hausdorff.

By Proposition 7.8, in order to prove ❶, it is sufficient to show the following:

❸ *All W^i are* CAT(0)*.*

This claim is proved by induction. Base: $W^1 = V^1$ is CAT(0) as a convex subset in $\mathbb{E}^m$.

Step: Assume that W^i is CAT(0). According to ❷,

$$V^{i+1} \cap W^i = V^{i+1} \cap V^i.$$

Moreover, this is a convex set in $\mathbb{E}^m$ and therefore it is a convex set in W^i and in V^{i+1}. By the Reshetnyak gluing theorem, W^{i+1} is CAT(0). Hence the claim follows. $\triangle$

Note that we have proved the following:

❹ *K' is* CAT(0) *if K is strongly two-convex, that is, $k_2(p) > 0$ at any point $p \in S$.*

It remains to show that p admits a CAT(0) neighborhood in the case $k_2(p) = 0$.

Choose a coordinate system $(x_1, \dots, x_m)$ as above, so that the $(x_1, \dots, x_{m-1})$-coordinate hyperplane is the tangent subspace to S at p.

Fix $\varepsilon > 0$ so that a neighborhood of p in S is the graph

$$x_m = f(x_1, \dots, x_{m-1})$$

of a function f defined on the open ball B of radius ε centered at the origin in the $(x_1, \dots, x_{m-1})$-hyperplane. Fix a smooth positive strongly convex function $\varphi\colon B \to \mathbb{R}_+$ such that $\varphi(x) \to \infty$ as x approaches the boundary of B. Note that for $\delta > 0$, the subgraph K_δ defined by the inequality

$$x_m \leqslant f(x_1, \dots, x_{m-1}) - \delta{\cdot}\varphi(x_1, \dots, x_{m-1})$$

is strongly two-convex. By ❹, K_δ is CAT(0).

Finally as $\delta \to 0$, the closed ε-neighborhoods of p in K_δ converge to the closed ε-neighborhood of p in K. By Proposition 7.8, the ε-neighborhood of p is CAT(0). $\square$

D Open plane sets

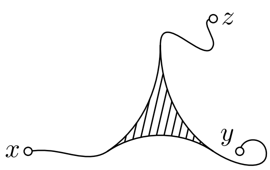
In this section, we consider inheritance of upper curvature bounds by subsets of the Euclidean plane.

12.11. Theorem. *Let Ω be an open simply connected subset of $\mathbb{E}^2$. Equip Ω with its induced length metric and denote its completion by K. Then K is* CAT(0).

The assumption that the set Ω is open is not critical; instead one can assume that the induced length metric takes finite values at all points of Ω. We sketch the proof given by Richard Bishop [26] and leave the details to be finished as an exercise. A generalization of this result is proved by Alexander Lytchak and Stefan Wenger [93, Proposition 12.1]; this paper also contains a far-reaching application.

Sketch of proof. It is sufficient to show that any triangle in K is thin, as defined in 7.18.

Note that K admits a length-preserving map to $\mathbb{E}^2$ that extends the embedding $\Omega \hookrightarrow \mathbb{E}^2$. Therefore each triangle $[xyz]$ in K can be mapped to the plane in a length-preserving way. Since Ω is simply connected, any open region, say Δ, that is surrounded by the image of $[xyz]$ lies completely in Ω.



Note that in each triangle $[xyz]$ in K, the sides $[xy]$, $[yz]$ and $[zx]$ intersect each other along a geodesic starting at a common vertex, possibly a one-point geodesic. In other words, every triangle in K looks like the one in the diagram.

Indeed, assuming the contrary, there will be a l u n e in K bounded by two minimizing geodesics with common ends but no other common points. The image of this lune in the plane must have concave sides, since otherwise one could shorten the sides by pushing them into the interior. Evidently, there is no plane lune with concave sides, a contradiction.

Note that it is sufficient to consider only simple triangles $[xyz]$, that is, triangles whose sides $[xy]$, $[yz]$ and $[zx]$ intersect each other only at the common vertices. If this is not the case, chopping the overlapping part of sides reduces to the injective case (this is formally stated in Exercise 12.12).

Again, the open region, say Δ, bounded by the image of $[xyz]$ has concave sides in the plane, since otherwise one could shorten the sides by pushing them into Ω. It remains to solve Exercise 12.13. $\square$

12.12. Exercise. *Assume that $[pq]$ is a common part of the two sides $[px]$ and $[py]$ of the triangle $[pxy]$. Consider the triangle $[qxy]$ whose sides are formed by arcs of the sides of $[pxy]$. Show that if $[qxy]$ is thin, then so is $[pxy]$.*

12.13. Exercise. *Assume S is a closed plane region whose boundary is a plane triangle T with concave sides. Equip S with the induced length metric. Show that the triangle T is thin in S.*

Here is a spherical analog of Theorem 12.11, which can be proved along the same lines. It will be used in the next section.

12.14. Proposition. *Let Θ be an open connected subset of the unit sphere $\mathbb{S}^2$ that does not contain a closed hemisphere. Equip Θ with the induced length metric. Let $\tilde{\Theta}$ be a metric cover of Θ such that any closed curve in $\tilde{\Theta}$ shorter than $2{\cdot}\pi$ is contractible.*

Show that the completion of $\tilde{\Theta}$ *is* CAT(1).

12.15. Exercise. *Prove the following partial case of the proposition:*
Let K *be closed subset of the unit sphere* $\mathbb{S}^2$ *that does not contain a closed hemisphere. Suppose* K *is simply connected and bounded by a simple Lipschitz curve. Show that* K *with induced length metric is* CAT(1).

E Shefel's theorem

In this section, we will formulate our version of a theorem of Samuel Shefel (12.17) and prove a couple of its corollaries.

It seems that Shefel was very intrigued by the survival of metric properties under affine transformation. To describe an instance of such phenomena, note that two-convexity survives under affine transformations of a Euclidean space. Therefore, as a consequence of the smooth two-convexity theorem (12.10), the following holds.

12.16. Corollary. *Let* K *be closed connected subset of Euclidean space equipped with the induced length metric. Assume* K *is* CAT(0) *and bounded by a smooth hypersurface. Then any affine transformation of* K *is also* CAT(0).

By Corollary 12.19, an analogous statement holds for sets bounded by Lipschitz surfaces in the three-dimensional Euclidean space. In higher dimensions this is no longer true.

12.17. Two-convexity theorem. *Let* Ω *be a connected open set in* $\mathbb{E}^3$. *Equip* Ω *with the induced length metric and denote by* $\tilde{K}$ *the completion of the universal metric cover of* Ω. *Then* $\tilde{K}$ *is* CAT(0) *if and only if* Ω *is two-convex.*

The proof of this statement will be given in the following three sections. First we prove its polyhedral analog, then we prove some properties of two-convex hulls in three-dimensional Euclidean space and only then do we prove the general statement.

The following exercise shows that the analogous statement does not hold in higher dimensions.

12.18. Exercise. *Let* Π_1, Π_2 *be two planes in* $\mathbb{E}^4$ *intersecting at a single point. Let* $\tilde{K}$ *be the completion of the universal metric cover of* $\mathbb{E}^4 \setminus (\Pi_1 \cup \Pi_2)$.
Show that $\tilde{K}$ *is* CAT(0) *if and only if* $\Pi_1 \perp \Pi_2$.

Before coming to the proof of the two-convexity theorem, let us formulate a few corollaries. The following corollary is a generalization of the smooth two-convexity theorem (12.10) for three-dimensional Euclidean space.

12.19. Corollary. *Let K be a closed subset in $\mathbb{E}^3$ bounded by a Lipschitz hypersurface. Then K with the induced length metric is* CAT(0) *if and only if the interior of K is two-convex and simply connected.*

Proof. Set $\Omega = \operatorname{Int} K$. Since K is simply connected and bounded by a surface, Ω is also simply connected.

Apply the two-convexity theorem to Ω. Note that the completion of Ω equipped with the induced length metric is isometric to K with the induced length metric. Hence the result. □

Note that the Lipschitz condition is used just once to show that the completion of Ω is isometric to K with the induced length metric. This property holds for a wider class of hypersurfaces; for instance Alexander horned ball might have CAT(0) induced length metric.

Let U be an open set in $\mathbb{R}^2$. A continuous function $f\colon U \to \mathbb{R}$ is called saddle if for any linear function $\ell\colon \mathbb{R}^2 \to \mathbb{R}$, the difference $f - \ell$ does not have local maxima or local minima in U. Equivalently, the open subgraph and epigraph of f

$$\left\{\, (x,y,z) \in \mathbb{E}^3 \;:\; z < f(x,y),\ (x,y) \in U \,\right\},$$
$$\left\{\, (x,y,z) \in \mathbb{E}^3 \;:\; z > f(x,y),\ (x,y) \in U \,\right\}$$

are two-convex.

12.20. Theorem. *Let $f\colon \mathbb{D} \to \mathbb{R}$ be a Lipschitz function which is saddle in the interior of the closed unit disc $\mathbb{D}$. Then the graph*

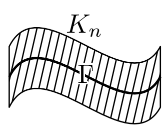
$$\Gamma = \left\{\, (x,y,z) \in \mathbb{E}^3 \;:\; z = f(x,y) \,\right\},$$

equipped with induced length metric is CAT(0).

Proof. Since the function f is Lipschitz, its graph Γ with the induced length metric is bi-Lipschitz equivalent to $\mathbb{D}$ with the Euclidean metric.

Consider the sequence of sets

$$K_n = \left\{\, (x,y,z) \in \mathbb{E}^3 \;:\; z \lesseqgtr f(x,y) \pm \tfrac{1}{n},\ (x,y) \in \mathbb{D} \,\right\}.$$



Note that each K_n is closed and simply connected.
By definition K is also two-convex. Moreover the boundary of K_n is a Lipschitz surface.

Equip K_n with the induced length metric. By Corollary 12.19, K_n is CAT(0). It remains to note that $K_n \to \Gamma$ in the sense of Gromov–Hausdorff, and apply Proposition 7.8. □

F Polyhedral case

Now we are back to the proof of the two-convexity theorem (12.17).

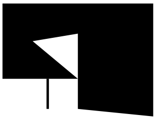
Recall that a subset P of $\mathbb{E}^m$ is called a polytope if it can be presented as a union of a finite number of simplices. Similarly, a spherical polytope is a union of a finite number of simplices in $\mathbb{S}^m$.

Note that any polytope admits a finite triangulation. Therefore any polytope equipped with the induced length metric forms a Euclidean polyhedral space as defined in 10.6.

12.21. Lemma. *The two-convexity theorem (12.17) holds if the set Ω is the interior of a polytope.*

The statement might look obvious, but there is a hidden obstacle in the proof that is related to the following. Let P be a polytope and Ω its interior, both considered with the induced length metrics. Typically, the completion K of Ω is isometric to P — in this case, the lemma follows easily from 10.7.

However in general we only have a locally distance-preserving map $K \to P$; it does not have to be onto and it may not be injective. An example can be guessed from the picture. Nevertheless, is easy to see that K is always a polyhedral space.



The proof uses the following two exercises.

12.22. Exercise. *Show that any closed path of length $< 2{\cdot}\pi$ in the units sphere $\mathbb{S}^2$ lies in an open hemisphere.*

12.23. Exercise. *Assume Ω is an open subset in $\mathbb{E}^3$ that is not two-convex. Show that there is a plane W such that the complement $W \setminus \Omega$ contains an isolated point and a small circle around this point in W is contractible in Ω.*

Proof of 12.21. The "only if" part can be proved in the same way as in the smooth two-convexity theorem (12.10) with additional use of Exercise 12.23.

If part. Assume that Ω is two-convex. Denote by $\tilde\Omega$ the universal metric cover of Ω. Let $\tilde K$ and K be the corresponding completions of $\tilde\Omega$ and Ω.

The main step is to show that $\tilde K$ is CAT(0).

Note that K is a polyhedral space and the covering $\tilde\Omega \to \Omega$ extends to a covering map $\tilde K \to K$ which might be branching at some vertices.[1]

[1]For example, if $K = \{\, (x,y,z) \in \mathbb{E}^3 \;:\; |z| \leqslant |x| + |y| \leqslant 1 \,\}$ and p is the origin, then Σ_p, the space of directions at p, is not simply connected and $\tilde K \to K$ branches at p.

Fix a point $\tilde{p} \in \tilde{K} \setminus \tilde{\Omega}$; denote by p the image of $\tilde{p}$ in K. Note that $\tilde{K}$ is a ramified cover of K and hence is locally contractible. Thus, any loop in $\tilde{K}$ is homotopic to a loop in $\tilde{\Omega}$ which is simply connected. Therefore $\tilde{K}$ is simply connected too.

Thus, by the globalization theorem (9.6), it is sufficient to show that

❶ *a small neighborhood of $\tilde{p}$ in $\tilde{K}$ is* CAT(0).

Recall that $\Sigma_{\tilde{p}} = \Sigma_{\tilde{p}}\tilde{K}$ denotes the space of directions at $\tilde{p}$. Note that a small neighborhood of $\tilde{p}$ in $\tilde{K}$ is isometric to an open set in the cone over $\Sigma_{\tilde{p}}\tilde{K}$. By Exercise 7.11, ❶ follows once we can show that

❷ $\Sigma_{\tilde{p}}$ *is* CAT(1).

By rescaling, we can assume that every face of K which does not contain p lies at distance at least 2 from p. Denote by $\mathbb{S}^2$ the unit sphere centered at p, and set $\Theta = \mathbb{S}^2 \cap \Omega$. Note that $\Sigma_p K$ is isometric to the completion of Θ and $\Sigma_{\tilde{p}}\tilde{K}$ is the completion of the regular metric covering $\tilde{\Theta}$ of Θ induced by the universal metric cover $\tilde{\Omega} \to \Omega$.

By 12.14, it remains to show the following:

❸ *Any closed curve in $\tilde{\Theta}$ shorter than $2{\cdot}\pi$ is contractible.*

Fix a closed curve $\tilde{\gamma}$ of length $< 2{\cdot}\pi$ in $\tilde{\Theta}$. Its projection γ in $\Theta \subset \mathbb{S}^2$ has the same length. Therefore, by Exercise 12.22, γ lies in an open hemisphere. Then for a plane Π passing close to p, the central projection γ' of γ to Π is defined and lies in Ω. By construction of $\tilde{\Theta}$, the curve γ and therefore γ' are contractible in Ω. From two-convexity of Ω and Proposition 12.8, the curve γ' is contractible in $\Pi \cap \Omega$.

It follows that γ is contractible in Θ and therefore $\tilde{\gamma}$ is contractible in $\tilde{\Theta}$. □

G Two-convex hulls

The following proposition describes a construction which produces the two-convex hull $\mathrm{Conv}_2\,\Omega$ of an open set $\Omega \subset \mathbb{E}^3$. This construction is very close to the one given by Samuel Shefel [122].

12.24. Proposition. *Let $\Pi_1, \Pi_2 \dots$ be an everywhere dense sequence of planes in $\mathbb{E}^3$. Given an open set Ω, consider the recursively defined sequence of open sets $\Omega = \Omega_0 \subset \Omega_1 \subset \dots$ such that Ω_n is the union of Ω_{n-1} and all the bounded components of $\mathbb{E}^3 \setminus (\Pi_n \cup \Omega_{n-1})$. Then*

$$\mathrm{Conv}_2\,\Omega = \bigcup_n \Omega_n.$$

Proof. Set

❶
$$\Omega' = \bigcup_n \Omega_n.$$

Note that Ω' is a union of open sets; in particular, Ω' is open.

Let us show that

❷
$$\mathrm{Conv}_2\,\Omega \supset \Omega'.$$

Suppose we already know that $\mathrm{Conv}_2\,\Omega \supset \Omega_{n-1}$. Fix a bounded component $\mathfrak{C}$ of $\mathbb{E}^3\setminus(\Pi_n \cup \Omega_{n-1})$. It is sufficient to show that $\mathfrak{C} \subset \mathrm{Conv}_2\,\Omega$.

By 12.5, $\mathrm{Conv}_2\,\Omega$ is open. Therefore, if $\mathfrak{C} \not\subset \mathrm{Conv}_2\,\Omega$, then there is a point $p \in \mathfrak{C} \setminus \mathrm{Conv}_2\,\Omega$ lying at maximal distance from Π_n. Denote by W_p the plane containing p which is parallel to Π_n.

Note that p lies in a bounded component of $W_p \setminus \mathrm{Conv}_2\,\Omega$. In particular p can be surrounded by a simple closed curve γ in $W_p \cap \cap \mathrm{Conv}_2\,\Omega$. Since p lies at maximal distance from Π_n, the curve γ is null-homotopic in $\mathrm{Conv}_2\,\Omega$. Therefore $p \in \mathrm{Conv}_2\,\Omega$, a contradiction.

By induction, $\mathrm{Conv}_2\,\Omega \supset \Omega_n$ for each n. Therefore ❶ implies ❷.

It remains to show that Ω' is two-convex. Assume the contrary; that is, there is a plane Π and a simple closed curve $\gamma\colon \mathbb{S}^1 \to \Pi \cap \Omega'$ which is null-homotopic in Ω', but not null-homotopic in $\Pi \cap \Omega'$.

By approximation we can assume that $\Pi = \Pi_n$ for a large n, and that γ lies in Ω_{n-1}. By the same argument as in the proof of Proposition 12.8 using the loop theorem, we can assume that there is an embedding $\varphi\colon \mathbb{D} \to \Omega'$ such that $\varphi|_{\partial\mathbb{D}} = \gamma$ and $\varphi(D)$ lies entirely in one of the half-spaces bounded by Π. By the n-step of the construction, the entire bounded domain U bounded by Π_n and $\varphi(D)$ is contained in Ω' and hence γ is contractible in $\Pi \cap \Omega'$, a contradiction. □

12.25. Key lemma. *The two-convex hull of the interior of a polytope in $\mathbb{E}^3$ is also the interior of a polytope.*

Proof. Fix a polytope P in $\mathbb{E}^3$. Set $\Omega = \mathrm{Int}\,P$. We may assume that Ω is dense in P (if not, redefine P as the closure of Ω). Denote by $F_1, \dots, F_m$ the facets of P. By subdividing F_i if necessary, we may assume that all F_i are convex polygons.

Set $\Omega' = \mathrm{Conv}_2\,\Omega$ and let P' be the closure of Ω'. Further, for each i, set $F_i' = F_i \setminus \Omega'$. In other words, F_i' is the subset of the facet F_i which remains on the boundary of P'.

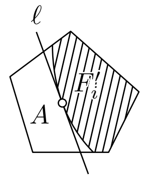
From the construction of the two-convex hull (12.24) we have:

❸ *F_i' is a convex subset of F_i.*

Further, since Ω' is two-convex we obtain the following:

❹ *Each connected component of the complement $F_i \setminus F_i'$ is convex.*

Indeed, assume a connected component A of $F_i \setminus F_i'$ fails to be convex. Then there is a supporting line ℓ to F_i' touching F_i' at a single point in the interior of F_i. Then one could rotate the plane of F_i slightly around ℓ and move it parallelly to cut a "cap" from the complement of Ω. The latter means that Ω is not two-convex, a contradiction. $\triangle$



From ❸ and ❹, we conclude

❺ *F_i' is a convex polygon for each i.*

Consider the complement $\mathbb{E}^3 \setminus \Omega$ equipped with the length metric. By construction of the two-convex hull (12.24), the complement $L = \mathbb{E}^3 \setminus (\Omega' \cup P)$ is locally convex; that is, any point of L admits a convex neighborhood.

Summarizing: (1) Ω' is a two-convex open set, (2) the boundary $\partial\Omega'$ contains a finite number of polygons F_i' and the remaining part S of the boundary is locally concave. It remains to show that (1) and (2) imply that S and therefore $\partial\Omega'$ are piecewise linear.

12.26. Exercise. *Prove the last statement.* $\square$

H Proof of Shefel's theorem

Proof of 12.17. The "only if" part can be proved in the same way as in the smooth two-convexity theorem (12.10) with the additional use of Exercise 12.23.

If part. Suppose Ω is two-convex. We need to show that $\tilde{K}$ is CAT(0).

Fix a quadruple of points $x^1, x^2, x^3, x^4 \in \tilde{\Omega}$. Let us show that CAT(0) comparison holds for this quadruple.

Fix $\varepsilon > 0$. Choose six broken lines in $\tilde{\Omega}$ connecting all pairs of points x^1, x^2, x^3, x^4, where the length of each broken line is at most ε bigger than the distance between its ends in the length metric on $\tilde{\Omega}$. Denote by X the union of these broken lines. Choose a polytope P in Ω such that its interior $\operatorname{Int} P$ contains the projections of all six broken lines and discs which contract all the loops created by them (it is sufficient to take 3 discs).

Denote by Ω' the two-convex hull of the interior of P. According to the key lemma (12.25), Ω' is the interior of a polytope.

Equip Ω' with the induced length metric. Consider the universal metric cover $\tilde{\Omega}'$ of Ω'. (The covering $\tilde{\Omega}' \to \Omega'$ might be nontrivial — even if $\operatorname{Int} P$ is simply connected, its two-convex hull Ω' might not be simply connected.) Denote by $\tilde{K}'$ the completion of $\tilde{\Omega}'$.

By Lemma 12.21, $\tilde{K}'$ is CAT(0).

By construction of $\operatorname{Int} P$, the embedding $\operatorname{Int} P \hookrightarrow \Omega'$ admits a lift $\iota\colon X \hookrightarrow \tilde{K}'$. By construction, ι almost preserves the distances between the points x^1, x^2, x^3, x^4; namely

$$|\iota(x^i) - \iota(x^j)|_L \lesseqgtr |x^i - x^j|_{\operatorname{Int} P} \pm \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and CAT(0) comparison holds in $\tilde{K}'$, we get that CAT(0) comparison holds in Ω for x^1, x^2, x^3, x^4.

The statement follows since the quadruple $x^1, x^2, x^3, x^4 \in \tilde{\Omega}$ is arbitrary. $\square$

12.27. Exercise. *Assume $K \subset \mathbb{E}^m$ is a closed set bounded by a Lipschitz hypersurface. Equip K with the induced length metric. Show that if K is* CAT(0)*, then K is two-convex.*

I Remarks

Under the name $(n-2)$-convex sets, two-convex sets in $\mathbb{E}^n$ were introduced by Mikhael Gromov [64]. In addition to the inheritance of upper curvature bounds by two-convex sets discussed in this lecture, these sets appear as the maximal open sets with vanishing curvature in Riemannian manifolds with non-negative or non-positive sectional curvature [see Lemma 5.8 in 42, 11 and 104].

Two-convex sets could be defined using homology instead of homotopy, as in Gromov's formulation of the Leftschetz theorem [64, §$\tfrac{1}{2}$]. Namely, we can say that K is two-convex if the following condition holds: if a one-dimensional cycle z has support in the intersection of K with a plane W and bounds in K, then it bounds in $K \cap W$.

The resulting definition is equivalent to the one used above. But unlike our definition it can be generalized to define k-convex sets in $\mathbb{E}^m$ for $k > 2$. With this homological definition one can also avoid the use of the loop theorem, whose proof is quite involved. Nevertheless, we chose the definition using homotopies since it is easier to visualize.

Both definitions work well for open sets; for general sets one should be able to give a similar definition using an appropriate homotopy/homology theory.

In [1], Stephanie Alexander, David Berg and Richard Bishop gave the exact upper bound on Alexandrov's curvature for the Riemannian manifolds with boundary. This theorem includes the smooth two-convexity theorem (12.10) as a partial case. Namely they show the following.

12.28. Theorem. *Let M be a Riemannian manifold with boundary ∂M. A direction tangent to the boundary will be called concave if there is a short geodesic in this direction which leaves the boundary and goes into the interior of M. A sectional direction (that is, a 2-plane) tangent to the boundary will be called* concave *if all the directions in it are concave.*

Denote by κ an upper bound of sectional curvatures of M and sectional curvatures of ∂M in the concave sectional directions. Then M is locally $\mathrm{CAT}(\kappa)$.

12.29. Corollary. *Let M be a Riemannian manifold with boundary ∂M. Assume that all the sectional curvatures of M and ∂M are bounded above by κ. Then M is locally* $\mathrm{CAT}(\kappa)$.

Theorem 12.20 is from Shefel's original paper [123]. It is related to Alexandrov's theorem about ruled surfaces [7].

Let D be an embedded closed disc in $\mathbb{E}^3$. We say that D is saddle if each connected component which any plane cuts from D contains a point on the boundary ∂D. If D is locally described by a Lipschitz embedding, then this condition is equivalent to saying that D is two-convex.

12.30. Shefel's conjecture. *Any saddle surface in $\mathbb{E}^3$ equipped with the length-metric is locally* $\mathrm{CAT}(0)$.

The conjecture is open even for the surfaces described by a bi-Lipschitz embedding of a disc. From another result of Samuel Shefel [123], it follows that a saddle surface satisfies the isoperimetric inequality $a \leqslant C\cdot\ell^2$ where a is the area of a disc bounded by a curve of length ℓ and $C = \frac{1}{3\cdot\pi}$. By a result of Alexander Lytchak and Stefan Wenger [93], Shefel's conjecture is equivalent to the isoperimetric inequality with the optimal constant $C = \frac{1}{4\cdot\pi}$. [For more on the subject, see 111, and the references therein.]

Part III

Metric geometry on manifolds

Lecture 13

Besicovitch inequality

We will focus on Riemannian spaces — these are specially nice length metrics on manifolds. These spaces are also most important in applications.

As it will be indicated in Section 13F, most of the statements of this and the following lecture have counterparts for general length metrics on manifolds.

A Riemannian spaces

Let M be a smooth connected manifold. A metric tensor on M is a choice of positive definite quadratic forms g_p on each tangent space T_pM that depends continuously on the point; that is, in any local coordinates of M the components of g are continuous functions.

A Riemannian manifold (M, g) is a smooth manifold M with a choice of metric tensor g on it.

The g-length of a Lipschitz curve $\gamma\colon [a,b] \to M$ is defined by

$$\operatorname{length}_g \gamma = \int\limits_a^b \sqrt{g(\gamma'(t),\gamma'(t))}\cdot dt.$$

The g-length induces a metric metric on M; it is defined as the greatest lower bound to lengths of Lipschitz curves connecting two given points; the distance between a pair of points $x, y \in M$ will be denoted by

$$|x-y|_g \quad \text{or} \quad \operatorname{dist}_x(y)_g.$$

The corresponding metric space $\mathcal{M}$ will be called Riemannian.

The following exercise implies that *isometry between Riemannian spaces might be not induced by a diffeomorphism.*

13.1. Exercise. *Construct a continuous Riemannian metric g on $\mathbb{R}^2$ such that the corresponding Riemannian space admits an isometry to the Euclidean palne but the induced map $\iota\colon \mathbb{R}^2 \to \mathbb{R}^2$ is not differentiable at some point.*

The exercise above shows that in general the smooth structure is not uniquely defined on Riemannian space. Therefore in general case one has to distinguish between Riemannian manifold and the corresponding Riemannian space altho there is almost no difference.[1]

The following observation states the key property of Riemannian spaces; it will be used to extend results from Euclidean space to Riemannian spaces.

13.2. Observation. *For any point p in a Riemannian space $\mathcal{M}$ and any $\varepsilon > 0$ there is a $e^{\mp\varepsilon}$-bilipschitz chart $s\colon W \to V$ from an open subset W of the n-dimensional Euclidean space to some neighborhood $V \ni p$.*

Proof. Choose a chart $s\colon U \to \mathcal{M}$ that covers p. Note that there is a linear transformation L such that for the metric tensor in the chart $s \circ L$ is coincides with the standard Euclidean tensor at the point $x = (s \circ L)^{-1}(p)$.

Since the metric tensor is continuous, the restriction of $s \circ L$ to a small neighborhood of x is $e^{\mp\varepsilon}$-bilipschitz. □

B Volume and Hausdorff measure

Let (M, g) be an n-dimensional Riemannian manifold. If a Borel set $R \subset M$ is covered by one chart $\iota\colon U \to M$, then its volume (briefly, $\operatorname{vol} R$ or $\operatorname{vol}_n R$) is defined by

$$\operatorname{vol} R := \int\limits_{\iota^{-1}(R)} \sqrt{\det g}.$$

In the general case we can subdivide R into a countable collection of regions $R_1, R_2 \dots$ such that each region R_i is covered by one chart $\iota_i\colon U_i \to M$ and define

$$\operatorname{vol} R := \operatorname{vol} R_1 + \operatorname{vol} R_2 + \dots$$

[1]In fact a straightforward smoothing procedure shows that isometry between Riemannian spaces can be approximated by diffeomorphisms between underlying manifolds; in particular, these manifolds are diffeomorphic. Also, if the metric tensor is smooth, then it is not hard to show that Riemannian space remembers everything about the Riemannian manifold which includes its smooth structure; it is a part of the so-called Myers–Steenrod theorem [101].

The chain rule for multiple integrals implies that the right-hand side does not depend on the choice of subdivision and the choice of charts.

Similarly, we define integral along (M, g). Any Borel function $u\colon M \to \mathbb{R}$, can be presented as a sum $u_1 + u_2 + \cdots$ such that the support of each function u_i can be covered by one chart $\iota_i\colon U_i \to M$ and set

$$\int\limits_{p\in\mathcal{M}} u(p) := \sum_i \left[\int\limits_{x\in U_i} u_i\circ s(x)\cdot\sqrt{\det g}\right].$$

In particular

$$\operatorname{vol} R = \int\limits_{p\in R} 1.$$

Let $\mathcal{X}$ be a metric space and $R\subset\mathcal{X}$. The α-dimensional Hausdorff measure of R is defined by

$$\operatorname{haus}_\alpha R := \lim_{\varepsilon\to 0}\ \inf\left\{\sum_{n\in\mathbb{N}}(\operatorname{diam} A_n)^\alpha \ :\ \begin{array}{c}\operatorname{diam} A_n<\varepsilon \text{ for}\\ \text{for each } n, \text{all } A_n\\ \text{are closed, and}\\ \bigcup\limits_{n\in\mathbb{N}} A_n \supset R.\end{array}\right\}.$$

For properties of Hausdorff measure we refer to the classical book of Herbert Federer [57]; in particular, $\operatorname{haus}_\alpha$ is indeed a measure and $\operatorname{haus}_\alpha$-measurable sets include all Borel sets.

The following observation follows from 13.2 and Rademacher's theorem:

13.3. Observation. *Suppose that a Borel set R in an n-dimensional Riemannian space $\mathcal{M}$ is subdivided into a countable collection of subsets R_i such that each R_i is covered by an $e^{\mp\varepsilon}$-bilipschitz charts s_i. Then*

$$\operatorname{vol}_n R \lesseqgtr e^{\pm n\cdot\varepsilon}\cdot\sum_i \operatorname{vol}_n[s_i^{-1}(R_i)]$$

and

$$\operatorname{haus}_n R \lesseqgtr e^{\pm n\cdot\varepsilon}\cdot\sum_i \operatorname{haus}_n[s_i^{-1}(R_i)]$$

According to Haar's theorem, a measure on n-dimensional Euclidean space that is invariant with respect to parallel translations is proportional to volume. Observe that

◇ A ball in n-dimensional Euclidean space of diameter 1 has unit Hausdorff measure.

◇ A unit cube in n-dimensional Euclidean space has unit volume.

Therefore, for any Borel region $R \subset \mathbb{E}^n$, we have

$$\text{vol}_n R = \tfrac{\omega_n}{2^n} \cdot \text{haus}_n R, \tag{❶}$$

where ω_n denotes the volume of a unit ball in the n-dimensional Euclidean space.

Applying ❶ together with 13.3, we get that the inequalities

$$\text{vol}_n R \lesseqgtr e^{\pm 2 \cdot n \cdot \varepsilon} \cdot \tfrac{\omega_n}{2^n} \cdot \text{haus}_n R$$

hold for any $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary, we get that ❶ holds in n-dimensional Riemannian spaces. More precisely:

13.4. Proposition. *The identity*

$$\text{vol}_n R = \tfrac{\omega_n}{2^n} \cdot \text{haus}_n R$$

holds for any Borel region R in an n-dimensional Riemannian space.

Since the Hausdorff measure is defined in pure metric terms, the proposition gives another way to prove that the volume does not depend on the choice of chars and subdivision of R.

The identity in this proposition will be used to define volume of any dimension. Namely, given an integer $k \geqslant 0$, the k-volume is defined by

$$\text{vol}_k := \tfrac{\omega_k}{2^k} \cdot \text{haus}_k.$$

By 13.4, if A is a subset of k-dimensional submanifold $\mathcal{N} \subset \mathcal{M}$, then the two definitions of $\text{vol}_k A$ agree; but the latter definition works for a wider class of sets.

13.5. Exercise. *Let $f \colon \mathcal{M} \to \mathcal{N}$ be a short volume-preserving map between n-dimensional Riemannian spaces. Prove the following statements and use them to conclude that f is locally distance-preserving.*

(a) f is injective; that is, if $f(x) = f(y)$, then $x = y$.

(b) For any $c < 1$, the map f is locally $[c, 1]$-bilipschitz; that is, for any point in $\mathcal{M}$ there is a neighborhood Ω and $\varepsilon > 0$ such that the inequality

$$c \leqslant \frac{|f(x) - f(y)|_{\mathcal{N}}}{|x - y|_{\mathcal{M}}} \leqslant 1$$

holds for any pair of distinct points $x, y \in \Omega$.

C Area and coarea formulas

Suppose that $f\colon \mathcal{M}\to\mathcal{N}$ is a Lipschitz map between n-dimensional Riemannian spaces $\mathcal{M}$ and $\mathcal{N}$. Then by Rademacher's theorem the differential $d_pf\colon \mathrm{T}_p\mathcal{M}\to\mathrm{T}_{f(p)}\mathcal{N}$ is defined at almost all $p\in\mathcal{M}$; that is, the differential defined at all points $p\in\mathcal{M}$ with exception of a subset with vanishing volume.

The differential is a linear map; it defines the Jacobian matrix Jac_pf in orthonormal frames of T_p and $\mathrm{T}_{f(p)}\mathcal{N}$. The determinant of Jac_pf will be denoted by jac_p. Note that the absolute value $|\mathrm{jac}_p|$ does not depend on the choice of the orthonormal frames.

The identity in the following proposition is called area formula.

13.6. Proposition. *Let $f\colon \mathcal{M}\to\mathcal{N}$ be a Lipschitz map between n-dimensional Riemannian spaces $\mathcal{M}$. Then for any Borel function $u\colon\mathcal{M}\to\mathbb{R}$ the following equality holds:*

$$\int\limits_{p\in\mathcal{M}} u(p)\cdot|\mathrm{jac}_pf| = \int\limits_{q\in\mathcal{N}}\sum_{p\in f^{-1}(q)} u(p).$$

Proof. If $\mathcal{M}$ and $\mathcal{N}$ are isometric to the n-dimensional Euclidean space, then the statement follows from the standard area formula [57, 3.2.3].

Note that Jacobian of a $e^{\mp\varepsilon}$-bilipschitz map between n-dimensional Riemannian manifolds (if defined) has determinant in the range $e^{\mp n\cdot\varepsilon}$. Applying 13.3 and the area formula in $\mathbb{E}^n$, we get the following approximate version of the needed identity for any $u\geqslant 0$:

$$\int\limits_{p\in\mathcal{M}} u(p)\cdot|\mathrm{jac}_pf| \lesseqgtr e^{\pm3\cdot n\cdot\varepsilon}\int\limits_{q\in\mathcal{N}}\sum_{p\in f^{-1}(q)} u(p).$$

Since $\varepsilon>0$ is arbitrary, we get that the area formula holds if $u\geqslant 0$. Finally, since both sides of the area formula are linear in u, it holds for any u. $\square$

The following inequality is called area inequality:

13.7. Corollary. *Let $f\colon\mathcal{M}\to\mathcal{N}$ be a locally Lipschitz map between n-dimensional Riemannian spaces. Then*

$$\int\limits_{p\in A}|\mathrm{jac}_pf|\geqslant \mathrm{vol}[f(A)]$$

for any Borel subset $A\subset M$.

In particular, if $|\mathrm{jac}_p f| \leqslant 1$ *almost everywhere in* A*, then*

$$\mathrm{vol}\, A \geqslant \mathrm{vol}[f(A)].$$

Proof. Apply the area formula to the characteristic function of A. □

Suppose that $f\colon \mathcal{M} \to \mathbb{R}$ is a Lipschitz function defined on an n-dimensional Riemannian space $\mathcal{M}$. Then by Rademacher's theorem, the differential $d_p f\colon \mathrm{T}_p\mathcal{M} \to \mathbb{R}$ and the gradient $\nabla_p f \in \mathrm{T}_p\mathcal{M}$ are defined at almost all $p \in \mathcal{M}$.

The identity in the following proposition is a partial case of the so-called coarea formula. (The general coarea formula deals with the maps to the spaces of arbitrary dimension, not necessary 1.)

13.8. Proposition. *Let* $f\colon \mathcal{M} \to \mathbb{R}$ *be a Lipschitz function defined on an* n*-dimensional Riemannian space* $\mathcal{M}$*. Suppose that the level sets* $L_x := f^{-1}(x)$ *are equipped with* $(n-1)$*-dimensional volume* $\mathrm{vol}_{n-1} := \frac{\omega_{n-1}}{2^{n-1}}\cdot \mathrm{haus}_{n-1}$. *Then for any Borel function* $u\colon \mathcal{M} \to \mathbb{R}$ *the following equality holds*

$$\int\limits_{p\in\mathcal{M}} u(p)\cdot|\nabla_p f| = \int\limits_{-\infty}^{+\infty} \left(\int\limits_{p\in L_x} u(p)\right)\cdot dx.$$

The following corollary is a partial case of the so-called coarea inequality;

13.9. Corollary. *Let* $\mathcal{M}$*,* f*, and* L_x *be as in 13.8.*

Suppose that f *is 1-Lipschitz. Then for any Borel subset* $A \subset M$ *we have*

❶ $$\mathrm{vol}_n A \geqslant \int\limits_{x\in\mathbb{R}} \mathrm{vol}_{n-1}[A \cap L_x]\cdot dx.$$

The right-hand side in ❶ is called coarea of the restriction $f|_A$.

Instead of proof of 13.8 and 13.9. If $\mathcal{M}$ is isometric to Euclidean space, then the statement follows from the standard coarea formula [57, 3.2.12]. The reduction to the Euclidean space is done the same way as in the proof of the area formula.

To prove the corollary, choose u to be the characteristic function of A and apply the coarea formula. □

D Besicovitch inequality

A closed connected region in a Riemannian manifold bounded by hypersurface will be called Riemannian manifold with boundary. We always assume that the hypersurface can be realized locally as a graph of Lipschitz function in a suitable chart. In this case, one can define g-length, g-distance, and g-volume the same way as we did for usual Riemannian manifolds.

13.10. Exercise. *Suppose that (M,g) is a compact Riemannian manifold with boundary. Observe that the interior (M°,g) of (M,g) is a usual Riemannian manifold. Show that the space of (M,g) is isometric to the completion of the space of (M°,g).*

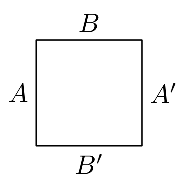
13.11. Theorem. *Let g be a continuous metric tensor on a unit n-dimensional cube $\square$. Suppose that the g-distances between the opposite faces of $\square$ are at least 1; that is, any Lipschitz curve that connects opposite faces has g-length at least 1. Then*

$$\operatorname{vol}(\square,g)\geqslant 1.$$

This is a partial case of the theorem proved by Abram Besicovitch [21].

Proof. We will consider the case $n=2$; the other cases are proved the same way.

Denote by A, A', and B, B' the opposite faces of the square $\square$. Consider two functions



$$\begin{aligned}f_A(x)&:=\min\{\operatorname{dist}_A(x)_g,1\},\\ f_B(x)&:=\min\{\operatorname{dist}_B(x)_g,1\}.\end{aligned}$$

Let $\boldsymbol{f}\colon\square\to\square$ be the map with coordinate functions f_A and f_B; that is, $\boldsymbol{f}(x):=(f_A(x),f_B(x))$.

❶ *The map $\boldsymbol{f}$ sends each face of $\square$ to itself.*

Indeed,

$$x\in A\quad\Longrightarrow\quad\operatorname{dist}_A(x)_g=0\quad\Longrightarrow\quad f_A(x)=0\quad\Longrightarrow\quad\boldsymbol{f}(x)\in A.$$

Similarly, if $x\in B$, then $\boldsymbol{f}(x)\in B$. Further,

$$x\in A'\quad\Longrightarrow\quad\operatorname{dist}_A(x)_g\geqslant 1\quad\Longrightarrow\quad f_A(x)=1\quad\Longrightarrow\quad\boldsymbol{f}(x)\in A'.$$

Similarly, if $x\in B'$, then $\boldsymbol{f}(x)\in B'$.

By ❶, it follows

$$\boldsymbol{f}_t(x) = t\cdot x + (1-t)\cdot\boldsymbol{f}(x)$$

defines a homotopy of maps of the pair of spaces $(\square, \partial\square)$ from $\boldsymbol{f}$ to the identity map; that is, $(t,x)\mapsto \boldsymbol{f}_t(x)$ is a continuous map and if $x\in\partial\square$, then $\boldsymbol{f}_t(x)\in\partial\square$ for any $t\in[0,1]$.

It follows that $\deg\boldsymbol{f}=1$; that is, $\boldsymbol{f}$ sends the fundamental class of $(\square,\partial\square)$ to itself.[2] In particular, $\boldsymbol{f}$ is onto.

Suppose that Jacobian matrix $\mathrm{Jac}_p\boldsymbol{f}$ of $\boldsymbol{f}$ is defined at $p\in\square$. Choose an orthonormal frame in T_p with respect to g and the standard frame in the target $\square$. Observe that the differentials d_pf_A and d_pf_B written in these frames are the rows of $\mathrm{Jac}_p\boldsymbol{f}$. Evidently $|d_pf_A|\leqslant 1$ and $|d_pf_B|\leqslant 1$. Since the determinant of a matrix is the volume of the parallelepiped spanned on its rows, we get

$$|\mathrm{jac}_p\boldsymbol{f}|\leqslant|d_pf_A|\cdot|d_pf_B|\leqslant 1.$$

Since $\boldsymbol{f}\colon\square\to\square$ is a Lipschitz onto map, the area inequality (13.7) implies that

$$\mathrm{vol}(\square,g)\geqslant\mathrm{vol}\,\square=1.$$

$\square$

If the g-distances between the opposite sides are $d_1,\dots,d_n$, then following the same lines one get that $\mathrm{vol}(\square,g)\geqslant d_1\cdots d_n$. Also note that in the proof we use topology of the n-cube only once, to show that the map f has degree one. Taking all this into account we get the following generalization of 13.11:

13.12. Theorem. *Let (M,g) be an n-dimensional Riemannian manifold with coonected boundary ∂M. Suppose that there is a degree 1 map $\partial M\to\partial\square$; denote by $d_1,\dots,d_n$ the g-distances between the inverse images of pairs of opposite faces of $\square$ in M. Then*

$$\mathrm{vol}(M,g)\geqslant d_1\cdots d_n.$$

13.13. Exercise. *Show that if equality holds in 13.12, then (M,g) is isometric to the rectangle $[0,d_1]\times\dots\times[0,d_n]$.*

13.14. Exercise. *Suppose g is a metric tensor on a regular hexagon ⬡ such that g-distances between the opposite sides are at least 1. Is there a positive lower bound on* area(⬡, g)*?*

13.15. Exercise. *Let g be a Riemannian metric on the cylinder $\mathbb{S}^1\times$ $\times[0,1]$. Suppose that*

[2]Here and further, we assume that homologies are taken with the coefficients in $\mathbb{Z}_2$, but you are welcome to play with other coefficients.

- ◇ *g-distance between pairs of points on the opposite boundary circles $\mathbb{S}^1 \times \{0\}$ and $\mathbb{S}^1 \times \{1\}$ is at least 1, and*
- ◇ *any curve γ in $\mathbb{S}^1 \times [0,1]$ that is homotopic to $\mathbb{S}^1 \times \{0\}$ has g-length at least 1.*

(a) Use Besicovitch inequality to show that

$$\operatorname{area}(\mathbb{S}^1 \times [0,1], g) \geqslant \tfrac{1}{2}.$$

(b) Modify the proof of Besicovitch inequality using coarea inequality (13.9) to prove the optimal bound

$$\operatorname{area}(\mathbb{S}^1 \times [0,1], g) \geqslant 1.$$

13.16. Exercise.

(a) Generalize 13.12 to noncontinuous metric tensor g described the following way: there are two Riemannian metric tensors g_1 and g_2 on M and a subset $V \subset M$ bounded by a Lipschitz hypersurface Σ such that $g = g_1$ at the points in V and $g = g_2$ otherwise.

(b) Use part (a) to prove the following: Let V be a compact set in the n-dimensional Euclidean space $\mathbb{E}^n$ bounded by a Lipschitz hypersurface Σ. Suppose g is a Riemannian metric on V such that

$$|p-q|_g \geqslant |p-q|_{\mathbb{E}^n}$$

for any two points $p, q \in \Sigma$. Show that

$$\operatorname{vol}(V, g) \geqslant \operatorname{vol}(V)_{\mathbb{E}^n}.$$

13.17. Exercise. *Suppose that sphere with Riemannian metric $(\mathbb{S}^2, g)$ admits an involution ι such that $|x - \iota(x)|_g \geqslant 1$.*

Show that

$$\operatorname{area}(\mathbb{S}^2, g) \geqslant \tfrac{1}{1000}.$$

Try to show that

$$\operatorname{area}(\mathbb{S}^2, g) \geqslant \tfrac{1}{2}, \quad \operatorname{area}(\mathbb{S}^2, g) \geqslant 1, \quad \textit{or} \quad \operatorname{area}(\mathbb{S}^2, g) \geqslant \tfrac{4}{\pi}$$

13.18. Advanced exercise. *Construct a metric tensor g on $\mathbb{S}^3$ such that (1) $\operatorname{vol}(\mathbb{S}^3, g)$ arbitrarily small and (2) there is an involution $\iota\colon \mathbb{S}^3 \to \mathbb{S}^3$ such that $|x - \iota(x)|_g \geqslant 1$ for any $x \in \mathbb{S}^3$.*

13.19. Exercise. *Let $g_1, g_2, \dots,$ and g_∞ be metrics on a fixed compact manifold M. Suppose that $\operatorname{dist}_{g_n}$ uniformly converges to $\operatorname{dist}_{g_\infty}$ as functions on $M \times M \to \mathbb{R}$. Show that*

$$\varliminf_{n\to\infty} \operatorname{vol}(M, g_n) \geqslant \operatorname{vol}(M, g_\infty).$$

Show that the inequality might be strict.

E Systolic inequality

Let $\mathcal{M}$ be a compact Riemannian space. The systole of $\mathcal{M}$ (briefly $\operatorname{sys}\mathcal{M}$) is defined to be the least length of a noncontractible closed curve in $\mathcal{M}$.

Let Λ be a class of closed n-dimensional Riemannian spaces. We say that a systolic inequality holds for Λ if there is a constant c such that

$$\operatorname{sys}\mathcal{M} \leqslant c\cdot \sqrt[n]{\operatorname{vol}\mathcal{M}}$$

for any $\mathcal{M} \in \Lambda$.

13.20. Exercise. *Use 13.11 or 13.15 to show that a systolic inequality holds for any Riemannian metric on the 2-torus $\mathbb{T}^2$.*

13.21. Exercise. *Use 13.11 to show that a systolic inequality holds for any Riemannian metric on the real projective plane $\mathbb{R}\mathrm{P}^2$.*

13.22. Exercise. *Use 13.12 to show that systolic inequality holds for any Riemannian metric on any closed surfaces of positive genus.*

13.23. Exercise. *Show that no systolic inequality holds for Riemannian metrics on $\mathbb{S}^2 \times \mathbb{S}^1$.*

In the following lecture we will show that systolic inequality holds for many manifolds, in particular for torus of arbitrary dimension.

F Generalization

The following proposition follows immediately from the definitions of Hausdorff measure (Section 13B).

13.24. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces, $A \subset \mathcal{X}$ and $f\colon \mathcal{X} \to \mathcal{Y}$ be a L-Lipschitz map. Then*

$$\operatorname{haus}_\alpha[f(A)] \leqslant L^\alpha\cdot \operatorname{haus}_\alpha A$$

for any α.

The following exercise provides a weak analog of the Besicovitch inequality that works for arbitrary metrics.

13.25. Exercise. *Let M be manifold with boundary and ρ is a semi-metric on M. Suppose ∂M admits a degree 1 map to the surface of the n-dimensional cube $\square$; denote by $d_1, \dots, d_n$ the ρ-distances between the inverse images of pairs of opposite faces of $\square$ in M. Then*

$$\operatorname{haus}_n(M,\rho) \geqslant d_1 \cdots d_n.$$

Recall that in n-dimensional Riemannian spaces we have

$$\tfrac{\omega_n}{2^n}\cdot\mathrm{haus}_n=\mathrm{vol}_n\,.$$

Note that $\frac{\omega_n}{2^n}<1$ if $n\geqslant 2$. Therefore, the conclusion in 13.25 is weaker than in 13.12 (the assumptions are weaker as well).

One can redefine systolic inequality on n-dimensional manifolds using the Hausdorff measure haus_n instead of the volume. It is straightforward to prove analogs of the exercises 13.20–13.23 with this definition.

13.26. Exercise. *Suppose that two embedded n-disks Δ_1,Δ_2 in a metric space $\mathcal{X}$ have identical boundaries. Assume that $\mathcal{X}$ is contractible and* $\mathrm{haus}_{n+1}\mathcal{X}=0$*. Show that $\Delta_1=\Delta_2$.*

G Remarks

The optimal constants in the systolic inequality are known only in the following three cases:

- For real projective plane $\mathbb{RP}^2$ the constant is $\sqrt{\pi/2}$ — the equality holds for a quotient of a round sphere by isometric involution. The statement was proved by Pao Ming Pu [116].
- For torus $\mathbb{T}^2$ the constant is $\sqrt{2}/\sqrt[4]{3}$ — the equality holds for a flat torus obtained from a regular hexagon by identifying opposite sides; this is the so-called Loewner's torus inequality.
- For the Klein bottle $\mathbb{RP}^2\#\mathbb{RP}^2$ the constant is $\sqrt{\pi}/2^{3/4}$ — the equality holds for a certain nonsmooth metric. The statement was proved by Christophe Bavard [16].

The proofs of these results use the so-called uniformization theorem available in the 2-dimensional case only. These proofs are beautiful, but they are too far from metric geometry. A good survey on the subject is written by Christopher Croke and Mikhail Katz [50].

An analog of Exercise 13.19 with Hausdorff measure instead of volume does not hold for general metrics on a manifold. In fact there is a nondecreasing sequence of metric tensors g_n on M, such that (1) $\mathrm{vol}(M,g_n)<1$ for any n and (2) dist_{g_n} converges to a metric on M with arbitrary large Hausdorff measure of any given dimension; such examples were constructed by Dmitri Burago, Sergei Ivanov, and David Shoenthal [39].

Lecture 14

Width and systole

This lecture is based on the paper of Alexander Nabutovsky [102].

A Partition of unity

14.1. Proposition. *Let $\{V_i\}$ be a finite open covering of a compact metric space $\mathcal{X}$. Then there are Lipschitz functions $\psi_i\colon \mathcal{X} \to [0,1]$ such that (1) if $\psi_i(x) > 0$, then $x \in V_i$ and (2) for any $x \in \mathcal{X}$ we have*

$$\sum_i \psi_i(x) = 1.$$

A collection of functions $\{\psi_i\}$ that meets the conditions in 14.1 is called a partition of unity subordinate to the covering $\{V_i\}$.

Proof. Denote by $\varphi_i(x)$ the distance from x to the complement of V_i; that is,

$$\varphi_i(x) = \operatorname{dist}_{\mathcal{X}\setminus V_i}(x).$$

Note φ_i is 1-Lipschitz for any i and $\varphi_i(x) > 0$ if and only if $x \in V_i$. Since $\{V_i\}$ is a covering, we have that

$$\Phi(x) := \sum_i \varphi_i(x) > 0 \quad \text{for any} \quad x \in \mathcal{X}.$$

Since $\mathcal{X}$ is compact, $\Phi > \delta$ for some $\delta > 0$. It follows that $x \mapsto \frac{1}{\Phi(x)}$ is a bounded Lipschitz function.

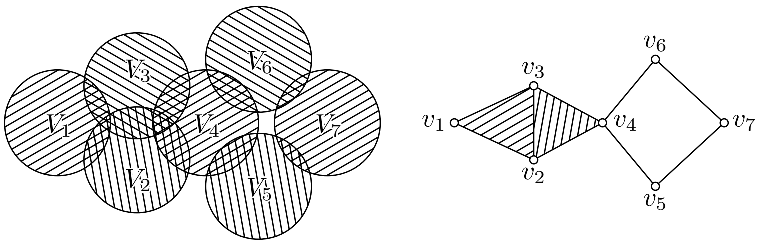
Set

$$\psi_k(x) = \frac{\varphi_k(x)}{\Phi(x)}.$$

Observe that by construction the functions ψ_i meet the conditions in the proposition. $\square$

B Nerves

Let $\{V_1, \dots, V_k\}$ be a finite open cover of a compact metric space $\mathcal{X}$. Consider an abstract simplicial complex $\mathcal{N}$, with one vertex v_i for each set V_i such that a simplex with vertices $v_{i_1}, \dots, v_{i_m}$ is included in $\mathcal{N}$ if the intersection $V_{i_1} \cap \dots \cap V_{i_m}$ is nonempty. The obtained simplicial



covering nerve

complex $\mathcal{N}$ is called the nerve of the covering $\{V_i\}$. Evidently $\mathcal{N}$ is a finite simplicial complex — it is a subcomplex of a simplex with the vertices $\{v_1, \dots, v_k\}$.

Note that the nerve $\mathcal{N}$ has dimension at most n if and only if the covering $\{V_1, \dots, V_k\}$ has multiplicity at most $n+1$; that is, any point $x \in \mathcal{X}$ belongs to at most $n+1$ sets of the covering.

Suppose $\{\psi_i\}$ is a partition of unity subordinate to the covering $\{V_1, \dots, V_k\}$. Choose a point $x \in \mathcal{X}$. Note that the set

$$\{v_{i_1}, \dots, v_{i_n}\} = \{\, v_i \ : \ \psi_i(x) > 0 \,\}$$

form vertices of a simplex in $\mathcal{N}$. Therefore

$$\boldsymbol{\psi}\colon x \mapsto \psi_1(x)\cdot v_1 + \psi_2(x)\cdot v_2 + \dots + \psi_k(x)\cdot v_n.$$

describes a Lipschitz map from $\mathcal{X}$ to the nerve $\mathcal{N}$ of $\{V_i\}$. In other words, $\boldsymbol{\psi}$ maps a point x to the point in $\mathcal{N}$ with barycentric coordinates $(\psi_1(x), \dots, \psi_k(x))$.

Recall that the star of a vertex v_i (briefly Star_{v_i}) is defined as the union of the interiors of all simplicies that have v_i as a vertex. Recall that $\psi_i(x) > 0$ implies $x \in V_i$. Therefore we get the following:

14.2. Proposition. *Let $\mathcal{N}$ be a nerve of an open covering $\{V_1, \dots$ $\dots, V_k\}$ of a compact metric space $\mathcal{X}$. Denote by v_i the vertex of $\mathcal{N}$ that corresponds to V_i.*

Then there is a Lipschitz map $\psi\colon \mathcal{X} \to \mathcal{N}$ such that $\psi(V_i) \subset \mathrm{Star}_{v_i}$ for every i.

C Width

Suppose A is a subset of a metric space $\mathcal{X}$. The radius of A (briefly $\operatorname{rad} A$) is defined as the least upper bound on the values $R > 0$ such that $\mathrm{B}(x, R) \supset A$ for some $x \in \mathcal{X}$.

14.3. Definition. *Let $\mathcal{X}$ be a metric space. The n-th width of $\mathcal{X}$ (briefly $\mathrm{width}_n \mathcal{X}$) is the least upper bound on values $R > 0$ such that $\mathcal{X}$ admits a finite open covering $\{V_i\}$ with multiplicity at most $n + 1$ and $\operatorname{rad} V_i < R$ for each i.*

Remarks.

- Observe that
$$\mathrm{width}_0 \mathcal{X} \geqslant \mathrm{width}_1 \mathcal{X} \geqslant \mathrm{width}_2 \mathcal{X} \geqslant \dots$$
for any compact metric space $\mathcal{X}$. Moreover, if $\mathcal{X}$ is connected, then
$$\mathrm{width}_0 \mathcal{X} = \operatorname{rad} \mathcal{X}.$$
- Usually width is defined using diameter instead of radius, but the results differ at most twice. Namely, if r is the n-th radius-width and d — the n-th diameter-width, then $r \leqslant d \leqslant 2{\cdot}r$.
- Note that Lebesgue covering dimension of $\mathcal{X}$ can be defined as the least number n such that $\mathrm{width}_n \mathcal{X} = 0$.
- Another closely related notion is the so-called macroscopic dimension on scale R; it is defined as the least number n such that $\mathrm{width}_n \mathcal{X} < R$.

14.4. Exercise. *Suppose $\mathcal{X}$ is a compact metric space such that any closed curve γ in $\mathcal{X}$ can be contracted in its R-neighborhood. Show that macroscopic dimension of $\mathcal{X}$ on scale $100{\cdot}R$ is at most 1.*

What about quasiconverse? That is, suppose a simply connected compact metric space $\mathcal{X}$ has macroscopic dimension at most 1 on scale R, is it true that any closed curve γ in $\mathcal{X}$ can be contracted in its $100{\cdot}R$-neighborhood?

The following exercise gives a good reason for the choice of term width; it also can be used as an alternative definition.

14.5. Exercise. *Suppose $\mathcal{X}$ is a compact metric space. Show that $\mathrm{width}_n \mathcal{X} < R$ if and only if there is a finite n-dimensional simplicial complex $\mathcal{N}$ and a continuous map $\psi\colon \mathcal{X} \to \mathcal{N}$ such that*
$$\operatorname{rad}[\psi^{-1}(s)] < R$$

for any $s \in \mathcal{N}$.

D Riemannian polyhedrons

A Riemannian polyhedron is defined as a finite simplicial complex with a metric tensor on each simplex such that the restriction of the metric tensor to a subsimplex coincides with the metric on the subsimplex.

The dimension of a Riemannian polyhedron is defined as the largest dimension in its triangulation. For Riemannian polyhedrons one can define length of curves and volume the same way as for Riemannian manifolds.

The obtained metric space will be called Riemannian polyhedron as well. A triangulation of Riemannian polyhedron will always be assumed to have the above property on the metric tensor.

Further we will apply the notion of width only to compact Riemannian polyhedrons. If $\mathcal{P}$ is an n-dimensional Riemannian polyhedron, then we suppose that

$$\operatorname{width}\mathcal{P} := \operatorname{width}_{n-1}\mathcal{P}.$$

Suppose that $\mathcal{P}$ is an n-dimensional Riemannian polyhedron; in this case we will use short cut vol for vol_n. Let us define volume profile of $\mathcal{P}$ as a function returning largest volume of r-ball in $\mathcal{P}$; that is, the volume profile of $\mathcal{P}$ is a function $\operatorname{VolPro}_{\mathcal{P}}\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\operatorname{VolPro}_{\mathcal{P}}(r) := \sup\{\,\operatorname{vol}\mathrm{B}(p,r) \;:\; p\in\mathcal{P}\,\}.$$

Note that $r \mapsto \operatorname{VolPro}_{\mathcal{P}}(r)$ is nondecreasing and

$$\operatorname{VolPro}_{\mathcal{P}}(r) \leqslant \operatorname{vol}\mathcal{P}$$

for any r. Moreover, if $\mathcal{P}$ is connected, then the equality $\operatorname{VolPro}_{\mathcal{P}}(r) = \operatorname{vol}\mathcal{P}$ holds for $r \geqslant \operatorname{rad}\mathcal{P}$.

Note that if $\mathcal{P}$ is a connected 1-dimensional Riemannian polyhedron, then

$$\operatorname{width}\mathcal{P} = \operatorname{width}_0\mathcal{P} = \operatorname{rad}\mathcal{P}.$$

14.6. Exercise. *Let $\mathcal{P}$ be a 1-dimensional Riemannian polyhedron. Suppose that* $\operatorname{VolPro}_{\mathcal{P}}(R) < R$ *for some* $R > 0$. *Show that*

$$\operatorname{width}\mathcal{P} < R.$$

Try to show that $c = \frac{1}{2}$ is the optimal constant for which the following inequality holds:

$$\operatorname{width}\mathcal{P} < c{\cdot}R.$$

E Volume profile bounds width

14.7. Theorem. *Let $\mathcal{P}$ be an n-dimensional Riemannian polyhedron. If the inequality*

$$R > n\cdot \sqrt[n]{\operatorname{VolPro}_{\mathcal{P}}(R)}$$

holds for some $R > 0$, then

$$\operatorname{width} \mathcal{P} \leqslant R.$$

Since $\operatorname{VolPro}_{\mathcal{P}}(R) \leqslant \operatorname{vol} \mathcal{P}$ for any $R > 0$, we get the following:

14.8. Corollary. *For any n-dimensional Riemannian polyhedron $\mathcal{P}$, we have*

$$\operatorname{width} \mathcal{P} \leqslant n\cdot \sqrt[n]{\operatorname{vol} \mathcal{P}}.$$

The proof of 14.7 will be given at the very end of this section, after discussing separating polyhedrons.

Let us start three technical statements. The first statement can be obtained by modifying a smoothing procedure for functions defined on Euclidean space.

A function f defined on a Riemannian polyhedron $\mathcal{P}$ is called piecewise smooth if there is a triangulation of $\mathcal{P}$ such that restriction of f to every simplex is smooth.

14.9. Smoothing procedure. *Let $\mathcal{P}$ be a Riemannian polyhedron and $f\colon \mathcal{P} \to \mathbb{R}$ be a 1-Lipschitz function. Then for any $\delta > 0$ there is a piecewise smooth 1-Lipschitz function $\tilde{f}\colon \mathcal{P} \to \mathbb{R}$ such that*

$$|\tilde{f}(x) - f(x)| < \delta$$

for any $x \in \mathcal{P}$.

The following statement can be proved by applying the classical Sard's theorem to each simplex of a Riemannian polyhedron.

14.10. Sard's theorem. *Let $\mathcal{P}$ be an n-dimensional Riemannian polyhedron and $f\colon \mathcal{P} \to \mathbb{R}$ be a piecewise smooth function. Then for almost all values $a \in \mathbb{R}$, the inverse image $f^{-1}\{a\}$ is a Riemannian polyhedron of dimension at most $n-1$ (we assume that $f^{-1}\{a\}$ is equipped with the induced length metric).*

The following statement can be proved by applying the coarea inequality (13.9) to the restriction of f to each simplex of the polyhedron and summing up the results.

14.11. Coarea inequality. *Let $\mathcal{P}$ be an n-dimensional Riemannian polyhedron and $f\colon \mathcal{P}\to\mathbb{R}$ be a piecewise smooth 1-Lipschitz function. Set $v=\mathrm{vol}_n(f^{-1}[r,R])$ and $a(t)=\mathrm{vol}_{n-1}(f^{-1}\{t\})$. Then*

$$\int\limits_r^R a(t)\cdot dt \geqslant v.$$

In particular, there is a subset of positive measure $S\subset[r,R]$ such that the inequality

$$a(t)\geqslant\frac{v}{R-r}$$

holds for any $t\in S$.

Separating subpolyhedrons

14.12. Definition. *Let $\mathcal{P}$ be an n-dimensional Riemannian polyhedron. An $(n-1)$-dimensional subpolyhedron $\mathcal{Q}\subset\mathcal{P}$ is called R-separating if for each connected component U of the complement $\mathcal{P}\setminus\mathcal{Q}$ we have*

$$\mathrm{rad}\, U < R.$$

14.13. Lemma. *Let $\mathcal{P}$ be an n-dimensional Riemannian polyhedron. Then given $R>0$ and $\varepsilon>0$ there is a R-separating subpolyhedron $\mathcal{Q}\subset\mathcal{P}$ such that for any $r_0<r_1\leqslant R$ we have*

$$\mathrm{VolPro}_{\mathcal{Q}}(r_0)<\tfrac{1}{r_1-r_0}\cdot\mathrm{VolPro}_{\mathcal{P}}(r_1)+\varepsilon.$$

The proof reminds the proof of the following statement about minimal surfaces: *if a point p lies on an compact area-minimizing surface Σ and $\partial\Sigma\cap\mathrm{B}(p,r)=\varnothing$, then*

$$\mathrm{area}(\Sigma\cap\mathrm{B}(p,r))\leqslant\tfrac12\cdot\mathrm{area}\,\mathbb{S}^2\cdot r^2.$$

Proof. Choose a small $\delta>0$. Applying the smoothing procedure (14.9), we can exchange each distance function dist_p on $\mathcal{P}$ by δ-close piecewise smooth 1-Lipschitz function, which will be denoted by $\widetilde{\mathrm{dist}}_p$.

By Sard's theorem (14.10), for almost all values $c\in(r_0+\delta,r_1-\delta)$, the level set

$$\tilde S_c(p)=\left\{\, x\in\mathcal{P}\ :\ \widetilde{\mathrm{dist}}_p(x)=c\,\right\}$$

is a Riemannian polyhedron of dimension at most $n-1$. Since δ is small, the coarea inequality (14.11) implies that c can be chosen so that in addition the following inequality holds:

$$\begin{aligned}\mathrm{vol}_{n-1}\tilde{S}_c(p) &\leqslant \tfrac{1}{r_1-r_0-2\cdot\delta}\cdot\mathrm{vol}_n[\mathrm{B}(p,r_1)] < \\ &< \tfrac{1}{r_1-r_0}\cdot\mathrm{VolPro}_{\mathcal{P}}(r_1)+\tfrac{\varepsilon}{2}.\end{aligned}$$

Suppose $\mathcal{Q}$ is an R-separating subpolyhedron in $\mathcal{P}$ with almost minimal volume; say its volume is at most $\frac{\varepsilon}{2}$-far from the greatest lower bound. Note that cutting from $\mathcal{Q}$ everything inside $\tilde{S}_c(p)$ and adding $\tilde{S}_c(p)$ produces a R-separating subpolyhedron, say $\mathcal{Q}'$.[1]

Since $\mathcal{Q}$ has almost minimal volume, we have

$$\mathrm{vol}_{n-1}[\mathcal{Q}\cap\mathrm{B}(p,r_0)_{\mathcal{P}}]-\tfrac{\varepsilon}{2}\leqslant\mathrm{vol}_{n-1}S_c(p).$$

Therefore

$$\text{❶}\qquad \mathrm{vol}_{n-1}[\mathcal{Q}\cap\mathrm{B}(p,r_0)_{\mathcal{P}}]\leqslant\tfrac{1}{r_1-r_0}\cdot\mathrm{VolPro}_{\mathcal{P}}(r_1)+\varepsilon.$$

Recall that $\mathcal{Q}$ is equipped with the induced length metric; therefore $|p-q|_{\mathcal{Q}}\geqslant|p-q|_{\mathcal{P}}$ for any $p,q\in\mathcal{Q}$; in particular,

$$\mathrm{B}(p,r_0)_{\mathcal{Q}}\subset\mathcal{Q}\cap\mathrm{B}(p,r_0)_{\mathcal{P}}$$

for any $p\in\mathcal{Q}$ and $r_0\geqslant 0$. Hence, ❶ implies the lemma. □

14.14. Lemma. *Let $\mathcal{Q}$ be an R-separating subpolyhedron in an n-dimensional Riemannian polyhedron $\mathcal{P}$. Then*

$$\mathrm{width}\,\mathcal{Q}\leqslant R\quad\Longrightarrow\quad\mathrm{width}\,\mathcal{P}\leqslant R.$$

Proof. Choose an open covering $\{V_1,\dots,V_k\}$ of $\mathcal{Q}$ as in the definition of width (14.3); that is, it has multiplicity at most n and $\mathrm{rad}\,V_i<R$ for any i.

Note that $\{V_1,\dots,V_k\}$ can be converted into an open covering of a small neighbourhood of $\mathcal{Q}$ in $\mathcal{P}$ without increasing the multiplicity. This can be done by setting

$$V_i'=\bigcup_{x\in V_i}\mathrm{B}(x,r_x),$$

where $r_x:=\frac{1}{10}\cdot\inf\{\,|x-y|\;:\;y\in\mathcal{Q}\setminus V_i\,\}$.

By adding to $\{V_i'\}$ all the components of $\mathcal{P}\setminus\mathcal{Q}$, we increase the multiplicity by at most 1 and obtain a covering of $\mathcal{P}$. The statement follows since $\dim\mathcal{P}=\dim\mathcal{Q}+1$. □

[1] If $\dim\tilde{S}_c(p)<n-1$, then it might happen that $\dim\mathcal{Q}'<n-1$; so, by the definition, $\mathcal{Q}'$ is not separating. It can be fixed by adding a tiny $(n-1)$-dimensional piece to $\mathcal{Q}'$.

Proof assembling

Proof of 14.7. We apply induction on the dimension $n = \dim \mathcal{P}$. The base case $n = 1$ is given in 14.6.

Suppose that the $(n-1)$-dimensional case is proved. Consider an n-dimensional Riemannian polyhedron $\mathcal{P}$ and suppose

$$n\cdot \sqrt[n]{\operatorname{VolPro}\mathcal{P}(R)} < R$$

for some $R > 0$. Let $\mathcal{Q}$ be an R-separating subpolyhedron in $\mathcal{P}$ provided by 14.13 for a small $\varepsilon > 0$.

Applying 14.13 for $r = \frac{n-1}{n}\cdot R$ and R, we have that

$$\begin{aligned}\operatorname{VolPro}_{\mathcal{Q}}(r) &< \frac{1}{R-r}\cdot \operatorname{VolPro}_{\mathcal{P}}(R) + \varepsilon < \\ &< \frac{n}{R}\cdot\left(\frac{R}{n}\right)^n = \\ &= \left(\frac{r}{n-1}\right)^{n-1};\end{aligned}$$

that is, $(n-1)\cdot \sqrt[n-1]{\operatorname{VolPro}\mathcal{Q}(r)} < r$. Since $\dim \mathcal{Q} = n-1$, by the induction hypothesis, we get that

$$\operatorname{width}\mathcal{Q} \leqslant r < R.$$

It remains to apply 14.14. □

F Width bounds systole

Recall that a topological space K is called aspherical if any continuous map $\mathbb{S}^k \to K$ for $k \geqslant 2$ is null-homotopic.

14.15. Theorem. *Suppose $\mathcal{M}$ is a compact aspherical n-dimensional Riemannian manifold. Then*

$$\operatorname{sys}\mathcal{M} \leqslant 6\cdot \operatorname{width}\mathcal{M}.$$

14.16. Lemma. *Let K be an aspherical space and $\mathcal{W}$ a connected CW-complex. Denote by $\mathcal{W}^k$ the k-skeleton of $\mathcal{W}$. Then any continuous map $f\colon \mathcal{W}^2 \to K$ can be extended to a continuous map $\bar f\colon \mathcal{W} \to K$*

Moreover, if $p \in \mathcal{W}$ is a 0-cell and $q \in K$. Then a continuous maps of pairs $\varphi_0, \varphi_1\colon (\mathcal{W}, p) \to (K, q)$ are homotopic if and only if

φ_0 and φ_1 induce the same homomorphism on fundamental groups $\pi_1(\mathcal{W}, p) \to \pi_1(K, q)$.

Proof. Since K is aspherical, any continuous map $\partial\mathbb{D}^n \to K$ for $n \geqslant 3$ is hull-homotopic; that is, it can be extended to a map $\mathbb{D}^n \colon \to K$.

It makes it possible to extend f to $\mathcal{W}^3$, $\mathcal{W}^4$, and so on. Therefore f can be extended to whole $\mathcal{W}$.

The only-if part of the second part of lemma is trivial; it remains to show the if part.

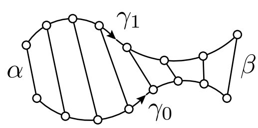
Sine $\mathcal{W}$ is connected, we can assume that p forms the only 0-cell in $\mathcal{W}$; otherwise, we can collapse a maximal subtree of the 1-skeleton in $\mathcal{W}$ to p. Therefore, $\mathcal{W}^1$ is formed by loops that generate $\pi_1(\mathcal{W}, p)$.

By assumption, the restrictions of φ_0 and φ_1 to $\mathcal{W}^1$ are homotopic. In other words the homotopy $\Phi \colon [0,1] \times \mathcal{W}$ is defined on the 2-skeleton of $[0,1] \times \mathcal{W}$. It remains to apply the first part of the lemma to the product $[0,1] \times \mathcal{W}$. ☐

14.17. Lemma. *Suppose γ_0, γ_1 are two paths between points in a Riemannian space $\mathcal{M}$ such that $|\gamma_0(t) - \gamma_1(t)|_{\mathcal{M}} < r$ for any $t \in [0,1]$. Let α be a shortest path from $\gamma_0(0)$ to $\gamma_1(0)$ and β be a shortest path from $\gamma_0(1)$ to $\gamma_1(1)$. If $2{\cdot}r < \operatorname{sys}\mathcal{M}$, then there is a homotopy γ_t from γ_0 to γ_1 such that $\alpha(t) \equiv \gamma_t(0)$ and $\beta(t) \equiv \gamma_t(1)$.*

Proof. Set $s = \operatorname{sys}\mathcal{M}$; since $2{\cdot}r < s$, we have that $\varepsilon = \frac{1}{10}(s - 2{\cdot}r) > 0$.

Note that we can assume that γ_0 and γ_1 are rectifiable; if not we can homotopy each into a broken geodesic line kipping the assumptions true.



Choose a fine partition $0 = t_0 < t_1 < \dots$ $\dots < t_n = 1$. Consider a sequence of shortest paths α_i from $\gamma_0(t_i)$ to $\gamma_1(t_i)$. We can assume that $\alpha_0 = \alpha$, $\alpha_n = \beta$, and each arc $\gamma_j|_{[t_{i-1},t_i]}$ has length smaller than ε. Therefore, every quadrilateral formed by concatenation of α_{i-1}, $\gamma_1|_{[t_{i-1},t_i]}$, reversed α_i, and reversed arc $\gamma_0|_{[t_{i-1},t_i]}$ has length smaller than s. It follows that this curve is contractible. Applying this observation for each quadrilateral, we get the statement. ☐

Proof of 14.15. Let $\mathcal{N}$ be the nerve of a covering $\{V_i\}$ of $\mathcal{M}$ and $\boldsymbol{\psi} \colon \mathcal{M} \to \mathcal{N}$ be the map provided by 14.2. As usual, we denote by v_i the vertex of $\mathcal{N}$ that corresponds to V_i. Observe that $\dim\mathcal{N} < n$; therefore, $\boldsymbol{\psi}$ kills the fundamental class of $\mathcal{M}$.

Let us construct a continuous map $f \colon \mathcal{N} \to \mathcal{M}$ such that $f \circ \boldsymbol{\psi}$ is homotopic to the identity map on $\mathcal{M}$. Note that once f is constructed, the theorem is proved. Indeed, since $\boldsymbol{\psi}$ kills the fundamental class $[\mathcal{M}]$ of $\mathcal{M}$, so does $f\circ\boldsymbol{\psi}$. Therefore, $[\mathcal{M}]=0$ — a contradiction.

Set $R=\operatorname{width}\mathcal{M}$ and $s=\operatorname{sys}\mathcal{M}$. Assume we choose $\{V_i\}$ as in the definition of width (14.3). For each i choose a point $p_i\in\mathcal{M}$ such that $V_i\subset \mathrm{B}(p_i,R)$.

Set $f(v_i)=p_i$ for each i. It defines the map f on the 0-skeleton $\mathcal{N}^0$ of the nerve $\mathcal{N}$. Further, f will be defined step by step on the skeletons $\mathcal{N}^1,\mathcal{N}^2,\dots$ of $\mathcal{N}$.

Let us map each edge $[v_iv_j]$ in $\mathcal{N}$ to a shortest path $[p_ip_j]$. It defines f on $\mathcal{N}^1$. Note that image of each edge is shorter than $2\cdot R$.

Suppose $[v_iv_jv_k]$ is a triangle in $\mathcal{N}$. Note that perimeter of the triangle $[p_ip_jp_k]$ can not exceed $6\cdot R$. Since $6\cdot R<s$, the contour of $[p_ip_jp_k]$ is contractible. Therefore, we can extend f to each triangle of $\mathcal{N}$. It defines the map f on $\mathcal{N}^2$.

Finally, since $\mathcal{M}$ is aspherical, by 14.16, the map f can be extended to $\mathcal{N}^3$, $\mathcal{N}^4$ and so on.

It remains to show that $f\circ\boldsymbol{\psi}$ is homotopic to the identity map. Choose a CW structure on $\mathcal{M}$ with sufficiently small cells, so that each cell lies in one of V_i. Note that $\boldsymbol{\psi}$ is homotopic to a map $\boldsymbol{\psi}_1$ that sends $\mathcal{M}^k$ to $\mathcal{N}^k$ for any k. Moreover, we may assume that (1) if a 0-cell x of $\mathcal{M}$ maps to a v_i, then $x\in V_i$ and (2) each 1-cell of $\mathcal{M}$ maps to an edge or a vertex of $\mathcal{N}$. Choose a 1-cell e in $\mathcal{M}$; by the construction, $f\circ\boldsymbol{\psi}_1$ maps e to a shortest path $[p_ip_j]$ and e lies $\mathrm{B}(p_i,R)$. Observe that $[p_ip_j]$ is shorter than $2\cdot R$. It follows that the distance between points on $[p_ip_j]$ and e can not exceed $3\cdot R$. Choose a shortest path α_i from every 0 cell x_i of $\mathcal{M}$ to $p_j=f\circ\boldsymbol{\psi}_1(x_i)$. It defines a homotopy on $\mathcal{M}^0$. Since $6\cdot R<s$, 14.17 implies that this homotopy can be extended to $\mathcal{M}^1$. By 14.16, it can be extended to whole $\mathcal{M}$. $\square$

14.18. Exercise. *Analyze the proof of 14.15 and improve its inequality to*

$$\operatorname{sys}\mathcal{M}\leqslant 4\cdot\operatorname{width}\mathcal{M}.$$

14.19. Exercise. *Modify the proof of 14.15 to prove the following:*

Suppose that $\mathcal{M}$ is a closed n-dimensional Riemannian manifold with injectivity radius at least r; that is, if $|p-q|_{\mathcal{M}}<r$, then a shortest path $[pq]_{\mathcal{M}}$ is uniquely defined. Show that

$$\operatorname{width}\mathcal{M}\geqslant \tfrac{r}{2\cdot(n+1)}.$$

Use 14.8 to conclude that $\operatorname{vol}\mathcal{M}\geqslant\varepsilon_n\cdot r^n$ *for some* $\varepsilon_n>0$ *that depends only on n.*

The second statement in the exercise is a theorem of Marcel Berger [20]; an inequality with optimal constant (with equality for round sphere) was obtained by Marcel Berger and Jerry Kazdan [19].

G Essential manifolds

To generalize 14.15 further, we need the following definition.

14.20. Definition. *A closed manifold M is called essential if it admits a continuous map $\iota\colon M\to K$ to an aspherical CW-complex K such that ι sends the fundamental class of M to a nonzero homology class in K.*

Note that any closed aspherical manifold is essential — in this case one can take ι to be the identity map on M.

The real projective space $\mathbb{R}\mathrm{P}^n$ provides an interesting example of an essential manifold which is not aspherical. Indeed, the infinite dimensional projective space $\mathbb{R}\mathrm{P}^\infty$ is aspherical and for the natural embedding $\mathbb{R}\mathrm{P}^n\hookrightarrow\mathbb{R}\mathrm{P}^\infty$ the image $\mathbb{R}\mathrm{P}^n$ does not bound in $\mathbb{R}\mathrm{P}^\infty$. The following exercise provides more examples of that type:

14.21. Exercise. *Show that the connected sum of an essential manifold with any closed manifold is essential.*

14.22. Exercise. *Show that the product of two essential manifolds is essential.*

Assume that the manifold M is essential and $\iota\colon M\to K$ as in the definition. Following the proof of 14.15, we can homotope the map $f\circ\boldsymbol{\psi}\colon M\to M$ to the identity on the 2-skeleton of M; further since K is aspherical, we can homotope the composition $\iota\circ f\circ\boldsymbol{\psi}$ to ι. Existence of this extension implies that ι kills the fundamental class of M — a contradiction. So, taking 14.18 into account, we proved the following generalization of 14.15:

14.23. Theorem. *Suppose $\mathcal{M}$ is an essential Riemannian space. Then*

$$\operatorname{sys}\mathcal{M}\leqslant 4\cdot\operatorname{width}\mathcal{M}.$$

As a corollary from 14.23 and 14.8 we get the so-called Gromov's systolic inequality:

14.24. Theorem. *Suppose $\mathcal{M}$ is an essential n-dimensional Riemannian space. Then*

$$\operatorname{sys}\mathcal{M}\leqslant 4\cdot n\cdot\sqrt[n]{\operatorname{vol}\mathcal{M}}.$$

H Remarks

Theorem 14.24 was proved originally by Mikhael Gromov [62] with a worse constant. The given proof is a result of a sequence of simplifications given by Larry Guth [67], Panos Papasoglu [106], Alexander Nabutovsky and Roman Karasev [102].

The calculations could be done better; namely we could get

$$\operatorname{width} \mathcal{P} \leqslant c_n \cdot \sqrt[n]{\operatorname{vol} \mathcal{P}},$$

where $c_n = \sqrt[n]{n!/2} = \frac{n}{e} + o(n)$ [102]. As a result, we may get a stronger statement in 14.24:

$$\operatorname{sys} \mathcal{M} \leqslant 4 \cdot c_n \cdot \sqrt[n]{\operatorname{vol} \mathcal{M}}.$$

For any nonessential oriented manifold M there is a metric with fixed volume and arbitrary small systole. This statement is proved by Ivan Babenko [14].

A wide open conjecture says that for any n-dimensional essential manifold we have

➊
$$\frac{\operatorname{sys} \mathcal{M}}{\sqrt[n]{\operatorname{vol} \mathcal{M}}} \leqslant \frac{\operatorname{sys} \mathbb{RP}^n}{\sqrt[n]{\operatorname{vol} \mathbb{RP}^n}},$$

where we assume that the n-dimensional real projective space $\mathbb{RP}^n$ is equipped with a canonical metric. In other words, the ratio in the right-hand side of ➊ is the optimal constant in the Gromov's systolic inequality; this ratio grows as $O(\sqrt{n})$. (The ratio for n-dimensional flat torus grows as $O(\sqrt{n})$ as well.)

Semisolutions

1.2. Add four triangle inequalities (1.1d).

1.4. Consider the function

$$f(x) = \frac{\mathrm{dist}_A x}{\mathrm{dist}_A x + \mathrm{dist}_B x},$$

where $\mathrm{dist}_A x := \inf_{a \in A} |a - x|$. Show that f is continuous and satisfies the needed property.

1.5. Use 1.4 to construct an approximation of the needed function and pass to a limit or find a proof of the Tietze extension theorem.

1.6;*(a)*. Note that if $\mu(A) = \mu(B) = 0$, then $|A - B| = 0$. Therefore, 1.1b does not hold for bounded closed subsets. It is straightforward to check the remaining conditions in 1.1 hold true.

(b). Note that the distance from the empty set to the whole plane is infinite; so the value $|A-B|$ might be infinite. It is straightforward to check the remaining conditions in 1.1.

Remark. Metrics of the form $|A-B| = \mu(A \triangle B)$ are very special. In particular, they satisfy the so-called hypermetric inequalities; that is, for any sequence of sets $A_1, \dots, A_n$ and any sequence of integers $b_1, \dots, b_n$ such that $\sum_i b_i = 1$ we have

$$\sum_{i,j} b_i \cdot b_j \cdot |A_i - A_j| \leqslant 0.$$

Note that for $n = 3$ and $b_1 = b_2 = -b_3 = 1$ we get the usual triangle inequality. For more on the subject, see [55].

1.7. Choose $\delta > 0$ and an increasing linear bijection $\ell \colon [a, b] \to [c, d]$. Show that ℓ has arbitrarily close increasing piecewise-linear bijection $s \colon [a, b] \to [c, d]$ such that derivative at any point is either $< \delta$ or $> \frac{1}{\delta}$.

Start with the identity map $[0, 1] \to [0, 1]$; iterate the above construction for smaller and smaller δ and pass to the limit. This way we obtain an increasing bijection $x \leftrightarrow x'$ from $[0, 1]$ to itself such that for any $\varepsilon > 0$ there is a partition $0 = t_0 < t_1 < \dots < t_{2 \cdot n} = 1$ of $[0, 1]$ with

$$\varepsilon > |t_0 - t_1| + |t_1' - t_2'| + |t_2 - t_3| + \dots$$
$$\dots + |t_{2 \cdot n-2} - t_{2 \cdot n-1}| + |t_{2 \cdot n-1}' - t_{2 \cdot n}'|.$$

Make a conclusion.

1.8. Assume the statement is wrong. Then for any point $x \in \mathcal{X}$, there is a point $x' \in \mathcal{X}$ such that

$$|x - x'| < \rho(x)$$

and

$$\rho(x') \leqslant \frac{\rho(x)}{1 + \varepsilon}.$$

Consider a sequence $x_1, x_2, \dots \in \mathcal{X}$ such that $x_{n+1} = x_n'$. Show that this is a Cauchy sequence. Since $\mathcal{X}$ is complete, x_n converges; denote its limit by x_∞. Since ρ is a continuous function we get

$$\rho(x_\infty) = \lim_{n \to \infty} \rho(x_n) = 0.$$

The latter contradicts that $\rho > 0$.

1.9. Let $\bar{\mathcal{X}}$ be the completion of $\mathcal{X}$. By the definition, for any $y \in \bar{\mathcal{X}}$ there is a Cauchy sequence x_n in $\mathcal{X}$ that converges to y.

Choose a Cauchy sequence y_m in $\bar{\mathcal{X}}$. From above, we can choose points $x_{n,m} \in \mathcal{X}$ such that $x_{n,m} \to y_m$ for any m. Choose $z_m = x_{n_m,m}$ such that $|y_m - z_m| < \frac{1}{m}$. Observe that z_m is Cauchy. Therefore, its limit z_∞ lie in $\bar{\mathcal{X}}$. Finally, show that $x_m \to z_\infty$.

1.13. A compact ε-net N in $\mathcal{K}$ contains a finite ε net F. Show and use that F is a $2 \cdot \varepsilon$-net of $\mathcal{K}$.

1.15. Given a pair of points $x_0, y_0 \in \mathcal{K}$, consider two sequences $x_0, x_1, \dots$ and $y_0, y_1, \dots$

such that $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ for each n.

Since $\mathcal{K}$ is compact, we can choose an increasing sequence of integers n_k such that both sequences $(x_{n_i})_{i=1}^{\infty}$ and $(y_{n_i})_{i=1}^{\infty}$ converge. In particular, both are Cauchy; that is,

$$|x_{n_i} - x_{n_j}|_{\mathcal{K}} \to 0 \quad \text{and} \quad |y_{n_i} - y_{n_j}|_{\mathcal{K}} \to 0$$

as $\min\{i,j\} \to \infty$.

Since f is distance-noncontracting,

$$|x_0 - x_{|n_i - n_j|}| \leqslant |x_{n_i} - x_{n_j}|$$

for any i and j. Therefore, there is a sequence $m_i \to \infty$ such that

$$(*) \qquad x_{m_i} \to x_0 \quad \text{and} \quad y_{m_i} \to y_0$$

as $i \to \infty$.

Since f is distance-noncontracting, the sequence $\ell_n = |x_n - y_n|_{\mathcal{K}}$ is nondecreasing. By $(*)$, $\ell_{m_i} \to \ell_0$ as $m_i \to \infty$. It follows that

$$\ell_0 = \ell_1 = \dots$$

In particular,

$$|x_0 - y_0|_{\mathcal{K}} = \ell_0 = \ell_1 = |f(x_0) - f(y_0)|_{\mathcal{K}}$$

for any pair of points (x_0, y_0) in $\mathcal{K}$. That is, the map f is distance-preserving; hence f is injective. From $(*)$, we also get that $f(\mathcal{K})$ is everywhere dense. Since $\mathcal{K}$ is compact $f\colon \mathcal{K} \to \mathcal{K}$ is surjective — hence the result.

Remarks. This is a basic lemma in the introduction to Gromov–Hausdorff distance [see 7.3.30 in 34]. The presented proof is not quite standard, I learned it from Travis Morrison, a student in my MASS class at Penn State, Fall 2011.

Note that this exercise implies that *any surjective non-expanding map from a compact metric space to itself is an isometry.*

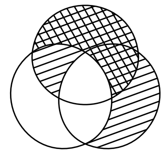
1.16. Check an infinite set with a discrete metric.

1.17. Set $B_p = B(x, \frac{\pi}{2})_{\mathbb{S}^2}$. The triangle inequality follows since

$$(*) \qquad (B_x \setminus B_y) \cup (B_y \setminus B_z) \supseteq B_x \setminus B_z.$$

The remaining conditions in Definition 1.1 are evident.

Observe that $B_x \setminus B_y$ does not overlap with $B_y \setminus B_z$ and we get equality in $(*)$ if and only if y lies on the great circle arc from x to z. Therefore, the second statement follows.



Remarks. This construction is due to Aleksei Pogorelov [114]. It is closely related to the construction given by David Hilbert [73] which was the motivating example for his fourth problem. See also the remark after the solution of 1.6.

1.18. We may assume that none of the points p, x, y, z lies on a geodesic between the other two.

Let K be the set in the tree covered by all six geodesics with the endpoints p, x, y, z. Observe that K looks like an H or like an X; make a conclusion.

*Remarks.*The value $\frac{1}{2}\cdot(|p-x|+|p-y|-|x-y|)$ is called Gromov's product of x and y with the origin at p; usually it is denoted by $(x|y)_p$.

Note that a four-point metric space admits an isometric embedding into a metric tree if and only if one of these two equivalent conditions holds. Moreover, a metric space admits an isometric embedding into a metric tree if every its four-point subspace admits such embedding.

1.19. Apply 1.18.

1.22. Note that $\mathcal{P}$ is complete. Choose $\varepsilon > 0$. Use 1.21 to show that the set of paths of length $> 1 - \varepsilon$ is open in $\mathcal{P}$; show that this set is also dense in $\mathcal{P}$. Apply Baire's theorem (1.10).

Remark. You might find it surprising that *most of the short maps from the sphere to the plane are length-preserving*; that is, they preserve lengths of all curves. The latter follows from the result of Bernd Kirchheim, Emanuele Spadaro, and László Székelyhidi [86]. (While most of the maps have this property, it is not at all easy to construct a single such example.)

1.24. *Formally speaking, a one-point space is a solution, but we will construct a nontrivial example.*

Recall that $c_0 \subset \ell^\infty$ denotes the space of all real sequences converging to zero. Consider the unit ball B in c_0; denote by ρ_0 the metric on B.

Let

$$\varphi(\boldsymbol{x}) = 2 + \tfrac{1}{2}\cdot x_1 + \tfrac{1}{4}\cdot x_2 + \tfrac{1}{8}\cdot x_3 + \dots,$$

where $\boldsymbol{x} = (x_1, x_2 \dots) \in B$. Consider another length metric ρ_1 on B that is different from ρ_0 by the conformal factor φ; that is, if $t \mapsto \boldsymbol{x}(t)$ for

$t \in [0, \ell]$ is a curve parametrized by ρ_0-length, then its ρ_1-length is defined by

$$\operatorname{length}_{\rho_1} \boldsymbol{x} := \int\limits_0^\ell \varphi \circ \boldsymbol{x}(t) \cdot dt.$$

Note that the metric ρ_1 is bilipschitz to ρ_0.

Assume $t \mapsto \boldsymbol{x}(t)$ and $t \mapsto \boldsymbol{x}'(t)$ are two curves parametrized by ρ_0-length that differ only in the m-th coordinate; denote them by $x_m(t)$ and $x'_m(t)$ respectively. Show that if $x'_m(t) \leqslant x_m(t)$ for any t and the function $x'_m(t)$ is locally 1-Lipschitz at all t such that $x'_m(t) < x_m(t)$, then

$$\operatorname{length}_{\rho_1} \boldsymbol{x}' \leqslant \operatorname{length}_{\rho_1} \boldsymbol{x}.$$

Moreover, this inequality is strict if $x'_m(t) < x_m(t)$ for some t.

Fix a curve $\boldsymbol{x}(t)$, $t \in [0, \ell]$, parametrized by ρ_0-length. We can choose large m so that $x_m(t)$ is sufficiently close to 0 for any t. In this case, it is easy to construct a function $t \mapsto x'_m$ that meets the above conditions. It follows that for any curve $\boldsymbol{x}(t)$ in (B, ρ_1), we can find a shorter curve $\boldsymbol{x}'(t)$ with the same endpoints. In particular, (B, ρ_1) has no geodesics.

Remark. This solution was suggested by Fedor Nazarov [103].

1.25. Choose a sequence of positive numbers $\varepsilon_n \to 0$ and a finite ε_n-net N_n of K for each n. We can assume that $\varepsilon_0 > \operatorname{diam} K$, and N_0 is a one-point set. If $|x - y| < \varepsilon_k$ for some $x \in N_{k+1}$ and $y \in N_k$, then connect them by a curve of length at most ε_k.

Let K' be the union of all these curves and K. Show that K' is compact and path-connected.

Source: This problem is due to Eugene Bilokopytov [23].

1.26. Choose a Cauchy sequence x_n in $(\mathcal{X}, \|*-*\|)$; it is sufficient to show that a subsequence of x_n converges.

Observe that the sequence x_n is Cauchy in $(\mathcal{X}, |*-*|)$; denote its limit by x_∞.

Passing to a subsequence, we can assume that $\|x_n - x_{n+1}\| < \frac{1}{2^n}$. It follows that there is a 1-Lipschitz path γ in $(\mathcal{X}, \|*-*\|)$ such that $x_n = \gamma(\frac{1}{2^n})$ for each n and $x_\infty = \gamma(0)$. Therefore,

$$\|x_\infty - x_n\| \leqslant \operatorname{length} \gamma|_{[0, \frac{1}{2^n}]} \leqslant \tfrac{1}{2^n}.$$

In particular, x_n converges to x_∞ in $(\mathcal{X}, \|*-*\|)$.

Source: [74, Corollary]; see also [111, Lemma 2.3].

1.28. Choose two points $x, y \in \mathcal{X}$; let $\ell = |x - y|$. Suppose $f\colon E \to \mathcal{X}$ is a distance-preserving map such that $0, \ell \in E \subset [0, \ell]$, $f(0) = x$, and $f(\ell) = y$.

Show that we can choose f so that E is maximal; that is, f cannot be extended to a distance-preserving map on a larger subset of $[0, \ell]$.

Show that there is no open interval (a, b) in the complement of E such that $a, b \in E$.

Apply the completeness of $\mathcal{X}$ to show that E is closed. Conclude that $E = [0, \ell]$.

1.31. Let U be the ε-neighborhood of $\mathrm{B}(x, R)_{\mathcal{X}}$. By the triangle inequality, $U \subset \mathrm{B}(x, R + \varepsilon)_{\mathcal{X}}$; this inclusion holds in any metric space.

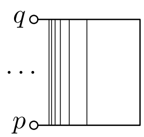
Choose $y \in \mathrm{B}(x, R + \varepsilon)_{\mathcal{X}}$, so $|x - y|_{\mathcal{X}} < R + \varepsilon$. Since $\mathcal{X}$ is a length space, there is a curve γ from x to y with length less than $R + \varepsilon$. Show and use that γ contains a point m such that $|x - m|_{\mathcal{X}} < R$ and $|y - m|_{\mathcal{X}} < \varepsilon$.

1.32. Consider the following subset of $\mathbb{R}^2$ equipped with the induced length metric

$$\mathcal{X} = \big((0, 1] \times \{0, 1\}\big) \cup \big(\{1, \tfrac12, \tfrac13, \dots\} \times [0, 1]\big)$$

Note that $\mathcal{X}$ is locally compact and geodesic.

Its completion $\bar{\mathcal{X}}$ is isometric to the closure of $\mathcal{X}$ equipped with the induced length metric. Note that $\bar{\mathcal{X}}$ is obtained from $\mathcal{X}$ by adding two points $p = (0, 0)$ and $q = (0, 1)$.



Observe that p admits no compact neighborhood in $\bar{\mathcal{X}}$ and there is no geodesic connecting p to q in $\bar{\mathcal{X}}$.

Source: [31, I.3.6(4)].

1.33. Choose a point $p \in \mathcal{X}$. Let r_0 be the least upper bound for radii $r > 0$ such that the closed ball $\overline{\mathrm{B}}[p, r]$ is compact. Suppose $\overline{\mathrm{B}}[p, r_0]$ is not compact. Choose a sequence of points $x_n \in \overline{\mathrm{B}}[p, r_0]$ that has no converging subsequence. Consider geodesic paths γ_n from p to x_n. Pass to a subsequence of x_n so that $\gamma_n(t)$ converges for any $t \in [0, 1)$; let $\gamma_\infty\colon [0, 1) \to \mathcal{X}$ be the limit curve. By the assumption, γ_∞ can be are extended to $[0, 1]$. Show that $x_\infty = \gamma_\infty(1)$ is a limit of x_n and arrive at a contradiction.

Further argue as in the proof of the Hopf–Rinow theorem.

Source: Communicated by Nina Lebedeva.

1.34. Suppose this number does not exist. Show that there are two point-arrays $(x_1,\dots\dots,x_n)$ and $(y_1,\dots,y_m)$ such that

$$(*)\qquad \min_{z\in\mathcal{X}}\{\,f(z)\,\} > \max_{z\in\mathcal{X}}\{\,h(z)\,\},$$

where

$$f(z)=\tfrac{1}{n}\cdot\sum_i|x_i-z|_{\mathcal{X}}$$

and

$$h(z)=\tfrac{1}{m}\cdot\sum_j|y_j-z|_{\mathcal{X}}.$$

Note that

$$\begin{aligned}\tfrac{1}{m}\cdot\sum_j f(y_j)&=\tfrac{1}{m\cdot n}\cdot\sum_{i,j}|x_i-y_j|_{\mathcal{X}}=\\&=\tfrac{1}{n}\cdot\sum_i h(x_i);\end{aligned}$$

that is, the average value of $f(y_j)$ coincides with the average value of $h(x_i)$. The latter contradicts $(*)$.

Remark. The value ℓ is uniquely defined; it is called the rendezvous value of $\mathcal{X}$. This is a result of Oliver Gross [66].

2.2. By the Fréchet lemma (2.1) we can identify $\mathcal{K}$ with a compact subset in ℓ^∞.

Denote by $\mathcal{L}$ the closed convex hull of $\mathcal{K}$; that is, $\mathcal{L}$ is the minimal convex closed set in ℓ^∞ that contains $\mathcal{K}$. (In other words, $\mathcal{L}$ is the minimal closed set containing $\mathcal{K}$ such that if $x,y\in\mathcal{L}$, then $t\cdot x+(1-t)\cdot y\in\mathcal{L}$ for any $t\in[0,1]$.)

Observe that $\mathcal{L}$ is a length space. It remains to show that $\mathcal{L}$ is compact.

By construction, $\mathcal{L}$ is a closed subset of ℓ^∞; in particular, it is complete. By 1.11*c*, it remains to show that $\mathcal{L}$ is totally bounded.

Recall that Minkowski sum $A+B$ of two sets A and B in a vector space is defined by

$$A+B:=\{\,a+b\;:\;a\in A,\;b\in B\,\}.$$

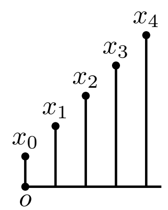
Observe that the Minkowski sum of two convex sets is convex.

Denote by $\bar B_\varepsilon$ the closed ε-ball in ℓ^∞ centered at the origin. Choose a finite ε-net N in $\mathcal{K}$ for some $\varepsilon>0$. Note that $P=\operatorname{Conv}N$ is a convex polyhedron; in particular, $\operatorname{Conv}N$ is compact.

Observe that $N+\bar B_\varepsilon$ is a closed ε-neighborhood of N. It follows that $N+\bar B_\varepsilon\supset K$, and therefore, $P+\bar B_\varepsilon\supset\mathcal{L}$. In particular, P is a $2{\cdot}\varepsilon$-net in $\mathcal{L}$; since P is compact and $\varepsilon>0$ is arbitrary, $\mathcal{L}$ is totally bounded (see 1.13).

Remark. Alternatively, one may use that *the injective envelope of a compact space is compact*; see 3.3*b*, 3.20, and 3.23.

2.3. Modify the proof of 2.1.



2.8. Consider the metric tree $\mathcal{T}$ shown on the diagram; it is a half-line $[0,\infty)$ with attached an interval of length $n+1$ to each integer $n\geqslant 0$. Denote by o the origin of the half-line and by x_n the endpoint of n^{th} interval.

Observe that if $m\neq n$, then

$$|x_m-x_n|_{\mathcal{T}}\geqslant|o-x_n|_{\mathcal{T}}+1.$$

Show and use that for any binary sequence ε_n there is an extension function f such that

$$f(x_n)=|o-x_n|_{\mathcal{T}}+\varepsilon_n.$$

Remark. An if-and-only-if condition on $\mathcal{X}$ that have separable $\mathcal{X}^\infty$ was found by Julien Melleray [96, 2.8]. A similar condition was used by Herbert Federer to describe metric spaces where Besicovitch covering lemma holds [57, 2.8.9].

2.13. Choose a separable space $\mathcal{X}$ that has an infinite number of geodesics between a pair of points with the given distance between them; say a square in $\mathbb{R}^2$ with ℓ^∞-metric will do. Apply to $\mathcal{X}$ universality of Urysohn space (2.12).

2.14. First let us prove the following claim:

◇ Suppose $f\colon K\to\mathbb{R}$ is an extension function defined on a compact subset K of the Urysohn space $\mathcal{U}$. Then there is a point $p\in\mathcal{U}$ such that $|p-x|=f(x)$ for any $x\in K$.

Without loss of generality, we may assume that $f>0$. Since K is compact, we may fix $\varepsilon>0$ such that $f(x)>\varepsilon$ for any $x\in K$.

Consider the sequence $\varepsilon_n=\frac{\varepsilon}{100\cdot 2^n}$. Choose a sequence of ε_n-nets $N_n\subset K$. Applying the universality of $\mathcal{U}$ recursively, we may choose a point p_n such that $|p_n-x|=f(x)$ for any $x\in N_n$ and $|p_n-p_{n-1}|=10{\cdot}\varepsilon_{n-1}$. Observe that the sequence p_n is Cauchy and its limit p meets $|p-x|=f(x)$ for any $x\in K$.

Now, choose a sequence x_n of points that is dense in $\mathcal{S}$. Applying the claim, we may extend the map from K to $K\cup\{x_1\}$, further to $K\cup\{x_1,x_2\}$, and so on. As a result, we extend the distance-preserving map f to the whole sequence x_n. It remains to extend it continuously to the whole space $\mathcal{S}$.

2.16. It is sufficient to show that any compact subspace $\mathcal{K}$ of the Urysohn space $\mathcal{U}$ can be contracted to a point.

Note that any compact space $\mathcal{K}$ can be extended to a contractible compact space $\mathcal{K}'$; for example, we may embed $\mathcal{K}$ into ℓ^∞ and pass to its convex hull, as it was done in 2.2.

By 2.20, there is an isometric embedding of $\mathcal{K}'$ that agrees with the inclusion $\mathcal{K}\hookrightarrow\mathcal{U}$. Since $\mathcal{K}$ is contractible in $\mathcal{K}'$, it is contractible in $\mathcal{U}$.

A better way. One can contract the whole Urysohn space using the following construction.

Note that points in $\mathcal{X}_\infty$ constructed in the proof of 2.7 can be multiplied by $t\in[0,1]$ — simply multiply each function by t. That defines a map

$$\lambda_t\colon\mathcal{X}_\infty\to\mathcal{X}_\infty$$

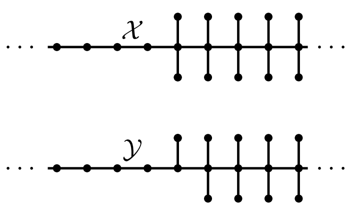
that rescales all distances by factor t. The map λ_t can be extended to the completion of $\mathcal{X}_\infty$, which is isometric to $\mathcal{U}_d$ (or $\mathcal{U}$).

Observe that the map λ_1 is the identity and λ_0 maps the whole space to a single point, say x_0 — this is the only point of $\mathcal{X}_0$. Further, note that $(t,p)\mapsto\lambda_t(p)$ is a continuous map; in particular, $\mathcal{U}_d$ and $\mathcal{U}$ are contractible.

As a bonus, observe that for any point $p\in\mathcal{U}_d$ the curve $t\mapsto\lambda_t(p)$ is a geodesic path from p to x_0.

Source: [65, (d) on page 82].

2.18. Consider two infinite metric trees as on the diagram.



Remark. A more sophisticated example: $\mathcal{X}=\ell^\infty$ and $\mathcal{Y}=L^\infty([0,1])$. Try to prove that it qualifies; see also [32].

2.19; *(a)* and *(b)*. Observe that L and M satisfy the definition of d-Urysohn space and apply the uniqueness (2.17). Note that

$$\ell=\operatorname{diam}L=\min\{2\cdot r,d\}.$$

(c). Use *(a)*, maybe twice.

2.21. Let p be the center of the sphere; without loss of generality, we can assume that $|p-x|\leqslant|p-y|$.

Consider function $f\colon\{p,x,y\}\to\mathbb{R}$ defined by $f(p)=1$, $f(x)=1+|p-x|$, and $f(y)=1+|p-y|-\varepsilon$. Suppose $\varepsilon>0$ is sufficiently small; show that f is an extension function on $\{p,x,y\}$.

By the extension property, there is a point $z\in\mathcal{U}$ such that $|p-z|=f(p)$, $|x-z|=f(x)$, and $|y-z|=f(y)$. Whence the statement follows.

Source: This problem is taken from a survey of Julien Melleray [96, Prop. 4.3], where it was attributed to Matatyahu Rubin.

2.22. Observe that the complement $\mathcal{V}=\mathcal{U}\setminus B$ is complete. Show that it $\mathcal{V}$ satisfies the extension property. Conclude that $\mathcal{V}$ is an Urysohn space and apply 2.17.

For the second part, observe that there is an isometry $\iota\colon\mathcal{U}\to\mathcal{V}$. Moreover, if p is the center of B, then we can assume that ι has a fixed point x such that $|p-x|>2$.

Consider the unit sphere S centered at x. The restriction of ι to S is an isometry of S. Use 2.21 to show that it cannot be extended to an isometry of $\mathcal{U}$.

Source: [96, Sec. 4.4].

2.23. Apply 2.17 and the construction in 2.11.

2.24; *(a).* The Euclidean plane is homogeneous in every sense.

(b). The Hilbert space ℓ^2 is finite-set-homogeneous, but not compact-set-homogeneous, nor countable-set-homogeneous.

(c). ℓ^∞ is one-point-homogeneous, but not two-point-homogeneous. Try to show that there is no isometry of ℓ^∞ such that

$$\begin{aligned}(0,0,0,\dots)&\mapsto(0,0,0,\dots),\\(1,1,1,\dots)&\mapsto(1,0,0,\dots).\end{aligned}$$

(d). ℓ^1 is one-point-homogeneous, but not two-point-homogeneous. Try to show that there is no isometry of ℓ^∞ such that

$$(0,0,0,\dots)\mapsto(0,0,0,\dots),$$
$$(2,0,0\dots)\mapsto(1,1,0,\dots).$$

2.25. Let $\mathcal{T}$ be a one-point-homogeneous metric tree. Note that all points in $\mathcal{T}$ have the same degree d; that is, for any point $t\in\mathcal{T}$ the set of connected components of the complement $\mathcal{T}\setminus\{t\}$ has the same cardinality d.

Show that if $d=0$, then $\mathcal{T}$ is a one-point space; there is no tree with $d=1$, and if $d=2$, then $\mathcal{T}\overset{iso}{=\!=}\mathbb{R}$.

Suppose $d\geqslant 3$. Choose a geodesic γ in $\mathcal{T}$. Show that the number of connected components of $\mathcal{T}\setminus\gamma$ has cardinality continuum. Observe and use that one can choose a point p_α in each connected component such that $|p_\alpha-p_\beta|_{\mathcal{T}}>1$ if $\alpha\ne\beta$.

2.26. Assume $F_{\mathcal{X}}$ is not an embedding. That is, there is a sequence of points $x_1,x_2,\dots$ and a point x_∞ such that $f_{x_n}\to f_{x_\infty}$ in $C(\mathcal{X},\mathbb{R})$ as $n\to\infty$, while $|x_n-x_\infty|_{\mathcal{X}}>\varepsilon$ for some fixed $\varepsilon>0$ and all n.

By 1.29, any pair of points $x,y\in\mathcal{X}$ can be connected by a minimizing geodesic $[xy]$. Choose $\bar x_n$ on a geodesic $[x_\infty x_n]$ such that $|x_\infty-\bar x_n|=\varepsilon$. Note that

$$f_{x_n}(x_\infty)-f_{x_n}(\bar x_n)=\varepsilon,$$
$$f_{x_\infty}(x_\infty)-f_{x_n}(\bar x_n)=-\varepsilon$$

for all n.

Since $\mathcal{X}$ is proper, we can pass to a subsequence of x_n so that the sequence $\bar x_n$ converges; denote its limit by $\bar x_\infty$. The above identities imply that

$$f_{x_n}(\bar x_\infty)\not\to f_{x_\infty}(\bar x_\infty)\quad\text{or}$$
$$f_{x_n}(x_\infty)\not\to f_{x_\infty}(x_\infty)$$

— a contradiction.

For the second part, take $\mathcal{Y}$ to be the set of nonnegative integers with the metric ρ defined by

$$\rho(m,n)=m+n$$

for $m\ne n$.

Source: I learned this example from Linus Kramer and Alexander Lytchak; it was also mentioned in the lectures of Anders Karlsson and attributed to Uri Bader [80, 2.3].

2.27. Suppose that our metric is $\sum a_S\cdot\delta_S$ with $a_S\geqslant 0$ for any $S\subset F$. Enumerate all the subsets $S_1,\dots,S_{2^n}$; set $S_i=F$ for all $i>2^n$. Consider the maps $x\mapsto(a_1,a_2,\dots)$ where $a_i=0$ if $x\in S_i$ and otherwise $a_i=1$. Observe that it defines a distance-preserving map $F\to\ell^1$.

The if part is proved. For the only-if part, check the statement for subsets of the real line, and use it.

2.28. Show that for any proper subset S in the vertex set there are three vertices x,y,z such that $|x-y|+|y-z|=|x-z|$ and either $x,z\in S$ and $y\notin S$, or $x,z\notin S$ and $y\in S$. Then apply 2.27.

2.29. For the first part, show and use that the quotient of $\mathbb{R}\mathrm{P}^2$ by the isotropy group of one point is isometric to a line segment.

For the second part, choose three points on a closed geodesic at equal distances from each other. Show and use that there is an isometric three-point set in $\mathbb{R}\mathrm{P}^2$ that does not lie on a closed geodesic.

Source: [40, V §2].

2.30. Denote by $\dim(x_1,\dots,x_m)$ the dimension of the minimal face of the cube that contains all the points $x_1,\dots,x_m\in Q$. Show and use that

$$\dim(x_1,\dots,x_m)=\dim(x_1',\dots,x_m')$$

for any isometry $x\mapsto x'$ of Q.

Source: [17, prop. 6 and 7].

3.1; *only-if part.* To check convexity, assume that B is a two-point subset. For closeness, assume that B is a countable set of A.

If part. Apply the Kirszbraun theorem together with the closest-point projection.

3.3. Choose an injective space $\mathcal{Y}$.

(a). Fix a Cauchy sequence x_n in $\mathcal{Y}$; we need to show that it has a limit $x_\infty\in\mathcal{Y}$. Consider metric on $\mathcal{X}=\mathbb{N}\cup\{\infty\}$ defined by

$$|m-n|_{\mathcal{X}}:=|x_m-x_n|_{\mathcal{Y}},$$
$$|m-\infty|_{\mathcal{X}}:=\lim_{n\to\infty}|x_m-x_n|_{\mathcal{Y}}.$$

Since the sequence is Cauchy, so is the sequence $\ell_n=|x_m-x_n|_{\mathcal{Y}}$ for any m. Therefore, the last limit is defined.

By construction, the map $n\mapsto x_n$ is distance-preserving on $\mathbb{N}\subset\mathcal{X}$. Since $\mathcal{Y}$ is injective, this map can be extended to ∞ as a short map; set $\infty \mapsto x_\infty$. Since $|x_n - x_\infty|_{\mathcal{Y}} \leqslant |n - -\infty|_{\mathcal{X}}$ and $|n-\infty|_{\mathcal{X}} \to 0$, we get that $x_n \to x_\infty$ as $n \to \infty$.

(b). Applying the definition of injective space, we get a midpoint for any pair of points in $\mathcal{Y}$. By *(a)*, $\mathcal{Y}$ is a complete space. It remains to apply Menger's lemma (1.27b).

(c). Let $k\colon \mathcal{Y} \hookrightarrow \ell^\infty(\mathcal{Y})$ be the Kuratowski embedding (2.4). Observe that $\ell^\infty(\mathcal{Y})$ is contractible; in particular, there is a homotopy $k_t\colon \mathcal{Y} \hookrightarrow \ell^\infty(\mathcal{Y})$ such that $k_0 = k$ and k_1 is a constant map. (In fact, one can take $k_t = (1-t)\cdot k$.)

Since k is distance-preserving and $\mathcal{Y}$ is injective, there is a short map $f\colon \ell^\infty(\mathcal{Y}) \to \mathcal{Y}$ such that the composition $f \circ k$ is the identity map on $\mathcal{Y}$. The composition $f \circ k_t\colon \mathcal{Y} \hookrightarrow \mathcal{Y}$ provides the needed homotopy.

3.4. By 2.4, the space $\mathcal{Y}$ can be considered as a subset in $\ell^\infty(\mathcal{Y})$. Let $r\colon \ell^\infty(\mathcal{Y}) \to \mathcal{Y}$ be the short retraction provided by the definition of injective space. Observe that $m(x,y) := r(\frac{x+y}{2})$ meets the condition.

Remark. The same argument can be used to construct the so-called geodesic bicombing on injective space — a useful tool introduced by Urs Lang [88, 3.6].

3.5. Suppose that a short map $f\colon A \to \mathcal{Y}$ is defined on a subset A of a metric space $\mathcal{X}$. We need to construct a short extension F of f. Without loss of generality, we may assume that $A \neq \varnothing$; otherwise, map the whole $\mathcal{X}$ to a single point. By Zorn's lemma, it is sufficient to enlarge A by a single point $x \notin A$.

(a). Suppose $\mathcal{Y} = \mathbb{R}$. Set

$$F(x) = \inf\{\, f(a) - |a - x| \;:\; a \in A\,\}.$$

Observe that F is short and $F(a) = f(a)$ for any $a \in A$.

(b). Suppose $\mathcal{Y}$ is a complete metric tree. Fix points $p \in \mathcal{X}$ and $q \in \mathcal{Y}$. Given a point $a \in A$, let $x_a \in \overline{\mathrm{B}}[f(a), |a-p|]$ be the point closest to $f(x)$. Note that $x_a \in [q\, f(a)]$ and either $x_a = q$ or x_a lies on distance $|a-p|$ from $f(a)$.

Note that the geodesics $[q\, x_a]$ are nested; that is, for any $a, b \in A$ we have either $[q\, x_a] \subset [q\, x_b]$ or $[q\, x_b] \subset [q\, x_a]$. Moreover, in the first case, we have $|x_b - f(a)| \leqslant |p - a|$ and in the second $|x_a - f(b)| \leqslant |p - b|$.

It follows that the closure of the union of all geodesics $[q\, x_a]$ for $a \in \mathcal{A}$ is a geodesic. Denote by x its endpoint; it exists since $\mathcal{Y}$ is complete. It remains to observe that $|x - f(a)| \leqslant |p - a|$ for any $a \in \mathcal{A}$; that is, one can take $f(p) = x$.

(c). Show and use that *any ℓ^∞-product of injective spaces is injective*; in particular, if $\mathcal{Y}$ and $\mathcal{Z}$ are injective, then so is the product $\mathcal{Y} \times \mathcal{Z}$ equipped with the metric

$$|(y,z)-(y',z')|_{\mathcal{Y}\times\mathcal{Z}} = \max\{\, |y-y'|_{\mathcal{Y}}, |z-z'|_{\mathcal{Z}}\,\}.$$

3.6; *(a).* Let $\mathcal{B} = \overline{\mathrm{B}}[o,R]_{\mathcal{Y}}$. Choose a metric space $\mathcal{X}$ with a subset A. Given a short map $f\colon A \to \mathcal{B}$ we need to find its short extension $\mathcal{X} \to \mathcal{B}$.

Since $\operatorname{diam}\mathcal{B} \leqslant 2\cdot R$, we may assume that $\operatorname{diam}\mathcal{X} \leqslant 2\cdot R$; if not pass to the metric defined by $|x-y| := \max\{\, |x-y|_{\mathcal{X}}, 2\cdot R\,\}$.

Let us add point w to $\mathcal{X}$ such that $|w - - x| = R$ for any $x \in \mathcal{X}$; denote the obtained space $\mathcal{X}'$. Let $f'\colon A \cup \{w\} \to \mathcal{B}$ be an extension of f by $w \mapsto o$; note that f' is short.

Since $\mathcal{Y}$ is injective, there is a short extension $F\colon \mathcal{X}' \to \mathcal{Y}$ of f'. Show and use that $F(\mathcal{X}') \subset \mathcal{B}$.

(b). Try to modify the argument in *(a)*. Namely, let $\mathcal{B} = \bigcap_\alpha \overline{\mathrm{B}}[o_\alpha, R_\alpha]_{\mathcal{Y}}$. Note that one may assume that $\operatorname{diam}\mathcal{X} \leqslant 2\cdot \inf_\alpha\{\, R_\alpha\,\}$. Consider the space $\mathcal{X}' = \mathcal{X} \cup \{w_\alpha\}$ such that $|w_\alpha - x| = R_\alpha$ for any $x \in \mathcal{X}$ and $|w_\alpha - w_\beta| = R_\alpha + R_\beta$ if $\alpha \neq \beta$. Further, consider an extension of f by $w_\alpha \mapsto o_\alpha$.

3.7. Let $\operatorname{diam}\mathcal{Y} = 2\cdot R$. We can assume that $R > 0$; otherwise, there is nothing to prove. Denote by $\mathcal{Z}$ a minimal (with respect to inclusion) intersection of closed R-balls in $\mathcal{Y}$ such that $s(\mathcal{Z}) \subset \mathcal{Z}$.

Consider the intersection

$$\mathcal{Y}' = \mathcal{Z} \cap \left(\bigcap_{p\in\mathcal{Z}} \overline{\mathrm{B}}[p,R]_{\mathcal{Y}}\right).$$

By 3.6b, $\mathcal{Y}'$ is injective. Use that $\mathcal{Z}$ is minimal to show that $s(\mathcal{Y}') \subset \mathcal{Y}'$. Show that $\operatorname{diam}\mathcal{Y}' \leqslant \frac{1}{2}\cdot\operatorname{diam}\mathcal{Y}$.

Consider a sequence of nested injective spaces $\mathcal{Y} = \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \dots$ such that $\mathcal{Y}_{n+1} = \mathcal{Y}'_n$. Choose a point $y_n \in \mathcal{Y}_n$ for each n. Show that the sequence y_n is Cauchy. By 3.3a, y_n converges, say to y_∞. Observe that y_∞ is a fixed point of s.

3.12; *only-if part.* Suppose r is extremal. By 3.11b, r is 1-Lipschitz. Since $\mathbb{S}^1$ is compact, 3.11d implies that for any $p\in\mathbb{S}^1$ there is $q\in\mathbb{S}^1$ such that

$$r(p)+r(q)=|p-q|_{\mathbb{S}^1}.$$

Therefore

$$\begin{aligned}\pi=|p-(-p)|_{\mathbb{S}^1}&\leqslant\\ &\leqslant r(p)+r(-p)=\\ &=r(p)+r(q)+r(-p)-r(q)\leqslant\\ &\leqslant|p-q|_{\mathbb{S}^1}+|q-(-p)|_{\mathbb{S}^1}=\\ &=\pi.\end{aligned}$$

So, we have equality in both places, and the only-if part follows.

If part. Assume r is a 1-Lipschitz function such that $r(p)+r(-p)=\pi$. Then

$$\begin{aligned}|p-q|_{\mathbb{S}^1}=|p-(-p)|_{\mathbb{S}^1}-|q-(-p)|_{\mathbb{S}^1}&\geqslant\\ \geqslant\pi-(r(-p)-r(q))&=\\ =r(p)+r(q).\end{aligned}$$

Therefore r is admissible.

Finally, if r is not extremal, then there is an admissible function $s\leqslant r$ such that $s(p)<r(p)$ for some p. The latter contradicts the equality $r(p)+r(-p)=\pi$.

Source: [139, Proposition 2.7].

3.13. Show and use that $s^*(x)\geqslant|x-y|-s(y)$ for any $x,y\in\mathcal{X}$.

Remarks. It is easy to check that $q\colon s\mapsto\frac12\cdot(s+s^*)$ is a short map on the space of admissible functions (with sup-norm). Moreover, iterating q and passing to the limit, we get a short retraction from the space of admissible functions to the space of extremal functions on $\mathcal{X}$ [see 3.1 in 88]. The existence of such a map also follows from 3.27.

3.15. Apply 3.14c.

Comment. Conditions under which gluings of injective spaces are injective were studied by Benjamin Miesch and Maël Pavón [98, 99].

3.16. Let $B=\overline{\mathrm{B}}[0,1]$ and $P\supset B$ be a parallelepiped of minimal volume. Choose coordinates so that P is described by inequalities $|x_i|\leqslant1$ for all i; let $e_1,\dots,e_m$ be the standard basis of these coordinates.

Let $B_i=\overline{\mathrm{B}}[(1+R)\cdot e_i,R]$ for some $R>0$. Show that $e_i\in B$ for any i; in particular $B\cap B_i\neq\varnothing$. Show P can be chosen so that $B_i\cap B_j\neq\varnothing$ for all i and j and all large $R>0$. Apply hyperconvexity to show that $e_1+\dots+e_m\in B$. The same way, show that $\pm e_1\pm\dots\pm e_m\in B$ for all choices of signs. Conclude that $B=P$.

3.17. Observe that closed balls are compact and apply the finite intersection property.

3.18. Denote by $\mathcal{U}_d$ the d-Urysohn space, so $\mathcal{U}_\infty$ is the Urysohn space.

The extension property implies finite hyperconvexity. It remains to show that $\mathcal{U}_d$ is not countably hyperconvex.

Suppose that $d<\infty$. Show that for any point $x\in\mathcal{U}_d$ there is a point $y\in\mathcal{U}_d$ such that $|x-y|_{\mathcal{U}_d}=d=\operatorname{diam}\mathcal{U}_d$. It follows that there is no point $z\in\mathcal{U}_d$ such that $|z-x|_{\mathcal{U}_d}\leqslant\frac d2$ for any $x\in\mathcal{U}_d$. Whence $\mathcal{U}_d$ is not countably hyperconvex.

Use 2.19b to reduce the case $d=\infty$ to the case $d<\infty$.

3.19. Choose $\varepsilon_0>0$. Let p_0 be a point provided by the definition of almost hyperconvexity; that is $|x_\alpha-p_0|\leqslant r_\alpha+\varepsilon_0$ for a given $\varepsilon_0>0$. We may assume that $\delta_0=\sup\{\,|x_\alpha-p_0|-r_\alpha\,\}>0$; otherwise, the problem is solved. Clearly, $\delta_0\leqslant\varepsilon_0$.

Applying hyperconvexity for $\varepsilon_1<\frac1{10}\cdot\delta_0$, we get a point p_1 such that $|x_\alpha-p_1|\leqslant r_\alpha+\varepsilon_1$ and $|p_0-p_1|\leqslant\delta_0+\varepsilon_1$. Again, we may assume that $\delta_1=\sup\{\,|x_\alpha-p_1|-r_\alpha,|p_0-p_1|\,\}>0$, and we have $\delta_1\leqslant\varepsilon_1$.

Continuing this way, we get a sequence $p_0,p_1,\dots$ that either terminates and in this case the problem is solved, or it is an infinite Cauchy sequence. In the latter case, its limit p_∞ satisfies $|x_\alpha-p_\infty|\leqslant r_\alpha$ for any α.

Comment. This solution reminds the proof of 2.9; a more exact statement was proved by Benjamin Miesch and Maël Pavón [100, 2.2]; namely, they show that almost n-hyperconvexity implies $(n-1)$-hyperconvexity.

3.20. Show and use that the functions in $\operatorname{Ext}\mathcal{X}$ are 1-Lipschitz and uniformly bounded.

3.21; *(a).* Use 3.11d to show that if f is extremal if and only if $f(v)=x$ and $f(w)=1-x$ for some $x\in[0,1]$. Conclude that $\operatorname{Ext}\mathcal{X}$ is isometric to the unit interval $[0,1]$.

(b). Let f be an extremal function. By 3.11d, at least two of the numbers $f(a)+f(b)$, $f(b)+f(c)$,

and $f(c) + f(a)$ are 1. It follows that for some $x \in [0, \frac{1}{2}]$, we have

$$f(a) = 1 \pm x, \quad f(b) = 1 \pm x, \quad f(c) = 1 \pm x,$$

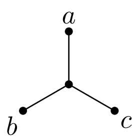
where we have one “minus” and two “pluses” in these three formulas.

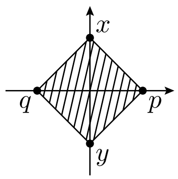
Suppose that

$$g(a) = 1 \pm y, \quad g(b) = 1 \pm y, \quad g(c) = 1 \pm y$$

is another extremal function. Then $|f - g| = |x - y|$ if g has “minus” at the same place as f and $|f - g| = |x + y|$ otherwise.

It follows that $\operatorname{Ext}\mathcal{X}$ is isometric to a *tripod* — three segments of length $\frac{1}{2}$ glued at one end.





(c). Assume f is an extremal function. Use 3.11d to show that

$$\begin{aligned} 2 &= f(x) + f(y) = \\ &= f(p) + f(q); \end{aligned}$$

in particular, two values $a = f(x) - 1$ and $b = f(p) - 1$ completely describe the function f. Since f is extremal, we also have that

$$(1 \pm a) + (1 \pm b) \geqslant 1$$

for all 4 choices of signs; equivalently,

$$|a| + |b| \leqslant 1.$$

It follows that $\operatorname{Ext}\mathcal{X}$ is isometric to the rhombus $|a| + |b| \leqslant 1$ in the (a, b)-plane with the metric induced by the ℓ^∞-norm.

Remarks. If $\mathcal{X}$ has n-points, then (evidently) $\operatorname{Ext}\mathcal{X}$ is a polyhedral complex in $(\mathbb{R}^n, \ell^\infty) = \ell^\infty(\mathcal{X})$; each face of the complex is defined by equalities and inequalities of the following type: $x_i + x_j \geqslant \text{const}$ and $x_i + x_j = \text{const}$. It is easy to see (and follows from 3.16) that each face is isometric to a convex polyhedron in $(\mathbb{R}^k, \ell^\infty)$ for some $k \leqslant n$; in fact $k \leqslant n/2$. The structure of the complex can be encoded by certain graphs with the vertex set $\mathcal{X}$ [see Section 4 in 88].

3.22. Recall that $x \mapsto \operatorname{dist}_x$ gives an isometric embedding $\mathcal{X} \hookrightarrow \ell^\infty(\mathcal{X})$; so we can identify $\mathcal{X}$ with a subset of $\ell^\infty(\mathcal{X})$. Further, $\operatorname{Ext}\mathcal{X}$ is a subset of $\ell^\infty(\mathcal{X})$. It is sufficient to show that $\operatorname{Ext}\mathcal{X} = G$.

Use 3.11d to show that $\operatorname{Ext}\mathcal{X} \subset G$.

Given $g \in G$, show that $g(x) = |g - x|_{\ell^\infty(\mathcal{X})}$. Conclude that g is admissible and apply 3.11d.

Source: Suggested by Rostislav Matveyev.

3.25. Recall that

$$|f - g|_{\operatorname{Ext}\mathcal{X}} = \sup\{\, |f(x) - g(x)| \,:\, x \in \mathcal{X} \,\}$$

and

$$|f - p|_{\operatorname{Ext}\mathcal{X}} = f(p)$$

for any $f, g \in \operatorname{Ext}\mathcal{X}$ and $p \in \mathcal{X}$.

Since $\mathcal{X}$ is compact we can find a point $p \in \mathcal{X}$ such that

$$\begin{aligned} |f - g|_{\operatorname{Ext}\mathcal{X}} &= |f(p) - g(p)| = \\ &= \big||f - p|_{\operatorname{Ext}\mathcal{X}} - |g - p|_{\operatorname{Ext}\mathcal{X}}\big|. \end{aligned}$$

Without loss of generality, we may assume that

$$|f - p|_{\operatorname{Ext}\mathcal{X}} = |g - p|_{\operatorname{Ext}\mathcal{X}} + |f - g|_{\operatorname{Ext}\mathcal{X}}.$$

Applying 3.11d, we can find a point $q \in \mathcal{X}$ such that

$$|q - p|_{\operatorname{Ext}\mathcal{X}} = |f - p|_{\operatorname{Ext}\mathcal{X}} + |f - q|_{\operatorname{Ext}\mathcal{X}},$$

whence the result.

Since $\operatorname{Ext}\mathcal{X}$ is injective (3.23), by 3.3b it has to be geodesic. It remains to note that the concatenation of geodesics $[pq]$, $[gf]$, and $[fq]$ is the required geodesic $[pq]$.

3.26. The only-if part follows since $\mathcal{X}$ is isometric to a subset of $\operatorname{Ext}\mathcal{X}$.

The if part means that

$$\text{❶} \quad \begin{aligned} |f - g| + |v - w| \leqslant \max\{\, &|f - v| + |g - w|, \\ &|f - w| + |g - v| \,\} + 2{\cdot}\delta \end{aligned}$$

for any $f, g, v, w \in \operatorname{Ext}\mathcal{X}$.

Suppose $\mathcal{X}$ is compact. Applying 3.25, we can choose $p, q, x, y \in \mathcal{X}$ such that

$$\text{❷} \quad \begin{aligned} |p - f| + |f - g| + |g - q| &= |p - q| \\ |x - v| + |v - w| + |w - y| &= |x - y| \end{aligned}$$

Since $\mathcal{X}$ is δ-hyperbolic, we have

$$\begin{aligned} |p - q| + |x - y| \leqslant \max\{\, &|p - x| + |q - y|, \\ &|p - y| + |q - x| \,\} + 2{\cdot}\delta. \end{aligned}$$

Show that this inequality, together with the triangle inequality and ❷ imply ❶.

For the noncompact case, prove an approximate version of ❷ and apply it the same way.

3.28. Show that there is a unique isometry of $\operatorname{Ext}\mathcal{X}$ that is identity on $\mathcal{X}$. Use it together with 3.27.

3.29. Show that there is a pair of short maps $\operatorname{Ext}\mathcal{X} \to \operatorname{Ext}\mathcal{U} \to \operatorname{Ext}\mathcal{X}$ such that their composition is the identity on $\mathcal{X}$. You may need to apply the Katětov extension (2.5a). Make a conclusion.

3.30. Apply 3.11d to show that for any $u \in \mathbb{S}^2_+$ the restriction $f_u := \operatorname{dist}_u|_{\mathbb{S}^1}$ is an extremal function on $\mathbb{S}^1$. Moreover, the function f_u uniquely determines u. Make a conclusion.

3.32. Observe that coordinate functions are monotonic on any geodesic in ℓ^1. Use it to show that ℓ^1 is a median space; that is, for any three points x, y, z there is a *unique* point m (it is called the median of x, y, and z) that lies on some geodesics $[xy]$, $[xz]$ and $[yz]$. Apply it to show that ℓ^1 is 3-hyperconvex.

The 4-hyperconvexity fails for the unit balls centered at four even vertices of the cube $([0,1]^3, \ell^1)$.

3.33. Choose three points $x, y, z \in \mathcal{X}$ and set $\mathcal{A} = \{x, z\}$. Let $f\colon \mathcal{A} \to \mathcal{A}$ be the identity map. Then $F(y) = x$ or $F(y) = z$. In both cases, the strong triangle inequality follows.

3.34; *main part.* Choose a maximal (with respect to inclusion) subset $A \supset \mathcal{K}$ that admits a short retraction $f\colon A \to \mathcal{K}$; it exists by Zorn's lemma. If A is the whole space, then the problem is solved. Otherwise, choose $p \notin A$.

Choose a sequence of points $a_n \in A$ such that $|a_n - p|$ converge to the exact lower bound on the distances from points in A to p. Since $\mathcal{K}$ is compact, we can pass to a subsequence of a_n such that $f(a_n)$ converges. Let

$$f(p) = \lim f(a_n).$$

It remains to check that

❸ $$|f(a) - f(p)| \leqslant |a - p|$$

for any $a \in A$. Choose $\varepsilon > 0$; note that

$$|a_n - p| < |a - p| + \varepsilon, \quad |f(a_n) - f(p)| < \varepsilon$$

for all large n. Therefore,

$$\begin{aligned}|f(a) - f(p)| &\leqslant \\ &\leqslant \max\{\,|f(a) - f(a_n)|, |f(a_n) - f(p)|\,\} \leqslant \\ &\leqslant |f(a) - f(a_n)| + \varepsilon \leqslant \\ &\leqslant |a - a_n| + \varepsilon \leqslant \\ &\leqslant \max\{\,|a - p|, |a_n - p|\,\} + \varepsilon < \\ &< |a - p| + 2\cdot\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get ❸.

Example. Consider set of $\{\infty, 1, 2, \dots\}$ with metric defined by

$$|m - n| := 1 + \frac{1}{\min\{m, n\}}$$

for $m \neq n$. Observe that the space is complete, the subset $\{1, 2, \dots\}$ is closed, but it is not a short retract of the ambient space.

3.35. Consider the space $\mathcal{K}^{\mathcal{X}}$ of all maps $\mathcal{X} \to \mathcal{K}$ equipped with the product topology.

Denote by $\mathfrak{S}_F$ the set of maps $h \in \mathcal{K}^{\mathcal{X}}$ such that the restriction $h|_F$ is short and agrees with f in $F \cap A$. Note that the sets $\mathfrak{S}_F \subset \mathcal{K}^{\mathcal{X}}$ are closed and any finite intersection of these sets is nonempty.

According to Tikhonov's theorem, $\mathcal{K}^{\mathcal{X}}$ is compact. By the finite intersection property, the intersection $\bigcap_F \mathfrak{S}_F$ for all finite sets $F \subset X$ is nonempty. Hence the statement follows.

Source: [111, 7.1].

4.3. Suppose that $|A - B|_{\operatorname{Haus}\mathcal{X}} < r$. Choose a pair of points $a, a' \in A$ on maximal distance from each other. Observe that there are points $b, b' \in B$ such that $|a - b|_{\mathcal{X}}, |a' - b'|_{\mathcal{X}} < r$. Whence

$$|a - a'|_{\mathcal{X}} - |b - b'|_{\mathcal{X}} \leqslant 2\cdot r$$

and therefore

$$\operatorname{diam} A - \operatorname{diam} B \leqslant 2\cdot|A - B|_{\operatorname{Haus}\mathcal{X}}.$$

Swap A and B and repeat the argument.

4.4; *(a).* Given a set $A \subset \mathbb{R}^2$, denote by A^r its closed r-neighborhood. Show and use that

$$(\operatorname{Conv} A)^r = \operatorname{Conv}(A^r).$$

(b). The answer is "no" in both parts.

For the first part let A be a unit disk and B a finite ε-net in A. Evidently, $|A - B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$, but $|\partial A - \partial B|_{\operatorname{Haus}\mathbb{R}^2} \approx 1$.

For the second part take A to be a unit disk and $B = \partial A$ to be its boundary circle. Note that $\partial A = \partial B$; in particular, $|\partial A - \partial B|_{\operatorname{Haus}\mathbb{R}^2} = 0$ while $|A - B|_{\operatorname{Haus}\mathbb{R}^2} = 1$.

Remark. A more interesting example for *(b)* is provided by the so-called *lakes of Wada* — an example of three (and more) disjoint open topological disks in the plane that have identical boundaries.

4.5. Checking two functions dist_A and dist_B leads to

$$|A - B| \leqslant \sup_f \{\, \max_{a \in A}\{f(a)\} - \max_{b \in B}\{f(b)\,\}.$$

Use 4.2 to prove the opposite inequality.

4.6. By 4.4a, it is sufficient to show that

$$|A-B|_{\mathrm{Haus}\,\mathbb{R}^n} = \sup_{|u|=1}\{|h_A(u)-h_B(u)|\}$$

for any nonempty compact convex sets $A, B \subset \subset \mathbb{R}^n$.

Prove the 1-dimensional case of this equality. Further, denote by A_ℓ the orthogonal projection of A to a line ℓ. Show and use that

$$|A-B|_{\mathrm{Haus}\,\mathbb{R}^n} = \sup_{\ell}\{|A_\ell - B_\ell|_{\mathrm{Haus}\,\ell}\},$$

where the least upper bound is taken for all lines ℓ.

4.7; *(a).* Given $t\in(0,1]$, consider the real interval $\tilde C_t = [\frac{1}{t}+t, \frac{1}{t}+1]$. Denote by C_t the image of $\tilde C_t$ under the covering map $\pi\colon \mathbb{R}\to\mathbb{S}^1=\mathbb{R}/\mathbb{Z}$.

Set $C_0=\mathbb{S}^1$. Note that the Hausdorff distance from C_0 to C_t is $\frac{t}{2}$. Therefore $\{C_t\}_{t\in[0,1]}$ is a family of compact subsets in $\mathbb{S}^1$ that is continuous in the sense of Hausdorff.

Assume there is a continuous section $c(t)\in \in C_t$, for $t\in[0,1]$. Since π is a covering map, we can lift the path c to a path $\tilde c\colon[0,1]\to\mathbb{R}$ such that $\tilde c(t)\in\tilde C_t$ for all t. In particular, $\tilde c(t)\to\infty$ as $t\to 0$, a contradiction.

(b). Consider path $c(t):=\min C_t$.

Source: Suggested by Stephan Stadler.

4.9. Show that $f\colon(p,r)\mapsto \overline{\mathrm{B}}[p,r]$ defines a continuous map $\mathcal{X}\times[0,\infty)\to\mathrm{Haus}\,\mathcal{X}$. Observe that $\overline{\mathrm{B}}[p,r]=\mathcal{X}$ for all $r\geqslant\operatorname{diam}\mathcal{X}$. Composing f with the retraction $r\colon \mathrm{Haus}\,\mathcal{X}\to\mathcal{X}$ we get a homotopy from the identity map to a constant map on $\mathcal{X}$, hence the statement.

By the way, is there a good description of such spaces? Note that if $\mathcal{X}$ is injective or discrete, then a short retraction $\mathrm{Haus}\,\mathcal{X}\to\mathcal{X}$ exists. On the other hand, the Euclidean plane does not have such a retraction [107, 115]. See also [110].

4.12. Show that for any $\varepsilon>0$ there is a positive integer N such that $\bigcup_{n\leqslant N}K_n$ is an ε-net in the union $\bigcup_n K_n$. Observe that $\bigcup_{n\leqslant N}K_n$ is compact. Apply 1.13 and 1.11c.

4.13; *if part.* Choose two compact sets $A,B\subset \subset\mathcal{X}$; suppose that $|A-B|_{\mathrm{Haus}\,\mathcal{X}}<r$.

Choose finite ε-nets $\{a_1,\dots,a_m\}\subset A$ and $\{b_1,\dots b_n\}\subset B$. For each pair a_i, b_j construct a constant-speed path $\gamma_{i,j}$ from a_i to b_j such that

$$\operatorname{length}\gamma_{i,j} < |a_i-b_j|+\varepsilon.$$

Set

$$C(t)=\{\,\gamma_{i,j}(t) \,:\, |a_i-b_j|_{\mathcal{X}}<r+\varepsilon\,\}.$$

Observe that $C(t)$ is finite; in particular, it is compact.

Show and use that

$$|A-C(t)|_{\mathcal{X}} < t{\cdot}r+10{\cdot}\varepsilon,$$
$$|C(t)-B|_{\mathcal{X}} < (1-t){\cdot}r+10{\cdot}\varepsilon.$$

Apply 4.12 and 1.27.

Only-if part. Choose points $p,q\in\mathcal{X}$. Show that the existence of ε-midpoints between $\{p\}$ and $\{q\}$ in $\mathrm{Haus}\,\mathcal{X}$ implies the existence of ε-midpoints between p and q in $\mathcal{X}$. Apply 1.27.

4.14; *(a)* Suppose that a sequence of compact subsets $K_n\subset\mathbb{R}^2$ converges to K_∞ in the sense of Hausdorff. Assume K_∞ is not connected; show that so is K_n for large n.

(b). Choose a finite subset F that is ε-close to K. Show that one can obtain a tree T by connecting some vertices of F by line segments of length smaller than $2{\cdot}\varepsilon$. Let γ be a curve that bounds a neighborhood of T. Show that if the neighborhood is sufficiently small, then γ is $2{\cdot}\varepsilon$-close to K.

Remarks. You might be surprised to learn that most connected compact sets in the pane are homeomorphic to each other — they are homeomorphic to the so-called pseudo-arc [24]; here the word *most* understood in the sense of 1F. In particular, most of the compact connected plane sets are not path-connected.

4.15. Let A be a compact convex set in the plane. Denote by A^r the closed r-neighborhood of A. Recall that by Steiner's formula we have

$$\operatorname{area}A^r=\operatorname{area}A+r\cdot\operatorname{perim}A+\pi{\cdot}r^2.$$

Taking the derivative and applying the coarea formula, we get

$$\operatorname{perim}A^r=\operatorname{perim}A+2{\cdot}\pi{\cdot}r.$$

Observe that if A lies in a compact set B bounded by a closed curve, then

$$\operatorname{perim}A\leqslant\operatorname{perim}B.$$

Indeed the closest-point projection $\mathbb{R}^2\to A$ is short and it maps ∂B onto ∂A.

It remains to use the following observation: if $A_n\to A_\infty$, then for any $r>0$ we have that the inclusions

$$A^r_\infty\supset A_n \quad\text{and}\quad A_\infty\subset A^r_n$$

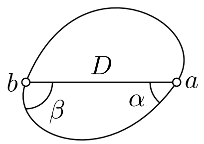
hold for all large n.

Remark. Observe that the inequality

$$\operatorname{area} A_\infty \geqslant \varlimsup_{n\to\infty} \operatorname{area} A_n$$

holds for any sequence of compact (not necessarily convex) sets $A_1, A_2, \dots$ in the plane that converges to A_∞ in the sense of Hausdorff.

4.17. Note that almost all points on ∂D have a defined tangent line. In particular, for almost all pairs of points $a, b \in \partial D$ the two angles α and β between the chord $[ab]$ and ∂D are defined.



The convexity of D' implies that $\alpha = \beta$; here we measure the angles α and β on one side from $[ab]$. Show that if the identity $\alpha = \beta$ holds for almost all chords, then D is a round disk.

4.19. Observe that all functions $\operatorname{dist}_{A_n}$ are Lipschitz. Suppose that for some (and therefore any) point x the sequence $\operatorname{dist}_{A_n}(x)$ is not bounded. Then we can pass to a subsequence of A_n so that $\operatorname{dist}_{A_n}(x) \to \infty$ for any x; in this case, A_n converges to the empty set.

Assume the sequence $\operatorname{dist}_{A_n}(x)$ is bounded for some (and therefore any) point x. Then, passing to a subsequence of A_n, we may assume that the sequence $\operatorname{dist}_{A_n}$ converges to some function f.

Set $A_\infty = f^{-1}\{0\}$. It remains to show that $f = \operatorname{dist}_{A_\infty}$.

5.3; *(a).* Apply the definition (5.1) for space $\mathcal{W}$ obtained from $\mathcal{X}$ by adding one point that lies at distance $\tfrac12 \cdot \operatorname{diam} \mathcal{X}$ from each point of $\mathcal{X}$.

(b). Given a point $x \in \mathcal{X}$, denote by $a\cdot x$ and $b\cdot x$ the corresponding points in $a\cdot\mathcal{X}$ and $b\cdot\mathcal{X}$ respectively. Show that there is a metric on $\mathcal{W} = a\cdot\mathcal{X} \sqcup b\cdot\mathcal{X}$ such that

$$|a\cdot x - b\cdot x|_{\mathcal{W}} = \tfrac{|b-a|}{2} \cdot \operatorname{diam} \mathcal{X}$$

for any x and the inclusions $a\cdot\mathcal{X} \hookrightarrow \mathcal{W}$, $b\cdot\mathcal{X} \hookrightarrow \mathcal{W}$ are distance-preserving. Conclude that $|\mathcal{X} - \mathcal{O}|_{\mathrm{GH}} \leqslant \tfrac12 \cdot \operatorname{diam} \mathcal{X}$. The opposite inequality follows from *(a).*

(c). Use *(a)* and *(b)* to show that the isometry class of $\mathcal{O}$ is completely determined by the following property

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} \leqslant \max\{\, |\mathcal{O} - \mathcal{X}|_{\mathrm{GH}}, |\mathcal{O} - \mathcal{Y}|_{\mathrm{GH}} \,\}.$$

for any $\mathcal{X}$ and $\mathcal{Y}$.

Remark. In fact, *the isometry group of space* GH *is trivial.* The latter was proved by George Lowther [79, 92].

5.4. Check a one-point set and the vertices of an equilateral triangle. You may use 5.3*a*.

5.5. Suppose that we can identify $\mathcal{A}_r$ and $\mathcal{B}_r$ with subspaces of a space $\mathcal{W}$ such that

$$|\mathcal{A}_r - \mathcal{B}_r|_{\operatorname{Haus} \mathcal{W}} < \tfrac{1}{10}$$

for large r; see the definition of Gromov–Hausdorff metric (5.1).

Set $n = \lceil r \rceil$. Note that there are $2\cdot n$ integer points in $\mathcal{A}_r$: $a_1 = (0,0)$, $a_2 = (1,0)$, $\dots$, $a_{2\cdot n} = (n,1)$. Choose a point $b_i \in \mathcal{B}_r$ that lies at the minimal distance from a_i. Note that $|b_i - b_j| > \tfrac45$ if $i \ne j$. It follows that $r > \tfrac45 \cdot (2\cdot n - 1)$. The latter contradicts $n = \lceil r \rceil$ for large r.

Remark. Try to show that $|\mathcal{A}_r - \mathcal{B}_r|_{\mathrm{GH}} = \tfrac12$ for all large r.

5.6. Suppose

$$|\mathcal{X} - \mathcal{Y}|_{\operatorname{Haus} \mathcal{U}} < \varepsilon.$$

Denote by $\hat{\mathcal{U}}$ the injective envelope of $\mathcal{U}$. According to 3.29, the inclusions $\mathcal{X} \hookrightarrow \mathcal{U}$ and $\mathcal{Y} \hookrightarrow \mathcal{U}$ can be extended to distance-preserving inclusions $\hat{\mathcal{X}} \hookrightarrow \hat{\mathcal{U}}$ and $\hat{\mathcal{Y}} \hookrightarrow \hat{\mathcal{U}}$. Therefore, we can and will consider $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$ as subspaces of $\hat{\mathcal{U}}$. It is sufficient to show that

❶ $$|\hat{\mathcal{X}} - \hat{\mathcal{Y}}|_{\operatorname{Haus} \hat{\mathcal{U}}} < 2\cdot\varepsilon.$$

Given $f \in \hat{\mathcal{U}}$, let us find $g \in \hat{\mathcal{X}}$ such that

❷ $$|f(u) - g(u)| < 2\cdot\varepsilon$$

for any $u \in \mathcal{U}$. Note that the restriction $f|_{\mathcal{X}}$ is admissible on $\mathcal{X}$. By 3.9, there is $g \in \hat{\mathcal{X}}$ such that

❸ $$g(x) \leqslant f(x)$$

for any $x \in \mathcal{X}$.

Recall that any extremal function is 1-Lipschitz; in particular, f and g are 1-Lipschitz on $\mathcal{U}$. Therefore, ❸ and $|\mathcal{X} - \mathcal{Y}|_{\mathcal{U}} < \varepsilon$ imply that

$$g(u) < f(u) + 2\cdot\varepsilon$$

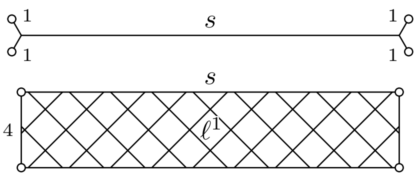
for any $u \in \mathcal{U}$. By 3.10, we also have

$$g(u) > f(u) - 2\cdot\varepsilon$$

for any $u \in \mathcal{U}$. Whence ❷ follows.

It follows that $\hat{\mathcal{Y}}$ lies in a $2{\cdot}\varepsilon$-neighborhood of $\hat{\mathcal{X}}$ in $\hat{\mathcal{U}}$. The same way we show that $\hat{\mathcal{X}}$ lies in a $2{\cdot}\varepsilon$-neighborhood of $\hat{\mathcal{Y}}$ in $\hat{\mathcal{U}}$. Hence ❶ follows.

Remark. This problem was discussed by Urs Lang, Maël Pavón, and Roger Züst [89, 3.1]. They also show that the constant 2 is optimal. To see this, look at the injective envelopes of two four-point metric spaces shown on the diagram and observe that the Gromov–Hausdorff distance between the 4-point metric spaces is 1, while the distance between their injective envelopes approaches 2 as $s \to \infty$.



5.8; *only-if part.* Let us identify $\mathcal{X}$ and $\mathcal{Y}$ with subspaces of a metric space $\mathcal{W}$ such that

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{Haus}\,\mathcal{W}} < \varepsilon.$$

Set $x \approx y$ if and only if $|x - y|_{\mathcal{W}} < \varepsilon$. It remains to check that $\approx$ is an ε-approximation.

If part. Show that we can assume that

$$R = \{\, (x, y) \in \mathcal{X} \times \mathcal{Y} \;:\; x \approx y \,\}$$

is a compact subset of $\mathcal{X} \times \mathcal{Y}$. Conclude that

$$\big||x - x'|_{\mathcal{X}} - |y - y'|_{\mathcal{Y}}\big| < 2{\cdot}\varepsilon'$$

for some $\varepsilon' < \varepsilon$.

Show that there is a metric on $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Y}$ such that the inclusions $\mathcal{X} \hookrightarrow \mathcal{W}$ and $\mathcal{Y} \hookrightarrow \mathcal{W}$ are distance-preserving and $|x - y|_{\mathcal{W}} = \varepsilon'$ if $x \approx y$. Conclude that

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{Haus}\,\mathcal{W}} \leqslant \varepsilon' < \varepsilon.$$

5.10; *(a).* Let $\approx$ be an ε-approximation provided by 5.8. For any $x \in \mathcal{X}$ choose a point $f(x) \in \mathcal{Y}$ such that $x \approx f(x)$. Show that $x \mapsto f(x)$ is an $2{\cdot}\varepsilon$-isometry.

(b). Let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Set $x \approx y$ if $|y - f(x)|_{\mathcal{Y}} < \varepsilon$. Show that $\approx$ is an ε-approximation. Apply 5.8.

5.13. Consider the product space $[0, \varepsilon] \times \mathbb{Z}_n$ with the natural ℓ^∞-product metric. Make three variations of it by changing the sizes of some segments.

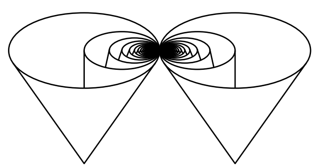
5.15; *(a).* Suppose $\mathcal{X}_n$ are simply-connected length metric space, $\mathcal{X}_n \xrightarrow{\mathrm{GH}} \mathcal{X}$, and there is a nontrivial covering map $f\colon \tilde{\mathcal{X}} \to \mathcal{X}$. We will arrive at a contradiction by showing that there is a nontrivial covering map $f_n\colon \tilde{\mathcal{X}}_n \to \mathcal{X}_n$ for large n.

Choose a base point $p \in \mathcal{X}$ and its inverse image $\tilde{p} \in \tilde{\mathcal{X}}$. Consider two paths $\alpha, \alpha'\colon [0,1] \to \mathcal{X}$ that start at p; denote by $\tilde{\alpha}, \tilde{\alpha}'\colon [0,1] \to \tilde{\mathcal{X}}$ their liftings. Show that there is $\varepsilon > 0$ such that if $|\alpha(t) - \alpha'(t)|_{\mathcal{X}} < \varepsilon$ for any t, then $|\tilde{\alpha}(1) - \tilde{\alpha}'(1)|_{\tilde{\mathcal{X}}} < \varepsilon$.

Now suppose n is large. Choose an $\frac{\varepsilon}{10}$-approximation $\approx$ for $\mathcal{X}_n$ and $\mathcal{X}$. Choose $q \in \mathcal{X}_n$ such that $q \approx p$. Show that for any path $\beta\colon [0,1] \to \mathcal{X}_n$ that starts at q there is a path $\alpha\colon [0,1] \to \mathcal{X}$ that starts at p such that $\alpha(t) \approx \beta(t)$ for any t. Observe that if α and α' are two choices of such paths, then $|\alpha(t) - \alpha'(t)|_{\mathcal{X}} < \varepsilon$.

Mimicking the standard construction of a covering map, we get the needed $f_n\colon \tilde{\mathcal{X}}_n \to \mathcal{X}_n$.

(b). Let $\mathcal{V}$ be a cone over Hawaiian earrings. Consider the *doubled cone* $\mathcal{W}$ — two copies of $\mathcal{V}$ with glued base points (see the diagram).



The space $\mathcal{W}$ can be equipped with a length metric (for example, the induced length metric from the shown embedding).

Show that $\mathcal{V}$ is simply-connected, but $\mathcal{W}$ is not; use the van Kampen theorem.

If we delete from the earrings all small circles and repeat the construction, then the obtained double cone becomes simply-connected and remains close to $\mathcal{W}$. That is, $\mathcal{W}$ is a Gromov–Hausdorff limit of simply-connected spaces.

Remark. Note that the limit space in *(b)*, does not admit a nontrivial covering.

5.16; *(a).* Suppose that a metric on $\mathbb{S}^2$ is close to the unit disk $\mathbb{D}^2$. Show that $\mathbb{S}^2$ contains a circle γ that is close to the boundary curve of $\mathbb{D}^2$. By the Jordan curve theorem, γ cuts $\mathbb{S}^2$ into two disks, say D_1 and D_2.

By 5.15a, the Gromov–Hausdorff limits of D_1 and D_2 have to contain the whole $\mathbb{D}^2$; otherwise, the limit would admit a nontrivial covering.

Consider points $p_1 \in D_1$ and $p_2 \in D_2$ that are close to the center of $\mathbb{D}^2$. On one hand, the distance $|p_1 - p_2|_{\mathbb{S}^2}$ has to be small. On the other hand, any curve from p_1 to p_2 must cross γ, so its length is about 2 (or larger) — a contradiction.

(b). Show that one can remove fine tunnels from the standard 3-ball in such a way that (1) the topology does not change, (2) the induced length metric is very close to the original one, and (3) the tunnels come sufficiently close to any point in the ball.

Consider the d o u b l i n g of the obtained ball along its boundary; that is, two copies of the ball with glued corresponding points on their boundaries. The obtained space is homeomorphic to $\mathbb{S}^3$. Observe that the obtained space is sufficiently close to the original ball.

Source: [34, Exercises 7.5.13 and 7.5.17].

5.18. Apply 1.14.

5.19. Let μ be a C-doubling measure on a space $\mathcal{X}$ from $\boldsymbol{Q}(C, D)$. Without loss of generality, we may assume that $\mu(\mathcal{X}) = 1$.

The doubling condition implies that

$$\mu[\mathrm{B}(p, \tfrac{D}{2^n})] \geqslant \tfrac{1}{C^n}$$

for any point $x \in \mathcal{X}$. It follows that

$$\operatorname{pack}_{\frac{D}{2^n}} \mathcal{X} \leqslant C^n.$$

By 1.14, for any $\varepsilon \geqslant \frac{D}{2^{n-1}}$, the space $\mathcal{X}$ admits an ε-net with at most C^n points. Whence $\boldsymbol{Q}(C, D)$ is uniformly totally bounded.

5.20. Since $\operatorname{diam} \overline{\mathrm{B}}[x, r] \leqslant r$, diameter of any space in $\boldsymbol{B}_{\mathcal{X}}$ is at most 2.

Suppose $\mathcal{X}$ is not doubling. Show that $n(x, r) = \operatorname{pack}_{\frac{r}{2}} \overline{\mathrm{B}}[x, r]$ is unbounded; that is, $n(x_n, r_n) \to \infty$ for some sequences x_n and $r_n > 0$. Conclude that $\boldsymbol{B}_{\mathcal{X}}$ is not uniformly totally bounded.

Suppose $\mathcal{X}$ is M-doubling. Show that

$$\operatorname{pack}_{\frac{r}{2^n}} \overline{\mathrm{B}}[x, r] \leqslant M^n,$$

and apply 5.18.

5.21; *(a).* Choose $\varepsilon > 0$. Since $\mathcal{Y}$ is compact, we can choose a finite ε-net $\{y_1, \dots, y_n\}$ in $\mathcal{Y}$.

Suppose $f\colon \mathcal{X} \to \mathcal{Y}$ be a distance-noncontracting map. Choose one point x_i in each nonempty subset $B_i = f^{-1}[\mathrm{B}(y_i, \varepsilon)]$. Note that the subset B_i has diameter at most $2{\cdot}\varepsilon$ and

$$\mathcal{X} = \bigcup_i B_i.$$

Therefore, the set of points $\{x_i\}$ is a $2{\cdot}\varepsilon$-net in $\mathcal{X}$.

(b). Let $\boldsymbol{Q}$ be a uniformly totally bounded family of spaces. Suppose that each space in $\boldsymbol{Q}$ has an $\frac{1}{2^n}$-net with at most M_n points; we may assume that $M_0 = 1$.

Consider the space $\mathcal{Y}$ of all infinite integer sequences $m_0, m_1, \dots$ such that $1 \leqslant m_n \leqslant M_n$ for any n. Given two sequences $\boldsymbol{\ell} = (\ell_1, \ell_2, \dots)$, and $\boldsymbol{m} = (m_1, m_2, \dots)$ of points in $\mathcal{Y}$, set

$$|\boldsymbol{\ell} - \boldsymbol{m}|_{\mathcal{Y}} = \tfrac{C}{2^n},$$

where n is the minimal index such that $\ell_n \neq m_n$ and C is a positive constant.

Observe that $\mathcal{Y}$ is compact. Indeed it is complete and the sequences with constant tails, starting from index n, form a finite $\frac{C}{2^n}$-net in $\mathcal{Y}$.

Given a space $\mathcal{X}$ in $\boldsymbol{Q}$, choose a sequence of $\frac{1}{2^n}$ nets $N_n \subset \mathcal{X}$ for each n. We can assume that $|N_n| \leqslant M_n$; let us label the points in N_n by $\{1, \dots, M_n\}$. Consider the map $f\colon \mathcal{X} \to \mathcal{Y}$ defined by $f : x \mapsto (m_1(x), m_2(x), \dots)$ where $m_n(x)$ is the label of a point in N_n that lies at the distance $< \frac{1}{2^n}$ from x.

If $\frac{1}{2^{n-2}} \geqslant |x - x'|_{\mathcal{X}} > \frac{1}{2^{n-1}}$, then $m_n(x) \neq$ $\neq m_n(x')$. It follows that $|f(x) - f(x')|_{\mathcal{Y}} \geqslant \frac{C}{2^n}$. In particular, if $C > 10$, then

$$|f(x) - f(x')|_{\mathcal{Y}} \geqslant |x - x'|_{\mathcal{X}}$$

for any $x, x' \in \mathcal{X}$. That is, f is a distance-noncontracting map $\mathcal{X} \to \mathcal{Y}$.

5.24. Let $\mathcal{K}$ be a compact space. Denote by $a(\mathcal{K})$ the largest diameter of connected component in a compact space $\mathcal{K}$. Further, let

$$b(\mathcal{K}) = \max_{p \in \mathcal{K}} \min_{q \neq p} \{|p - q|_{\mathcal{K}}\}.$$

Note that $b(\mathcal{K}) = 0$ if and only if $\mathcal{K}$ has no isolated points.

Show that if $a(\mathcal{K}) = b(\mathcal{K}) = 0$, then $\mathcal{K}$ is homeomorphic to the Cantor set.

Further show that the sets

$$A_\varepsilon = \{\, \mathcal{K} \in \mathrm{GH} : a(\mathcal{K}) < \varepsilon \,\}$$
$$B_\varepsilon = \{\, \mathcal{K} \in \mathrm{GH} : b(\mathcal{K}) < \varepsilon \,\}$$

are open and dense in GH. Apply 1.10.

5.25; *(a).* Show that (the isometry classes of) finite metric spaces with only rational distances form a countable dense subset in GH.

(b)+(c). Choose two compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$. Let $\mathcal{W} \supset \mathcal{X}', \mathcal{Y}'$ be as in 5.11; so $\mathcal{X}' \stackrel{iso}{=\!=} \mathcal{X}$, $\mathcal{Y}' \stackrel{iso}{=\!=} \mathcal{Y}$, and

$$\ell = |\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}} = |\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}}$$

for some $\ell \geqslant 0$.

We can assume that $\mathcal{W} = \mathcal{X}' \cup \mathcal{Y}'$, so $\mathcal{W}$ is compact. Choose a compact geodesic extension $\mathcal{G}$ of $\mathcal{W}$; it exists by 2.2 (or 3.20).

Given $t \in [0, \ell]$, consider the set

$$\mathcal{Z}_t = \{\, w \in \mathcal{G} \;:\; \mathrm{dist}_{\mathcal{X}'} w \leqslant t,\ \mathrm{dist}_{\mathcal{Y}'} w \leqslant \ell - t \,\}.$$

Observe that $t \mapsto \mathcal{Z}_t$ is a geodesic in $\mathrm{Haus}(\mathcal{G})$ from $\mathcal{X}'$ to $\mathcal{Y}'$. Conclude that $t \mapsto [\mathcal{Z}_t]$ is a geodesic in GH from $[\mathcal{X}]$ to $[\mathcal{Y}]$.

Source: [78].

5.26; *(a).* To check that $|{*} - {*}|_{\mathrm{GH}'}$ is a metric, it is sufficient to show that

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}'} = 0 \quad\Longrightarrow\quad \mathcal{X} \stackrel{iso}{=\!=} \mathcal{Y};$$

the remaining conditions are trivial.

If $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}'} = 0$, then there is a sequence of maps $f_n \colon \mathcal{X} \to \mathcal{Y}$ such that

$$|f_n(x) - f_n(x')|_{\mathcal{Y}} \geqslant |x - x'|_{\mathcal{X}} - \tfrac{1}{n}.$$

Choose a countable dense subset $S \subset \mathcal{X}$ and pass to a subsequence such that $f_n(x)$ converges for any $x \in S$; denote by $f_\infty \colon S \to \to \mathcal{Y}$ the limit map. Note that f_∞ is distance-noncontracting, and it can be extended to a distance-noncontracting map $f_\infty \colon \mathcal{X} \to \mathcal{Y}$.

The same way we can construct a distance-noncontracting map $g_\infty \colon \mathcal{Y} \to \mathcal{X}$.

By 1.15, the compositions $f_\infty \circ g_\infty \colon \mathcal{Y} \to \mathcal{Y}$ and $g_\infty \circ f_\infty \colon \mathcal{X} \to \mathcal{X}$ are isometries. Therefore, f_∞ and g_∞ are isometries as well.

(b). The implication

$$|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \to 0 \quad\Rightarrow\quad |\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}'} \to 0$$

follows from 5.10*a*.

Now suppose $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}'} \to 0$. Show that $\{\mathcal{X}_n\}$ is a uniformly totally bonded family.

If $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \not\to 0$, then we can pass to a subsequence such that $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \geqslant \varepsilon$ for some $\varepsilon > 0$. By the Gromov selection theorem, we can assume that $\mathcal{X}_n$ converges in the sense of Gromov–Hausdorff. From the first implication, the limit $\mathcal{X}'_\infty$ has to be isometric to $\mathcal{X}_\infty$; on the other hand, $|\mathcal{X}'_\infty - \mathcal{X}_\infty|_{\mathrm{GH}} \geqslant \varepsilon$ — a contradiction.

5.28. Apply 2.20 and 5.27.

6.2. Let $F = \{\, n \in \mathbb{N} : f(n) = n \,\}$; we need to show that $\omega(F) = 1$.

Consider an oriented graph Γ with vertex set $\mathbb{N} \setminus F$ such that m is connected to n if $f(m) = n$. Show that each connected component of Γ has at most one cycle. Use it to subdivide vertices of Γ into three sets S_1, S_2, and S_3 such that $f(S_i) \cap S_i = \varnothing$ for each i.

Conclude that $\omega(S_1) = \omega(S_2) = \omega(S_3) = 0$ and hence

$$\omega(F) = \omega(\mathbb{N} \setminus (S_1 \cup S_2 \cup S_3)) = 1.$$

Source: The presented proof was given by Robert Solovay [124], but the key statement is due to Miroslav Katětov [82].

6.6. Choose a nonprincipal ultrafilter ω and set $L(\boldsymbol{s}) = s_\omega$. It remains to observe that L is linear.

Remark. This construction identifies ultrafilters with vectors in $(\ell^\infty)^*$. Recall that $\ell^\infty = (\ell^1)^*$ and $\ell^1 \subsetneq (\ell^\infty)^*$. A principle ultrafilter is a basis vector in ℓ^1; nonprincipal ultrafilters lie in $(\ell^\infty)^* \setminus \ell^1$. The set of ultrafilters is the closure of basis vectors in ℓ^1 with respect to weak*-topology on $(\ell^\infty)^*$.

6.7. Apply 6.2.

6.11. Let γ be a path from p to q in a metric tree $\mathcal{T}$. Assume that γ passes thru a point x on distance ℓ from $[pq]$. Then

❶ $$\mathrm{length}\,\gamma \geqslant |p - q| + 2{\cdot}\ell.$$

Suppose that $\mathcal{T}_n$ is a sequence of metric trees that ω-converges to $\mathcal{T}_\omega$. By 6.10, the space $\mathcal{T}_\omega$ is geodesic.

The uniqueness of geodesics follows from ❶. Indeed, if for a geodesic $[p_\omega q_\omega]$ there is another geodesic γ_ω connecting its ends, then it has to pass thru a point $x_\omega \notin [p_\omega q_\omega]$. Choose sequences $p_n, q_n, x_n \in \mathcal{T}_n$ such that $p_n \to p_\omega$, $q_n \to q_\omega$, and $x_n \to x_\omega$ as $n \to \omega$. Then

$$\begin{aligned} |p_\omega - q_\omega| &= \mathrm{length}\,\gamma \geqslant \\ &\geqslant \lim_{n\to\omega} (|p_n - x_n| + |q_n - x_n|) \geqslant \\ &\geqslant \lim_{n\to\omega} (|p_n - q_n| + 2{\cdot}\ell_n) = \\ &= |p_\omega - q_\omega| + 2{\cdot}\ell_\omega. \end{aligned}$$

Since $x_\omega \notin [p_\omega q_\omega]$, we have that $\ell_\omega > 0$ — a contradiction.

It remains to show that any geodesic triangle $\mathcal{T}_\omega$ is a tripod. Consider the sequence of centers of tripods m_n for given sequences of points $x_n, y_n, z_n \in \mathcal{T}_n$. Observe that its ultralimit m_ω is the center of a tripod with ends at $x_\omega, y_\omega, z_\omega \in \mathcal{T}_\omega$.

6.12. Construct $\boldsymbol{X}$ and distance-preserving embeddings $\mathcal{X}_n \hookrightarrow \boldsymbol{X}$ that satisfy 5.29. Given $x_\infty \in \mathcal{X}_\infty$, choose a sequence $x_n \in \mathcal{X}_n$ such that $x_n \to x_\infty$ in $\boldsymbol{X}$. Let x_ω be the ω-limit of the sequence x_n in $\boldsymbol{X}$. Note that $x_\omega \in \mathcal{X}_\infty$. Show that the map $x_\infty \mapsto x_\omega$ is defined; that is, it does not depend on the choice of the sequence x_n. Further, show that the map $x_\infty \mapsto x_\omega$ is an isometry of $\mathcal{X}_\infty$. Make a conclusion.

6.13. Further, we consider $\mathcal{X}$ as a subset of $\mathcal{X}^\omega$.

(a). Follows directly from the definitions.

(b). Suppose $\mathcal{X}$ compact. Given a sequence $x_1, x_2, \ldots \in \mathcal{X}$, denote its ω-limit in $\mathcal{X}^\omega$ by x^ω and its ω-limit in $\mathcal{X}$ by x_ω.

Observe that $x^\omega = \iota(x_\omega)$. Therefore, ι is onto.

If $\mathcal{X}$ is not compact, we can choose a sequence x_n such that $|x_m - x_n| > \varepsilon$ for fixed $\varepsilon > 0$ and all $m \neq n$. Observe that

$$\lim_{n\to\omega} |x_n - y|_{\mathcal{X}} \geqslant \tfrac{\varepsilon}{2}$$

for any $y \in \mathcal{X}$. It follows that x_ω lies at the distance $\geqslant \frac{\varepsilon}{2}$ from $\mathcal{X}$.

(c). A sequence of points x_n in $\mathcal{X}$ will be called ω-bounded if there is a real constant C such that

$$|p - x_n|_{\mathcal{X}} \leqslant C$$

for ω-almost all n.

The same argument as in *(b)* shows that any ω-bounded sequence has its ω-limit in $\mathcal{X}$. Further, if (x_n) is not ω-bounded, then

$$\lim_{n\to\omega} |p - x_n|_{\mathcal{X}} = \infty;$$

that is, x_ω does not lie in the metric component of p in $\mathcal{X}^\omega$.

6.14. Let us show that cardinality of $\mathcal{X}^\omega$ is at least continuum — it is sufficient to construct a continuum family $\mathcal{A}$ sequences of points on $\mathcal{X}$ such that for any two sequences (a_n) and (b_n) in $\mathcal{A}$ the equality $a_n = b_n$ holds only for finitely many n.

To do this, let us identify points in $\mathcal{X}$ with nonnegative integers. Consider the set $\mathcal{A}$ of all sequences a_n such that $a_0 = 0$ and $a_{n+1} = a_n + \varepsilon_n \cdot 2^n$ where $\varepsilon_n \in \{0, 1\}$ for any n. Observe that $\mathcal{A}$ has cardinality continuum and distinct sequences in $\mathcal{A}$ have distinct ω-limits.

Show and use that the spaces $\mathcal{X}^\omega$ and $(\mathcal{X}^\omega)^\omega$ have discrete metrics and both have cardinality at most continuum.

A more conceptual construction of $\mathcal{A}$. Choose a compact metric space $\mathcal{K}$ with continuum points, say $\mathcal{K} = [0, 1]$. Identify $\mathcal{X}$ with a dense subset of $\mathcal{K}$. For any point $k \in \mathcal{K}$, choose a sequence $a_n \in \mathcal{X}$ that converges to k. Observe that the family of all these sequences meet the required condition.

6.15. Choose a bijection $\iota\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Given a set $S \subset \mathbb{N}$, consider the sequence $S_1, S_2, \ldots$ of subsets in $\mathbb{N}$ defined by $m \in S_n$ if $(m, n) = \iota(k)$ for some $k \in S$. Set $\omega_1(S) = 1$ if and only if $\omega(S_n) = 1$ for ω-almost all n. It remains to check that ω_1 meets the conditions of the exercise.

Comment. It turns out that $\omega_1 \neq \omega$ for any ι; see the post of Andreas Blass [28].

6.17. Arguing as in 6.16, we get a pair of points x and y in $\mathcal{X}$ such that

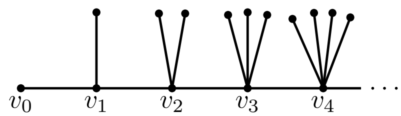
$$|p - x| + |x - y| + |y - q| = |p - q|$$

and there is no midpoint between x and y in $\mathcal{X}$ (possibly $p = x$ and $q = y$). Note that it is sufficient to show that there is a continuum of distinct midpoints in $\mathcal{X}^\omega$ between x and y in $\mathcal{X}$.

Since $\mathcal{X}$ is a length space, we can choose a $\frac{1}{n}$-midpoint $m_n \in \mathcal{X}$ between x and y. Note that the sequence m_n contains no converging subsequence. Conclude that we may pass to a subsequence of m_n such that $|m_i - m_j| > \varepsilon$ for a fixed $\varepsilon > 0$ and any $i \neq j$.

Argue as in 6.14 to show that there is a continuum of distinct ω-limits of subsequences of m_n; each such limit is a midpoint between x and y.

6.18. Consider the infinite metric $\mathcal{T}$ tree with unit edges shown on the diagram. Observe that $\mathcal{T}$ is proper.



Consider the vertex $v_\omega = \lim_{n\to\omega} v_n$ in the ultrapower $\mathcal{T}^\omega$. Observe that ω has an infinite degree. Conclude that $\mathcal{T}^\omega$ is not locally compact.

6.19. Consider a product of an infinite sequence of two-point spaces.

Remark. There are such examples of geodesic spaces with a cocompact isometric action of a finitely generated group [127].

6.20. Assume $\mathcal{L}$ is the Lobachevsky plane.

(a). Show that there is $\delta > 0$ such that sides of any geodesic triangle in $\mathcal{L}$ intersect a disk of radius δ. Conclude that any geodesic triangle in $\operatorname{Asym}\mathcal{L}$ is a tripod.

(b). Observe that $\mathcal{L}$ is one-point-homogeneous and use it.

(c). By *(b)*, it is sufficient to show that p_ω has a continuum degree.

Choose distinct geodesics $\gamma_1, \gamma_2 \colon [0,\infty) \to \to L$ that start at a point p. Show that the limits of γ_1 and γ_2 run in the different connected components of $(\operatorname{Asym}\mathcal{L})\setminus\{p_\omega\}$. Since there is a continuum of distinct geodesics starting at p, we get that the degree of p_ω is at least continuum.

On the other hand, the set of sequences of points in $\mathcal{L}$ has cardinality continuum. In particular, the set of points in $\operatorname{Asym}\mathcal{L}$ has cardinality at most continuum. It follows that the degree of any vertex is at most continuum.

The proof for the Lobachevsky space goes along the same lines.

For the infinite three-regular tree, part *(a)* follows from 6.11. The three-regular tree is only vertex-homogeneous; the latter is sufficient to prove *(b)*. No changes are needed in *(c)*.

Remark. The properties *(b)* and *(c)* describe the tree $\mathcal{T}$ up to isometry [56]. In particular, the asymptotic space of the Lobachevsky plane does not depend on the choice of the ultrafilter and the sequence $\lambda_n \to \infty$.

6.21. Denote by o_ω the point in $\mathrm{T}^\omega_o\mathcal{X}$ that corresponds to o. Argue as in 6.20*c* to show that $\mathrm{T}^\omega_o\mathcal{X}\setminus\{o_\omega\}$ has a continuum of connected components. Further, show that each connected component $\mathcal{W}_\alpha$ is isometric to $\mathbb{R}\times(0,\infty)$ with the metric described by

$$|(x_1,t_1)-(x_2,t_2)| = \\ = \min\{\,|(x_1,t_1)-(x_2,t_2)|_{\mathbb{R}^2}, t_1+t_2\,\}.$$

Conclude that the space $\mathrm{T}^\omega_o\mathcal{X}$ can be described as follows. Prepare continuum copies $\mathcal{W}_\alpha$ as above; denote by $(x,t)_\alpha$ the point in $\mathcal{W}_\alpha$ with coordinates (x,t). The tangent space is the disjoint union of single point o_ω and all $\mathcal{W}_\alpha$ with metric such that $|(x_1,t_1)_\alpha-(x_2,t_2)_\alpha|$ is the same as in $\mathcal{W}_\alpha$ and for the remaining pairs, we have $|o_\omega-(x,t)_\alpha| = t$ and $|(x_1,t_1)_\alpha - (x_2,t_2)_\beta| = t_1+t_2$ if $\alpha\neq\beta$.

7.4. Let us show that $\gamma \leqslant \alpha+\beta$; the rest of inequalities can be done the same way. Since $\gamma\leqslant\pi$, we may assume that $\alpha+\beta<\pi$.

Denote by γ_x, γ_y, and γ_z the geodesics $[px]$, $[py]$, and $[pz]$ parameterized from p by arc-length. By the triangle inequality, for any $\varepsilon>0$ and all sufficiently small $t,\tau,s\in\mathbb{R}_+$ we have

$$\begin{aligned}&|\gamma_x(t)-\gamma_z(\tau)| \leqslant \\ &\quad\leqslant |\gamma_x(t)-\gamma_y(s)|+|\gamma_y(s)-\gamma_z(\tau)| < \\ &\quad< t^2+s^2-2{\cdot}t{\cdot}s{\cdot}\cos(\alpha+\varepsilon)+ \\ &\quad\quad+\sqrt{s^2+\tau^2-2{\cdot}s{\cdot}\tau{\cdot}\cos(\beta+\varepsilon)}\leqslant\end{aligned}$$

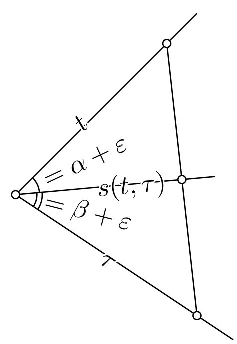
Below we define $s(t,\tau)$ so that for $s = = s(t,\tau)$, this chain of inequalities can be continued as follows:

$$\leqslant \sqrt{t^2+\tau^2-2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha+\beta+2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon>0$,

$$\gamma\leqslant\alpha+\beta+2{\cdot}\varepsilon.$$

Hence the result.



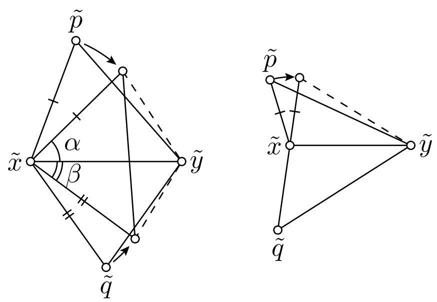
To define $s(t,\tau)$, consider three rays $\tilde\gamma_x$, $\tilde\gamma_y$, $\tilde\gamma_z$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde\gamma_x,\tilde\gamma_y)=\alpha+\varepsilon$, $\measuredangle(\tilde\gamma_y,\tilde\gamma_z)=\beta+\varepsilon$ and $\measuredangle(\tilde\gamma_x,\tilde\gamma_z)=\alpha+\beta+2{\cdot}\varepsilon$. We parametrize each ray by the distance from the starting point. Given two positive numbers $t,\tau\in\mathbb{R}_+$, let $s = s(t,\tau)$ be the number such that $\tilde\gamma_y(s)\in \in[\tilde\gamma_x(t)\ \tilde\gamma_z(\tau)]$. Clearly $s\leqslant\max\{t,\tau\}$, so t,τ,s may be taken sufficiently small.

Remark. Note that for the Euclidean space the statement implies that central angle defines a metric on unit sphere. This statement is not quite trivial; moreover, it is straightforward to modify Euclidean proof so it will work in Alexandrov settings.

7.5; *only-if part.* Let us start with two model triangles $[\tilde x\tilde y\tilde p]=\tilde\triangle(xyp)$ and $[\tilde x\tilde y\tilde q]=\tilde\triangle(xyq)$ such that $\tilde p$ and $\tilde q$ lie on the opposite sides of the line $\tilde x\tilde y$.

Suppose $[\tilde x\tilde y]$ intersects $[\tilde p\tilde q]$ at a point $\tilde z$. In this case by CAT(0) comparison we have that

$$|\tilde p-\tilde q|_{\mathbb E^2}=|\tilde p-\tilde z|_{\mathbb E^2}-|\tilde z-\tilde q|_{\mathbb E^2}\leqslant|p-q|_{\mathcal X}.$$



Let us fix points $\tilde x$ and $\tilde y$, and the distances from $\tilde x$ to the remaining three points and reduce the angles $\alpha=\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}]$ and $\beta=\measuredangle[\tilde x\,{}^{\tilde q}_{\tilde y}]$. It results in decreasing distances $|\tilde p-\tilde q|$, $|\tilde p-\tilde y|$, and $|\tilde q-\tilde y|$. If $\alpha=\beta=0$, then

$$\begin{aligned}|\tilde p-\tilde q|_{\mathbb E^2}&=\Big||\tilde x-\tilde p|_{\mathbb E^2}-|\tilde x-\tilde q|_{\mathbb E^2}\Big|=\\&=\Big||x-p|_{\mathcal X}-|x-q|_{\mathcal X}\Big|\geqslant\\&\geqslant|p-q|_{\mathcal X}.\end{aligned}$$

By the intermediate value theorem, there are intermediate values of α and β so that $|\tilde p-\tilde q|_{\mathbb E^2}=|p-q|_{\mathcal X}$. By construction, $|\tilde x-\tilde p|_{\mathbb E^2}=|x-p|_{\mathcal X}$, $|\tilde x-\tilde q|_{\mathbb E^2}=|x-q|_{\mathcal X}$, $|\tilde y-\tilde p|_{\mathbb E^2}\leqslant|y-p|_{\mathcal X}$, $|\tilde y-\tilde q|_{\mathbb E^2}\leqslant|y-q|_{\mathcal X}$.

Now suppose $[\tilde p\tilde q]$ does not intersect $[\tilde x\tilde y]$. Without loss of generality, we may assume that $[\tilde p\tilde q]$ crosses the line $\tilde x\tilde y$ behind $\tilde x$.

Let us rotate $\tilde p$ around $\tilde x$ so that $\tilde x$ will lie between $\tilde p$ and $\tilde q$. It will result in decreasing the distance $|\tilde p-\tilde y|$; by the triangle inequality we have that

$$\begin{aligned}|\tilde p-\tilde q|_{\mathbb E^2}&=|\tilde p-\tilde x|_{\mathbb E^2}+|\tilde x-\tilde q|_{\mathbb E^2}=\\&=|p-x|_{\mathcal X}+|x-q|_{\mathcal X}\geqslant\\&\geqslant|p-q|_{\mathcal X}.\end{aligned}$$

Repeating the argument above produces the needed configuration.

If part. Suppose $\tilde p,\tilde q,\tilde x,\tilde y\in\mathbb E^2$ satisfies the conditions

$$\begin{aligned}&|\tilde p-\tilde q|=|p-q|,&&|\tilde x-\tilde y|=|x-y|,\\&|\tilde p-\tilde x|\leqslant|p-x|,&&|\tilde p-\tilde y|\leqslant|p-y|,\\&|\tilde q-\tilde x|\leqslant|q-x|,&&|\tilde q-\tilde y|\leqslant|q-y|.\end{aligned}$$

Fix $\tilde z\in[\tilde x\tilde y]$. By the triangle inequality

$$|\tilde p-\tilde z|+|\tilde z-\tilde q|\geqslant|\tilde p-\tilde q|=|p-q|.$$

Note that if $|\tilde p'-\tilde x|\geqslant|\tilde p-\tilde x|$ and $|\tilde p'-\tilde y|\geqslant|\tilde p-\tilde y|$, then $|\tilde p'-\tilde z|\geqslant|\tilde p-\tilde z|$. In particular, if $[\tilde x\tilde y\tilde p']=\tilde\triangle(xyp)$ and $[\tilde x\tilde y\tilde q']=\tilde\triangle(xyq)$, then

$$|\tilde p'-\tilde z|+|\tilde q'-\tilde z|\geqslant|\tilde p-\tilde z|+|\tilde z-\tilde q|.$$

Whence the "if" part follows.

7.6. Set $\tilde\alpha=\tilde\measuredangle(p\,{}^x_y)$, $\tilde\beta=\tilde\measuredangle(p\,{}^y_z)$ and $\tilde\gamma=\tilde\measuredangle(p\,{}^z_x)$.

If $\mathcal X$ is CBB(0), then

$$\tilde\alpha+\tilde\beta+\tilde\gamma\leqslant2\cdot\pi.$$

Note that we can find α,β,γ such that

$$\tilde\alpha\leqslant\alpha\leqslant\pi,\quad\tilde\beta\leqslant\beta\leqslant\pi,\quad\tilde\gamma\leqslant\gamma\leqslant\pi,$$

and

$$\alpha+\beta+\gamma=2\cdot\pi.$$

Consider a model configuration $\tilde p$, $\tilde x$, $\tilde y$, $\tilde z\in\mathbb E^2$ such that

$$\begin{aligned}&|\tilde p-\tilde x|_{\mathbb E^2}=|p-x|_{\mathcal X},&&\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}]=\alpha,\\&|\tilde p-\tilde y|_{\mathbb E^2}=|p-y|_{\mathcal X},&&\measuredangle[\tilde p\,{}^{\tilde y}_{\tilde z}]=\beta,\\&|\tilde p-\tilde z|_{\mathbb E^2}=|p-z|_{\mathcal X},&&\measuredangle[\tilde p\,{}^{\tilde z}_{\tilde x}]=\gamma.\end{aligned}$$

Since increasing angle in a triangle increase the opposite side, we have

$$\begin{aligned}|x-y|_{\mathcal X}&\leqslant|\tilde x-\tilde y|_{\mathbb E^2},\\|y-z|_{\mathcal X}&\leqslant|\tilde y-\tilde z|_{\mathbb E^2},\\|z-x|_{\mathcal X}&\leqslant|\tilde z-\tilde x|_{\mathbb E^2}.\end{aligned}$$

Whence the "only-if" part follows.

Now suppse that we have a model configuration $\tilde p,\tilde x,\tilde y,\tilde z\in\mathbb E^2$ such that

$$\begin{aligned}&|p-x|_{\mathcal X}=|\tilde p-\tilde x|_{\mathbb E^2},&&|x-y|_{\mathcal X}\leqslant|\tilde x-\tilde y|_{\mathbb E^2},\\&|p-y|_{\mathcal X}=|\tilde p-\tilde y|_{\mathbb E^2},&&|y-z|_{\mathcal X}\leqslant|\tilde y-\tilde z|_{\mathbb E^2},\\&|p-z|_{\mathcal X}=|\tilde p-\tilde z|_{\mathbb E^2},&&|z-x|_{\mathcal X}\leqslant|\tilde z-\tilde x|_{\mathbb E^2}.\end{aligned}$$

Set

$$\alpha=\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}],\qquad\beta=\measuredangle[\tilde p\,{}^{\tilde y}_{\tilde z}],\qquad\gamma=\measuredangle[\tilde p\,{}^{\tilde z}_{\tilde x}].$$

Observe that

$$\alpha+\beta+\gamma\leqslant2\cdot\pi.$$

Since increasing a side in a triangle increase the opposite angle, we have that

$$\tilde\alpha \leqslant \alpha, \quad \tilde\beta \leqslant \beta, \quad \tilde\gamma \leqslant \gamma.$$

Whence the "if" part follows.

7.7. Set $\tilde\alpha = \tilde\measuredangle(p\,{}^x_q)$, $\tilde\beta = \tilde\measuredangle(p\,{}^y_q)$ and $\tilde\gamma = \tilde\measuredangle(p\,{}^x_y)$.

Note that the quadruple p, x, y, z is euclidean if

❶ $$\tilde\alpha + \tilde\beta + \tilde\gamma \leqslant 2\cdot\pi$$

and the triple of numbers $\tilde\alpha, \tilde\beta, \tilde\gamma$ satisfies all triangle inequalities. Without loss of generality we may assume that $\tilde\alpha \leqslant \tilde\beta \leqslant \tilde\gamma$; in this case, the triangle inequities hold if

❷ $$\tilde\gamma \leqslant \tilde\alpha + \tilde\beta.$$

Note that the inequality ❶ follow from CBB(0) comparison.

Consider two model triangles $[\tilde x\tilde y\tilde p] = \tilde\triangle(xyp)$ and $[\tilde x\tilde y\tilde q] = \tilde\triangle(xyq)$ such that $\tilde p$ and $\tilde q$ lie on the opposite sides of the line $\tilde x\tilde y$.

Suppose $[\tilde x\tilde y]$ intersects $[\tilde p\tilde q]$ at a point $\tilde z$. In this case by CAT(0) comparison we have that

$$|\tilde x - \tilde y|_{\mathbb{E}^2} = |\tilde x - \tilde z|_{\mathbb{E}^2} - |\tilde z - \tilde y|_{\mathbb{E}^2} \leqslant |x - y|_{\mathcal{X}}.$$

Which is equivalent to ❷.

If $[\tilde x\tilde y]$ crosses the line $[\tilde p\tilde q]$ behind $\tilde p$, then $\tilde\alpha + \tilde\beta > \pi$ and therefore ❷ follows from ❶.

Finally if $[\tilde x\tilde y]$ crosses the line $[\tilde p\tilde q]$ behind $\tilde q$, then by CBB(0) comparison with center at q, we have that

$$\tilde\measuredangle(q\,{}^x_p) + \tilde\measuredangle(q\,{}^y_p) + \tilde\measuredangle(q\,{}^x_y) \leqslant 2\cdot\pi$$

It follows that

$$|\tilde x - \tilde y|_{\mathbb{E}^2} \geqslant |x - y|_{\mathcal{X}}$$

and therefore

$$\tilde\gamma \leqslant \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}].$$

Since $\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] = \tilde\alpha + \tilde\beta$ we get ❷.

7.10. We will use the charcterization of CBB(0) space provided by 7.6; the rest is nearly identical to the proof of 7.9.

Fix a quadruple in $\mathcal{U} \times \mathcal{V}$:

$$p = (p_1, p_2), \qquad x = (x_1, x_2),$$
$$y = (y_1, y_2), \qquad z = (z_1, z_2).$$

For the quadruple p_1, x_1, y_1, z_1 in $\mathcal{U}$, construct model configurations $\tilde p_1, \tilde x_1, \tilde y_1, \tilde z_1$ in $\mathbb{E}^2$ provided by 7.6. Similarly, for the quadruple p_2, q_2, x_2, y_2 in $\mathcal{V}$ construct model configurations $\tilde p_2, \tilde x_2, \tilde y_2, \tilde z_2$ in $\mathbb{E}^2$

Consider four points in $\mathbb{E}^4 = \mathbb{E}^2 \times \mathbb{E}^2$

$$\tilde p = (\tilde p_1, \tilde p_2), \qquad \tilde x = (\tilde x_1, \tilde x_2),$$
$$\tilde y = (\tilde y_1, \tilde y_2), \qquad \tilde z = (\tilde z_1, \tilde z_2).$$

The inequalities in 7.6 imply that

$$|p - x|_{\mathcal{X}} = |\tilde p - \tilde x|_{\mathbb{E}^4}, \quad |x - y|_{\mathcal{X}} \leqslant |\tilde x - \tilde y|_{\mathbb{E}^4},$$
$$|p - y|_{\mathcal{X}} = |\tilde p - \tilde y|_{\mathbb{E}^4}, \quad |y - z|_{\mathcal{X}} \leqslant |\tilde y - \tilde z|_{\mathbb{E}^4},$$
$$|p - z|_{\mathcal{X}} = |\tilde p - \tilde z|_{\mathbb{E}^4}, \quad |z - x|_{\mathcal{X}} \leqslant |\tilde z - \tilde x|_{\mathbb{E}^4}.$$

It remains to observe that one can move $\tilde z$ into the plane of $\tilde p$, $\tilde x$, and $\tilde y$ keeping the distance $|\tilde p - \tilde z|_{\mathbb{E}^4}$ and nondecreasing the rest of distances.

7.13. Suppose that there are distinct geodesics. Then there are two points p and q on different geodesics such that $|p - x| = |q - x|$. Without loss of generality, we may assume that $|z - x| < |p - x|$; in other words z lies between p and x on the first geodesic and z lies between q and x on the second geodesic. Observe that

$$\tilde\measuredangle(z\,{}^x_p) = \tilde\measuredangle(z\,{}^x_q) = \pi.$$

By comparison, we have

$$\tilde\measuredangle(z\,{}^x_p) + \tilde\measuredangle(z\,{}^x_q) + \tilde\measuredangle(z\,{}^p_q) \leqslant 2\cdot\pi.$$

It follows that $\tilde\measuredangle(z\,{}^p_q) = 0$. Since $|z-p| = |z-q|$, it implies that $p = q$ — a contradiction.

7.16. Use 7.15a, to show that the map $(t, x) \mapsto \gamma_x(t)$ is continuous; that is $h_t(x) = \gamma_x(t)$ defines a homotopy.

It remains to observe that $h_1(x) = x$ and $h_0(x) = p$ for any x.

7.17. Suppose that a geodesic $[pq]$ is not expendable behind q. Denote by h_t the geodesic homotopy with the center at p; see 7.16.

Since $[pq]$ is not extendable, $q \notin \operatorname{Im} h_t$ for any $t < 1$. In particular, the local homology groups vanish at p; the latter does not hold for a manifold — a contradiction.

7.20. Apply 7.15b twice.

More precisely, consider a triangle $[xyz]$ in the space; let $[\tilde x\tilde y\tilde z] = \tilde\triangle(xyz)$. Choose points $p \in [xy]$ and $q \in [xz]$; consider the corresponding points $\tilde p \in [\tilde x\tilde y]$ and $\tilde q \in [\tilde x\tilde z]$. We need to show that

❸ $$|\tilde p - \tilde q|_{\mathbb{E}^2} \leqslant |p - q|_{\mathcal{X}}.$$

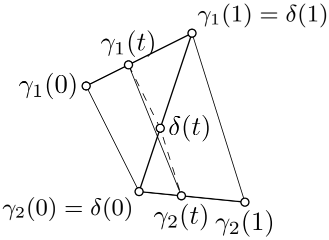
By 7.15b, we have

$$\tilde\measuredangle(x\,{}^p_q) \geqslant \tilde\measuredangle(x\,{}^y_z).$$

Whence ❸ follows.

7.21. It is sufficient to prove the Jensen inequality; that is,

$$|\gamma_1(t) - \gamma_2(t)| \leqslant (1-t)\cdot|\gamma_1(0) - \gamma_2(0)| + t\cdot|\gamma_1(1) - \gamma_2(1)|.$$



Let δ be the geodesic path from $\gamma_2(0)$ to $\gamma_1(1)$. From 7.15a, we have

$$|\gamma_1(t) - \delta(t)| \leqslant (1-t)\cdot|\gamma_1(0) - \delta(0)|$$
$$|\delta(t) - \gamma_2(t)| \leqslant t\cdot|\delta(1) - \gamma_2(1)|$$

It remains to sum it up and apply the triangle inequality.

Remark. Note that in the Euclidean space the proof is just as hard.

7.22. Let $p, q \in A_r$; that is, there are points $p^*, q^* \in A$ such that $|p - p^*|, |q - q^*| \leqslant r$. Consider a geodesic path γ from p to q and a geodesic path γ^* from p^* to q^*. Set $f(t) = |\gamma(t) - \gamma^*(t)|$.

Observe that $f(0), f(t) \leqslant r$. By 7.21, f is convex. Therefore $f(t) \leqslant r$ for any $t \in [0,1]$.

Since A is convex γ^* runs in A. Therefore $f(t) \geqslant \mathrm{dist}_A \circ \gamma(t)$; that is, γ runs in A_r.

7.23; *(a).* Assume there are two point $x, y \in K$ that minimize the distance to p; suppose $\ell = |p - x| = |p - y|$. Since K is convex, the geodesic $[xy]$ lies in K. Let m be a midpoint of $[xy]$.

Use thinness of $[pxy]$ to show that $|p - m| < \ell$. It follows that x does not minimize the distance to p — a contradiction.

(b). Let p^* and q^* be the closest point projections of p and q to K. Assume all four points p, q, p^*, q^* are distinct. Consider two model triangles $[\tilde p \tilde p^* \tilde q^*] = \tilde\triangle(pp^*q^*)$ and $[\tilde p \tilde q \tilde q^*] = \tilde\triangle(pqq^*)$ such that the points $\tilde p^*$ and $\tilde q$ lie on the opposite sides from the line $\tilde p \tilde q^*$.

Use thinness of $[pp^*q^*]$ and $[pqq^*]$ to show that $\measuredangle[\tilde p^*\,{}^{\tilde p}_{\tilde q^*}] \geqslant \frac{\pi}{2}$ and $\measuredangle[\tilde q^*\,{}^{\tilde p^*}_{\tilde q}] \geqslant \frac{\pi}{2}$. Finally observe that

$$|p - q|_{\mathcal U} = |\tilde p - \tilde q|_{\mathbb E^2} \geqslant |\tilde p^* - \tilde q^*|_{\mathbb E^2} = |p^* - q^*|_{\mathcal U}.$$

If some of the points p, q, p^*, q^* coincide, then the proof is easier.

7.24. Fix a closed, connected, locally convex set K.

Let us show that $f = \mathrm{dist}_K$ is convex in a neighborhood $\Omega \supset K$; that is, dist_K is convex along any geodesic completely contained in Ω. It is sufficient to show that for any a point $p \in K$ the function f is convex in a ball $B_p = \mathrm{B}(p, r_p)$ if $K \cap \overline{\mathrm{B}}[p, 2\cdot r_p]$ is convex.

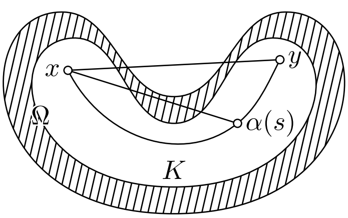
By 7.21 for any geodesic path γ_0 in B and any geodesic path γ_1 in K we have that the function $t \mapsto |\gamma_0(t) - \gamma_1(t)|$ is convex. We may choose γ_1 in such a way that its ends realize the distances from the ends of γ_0 to K; that is,

$$|\gamma_0(0) - \gamma_1(0)| = f \circ \gamma_0(0),$$
$$|\gamma_0(1) - \gamma_1(1)| = f \circ \gamma_0(1).$$

Observe that

$$|\gamma_0(t) - \gamma_1(t)| \geqslant f \circ \gamma_0(t)$$

for any t. Whence Jensen's inequality holds for $f \circ \gamma$ if γ is any geodesic in B_p.



Since K is locally convex, it is locally path-connected. Since K is connected, the latter implies that K is path-connected.

Fix two points $x, y \in K$. Let us connect x to y by a path $\alpha\colon [0,1] \to K$. By 7.12 and 7.21, the geodesic $[x\,\alpha(s)]$ is uniquely defined and depends continuously on s.

If $[xy] = [x\,\alpha(1)]$ does not completely lie in K, then there is a value $s \in [0,1]$ such that $[x\,\alpha(s)]$ lies in Ω, but does not completely lie in K. Therefore f is convex along $[x\alpha(s)]$. Note that $f(x) = f(\alpha(s)) = 0$ and $f \geqslant 0$, therefore $f(z) = 0$ for any $z \in [x\,\alpha(s)]$. In other words, $[x\,\alpha(s)] \subset K$ — a contradiction.

Remark. The statement generalizes a theorem of Heinrich Tietze [129]; our proof is nearly identical to the original.

8.5. If A is not convex, then there is a geodesic $[xy]$ with the ends in A and the remaining points outside of A. Observe that in the doubling, say $\mathcal{W}$, two copies of this geodesics connect the same pair of points x and y. By 7.12, $\mathcal{W}$ is not CAT(0).

8.12. By approximation, it is sufficient to consider the case when A and B have smooth boundary.

If $[xy] \cap A \cap B \neq \varnothing$, then $z_0 \in [xy]$ and $\dot{A}, \dot{B}$ can be chosen to be arbitrary half-spaces containing A and B respectively.

In the remaining case $[xy] \cap A \cap B = \varnothing$, we have $z_0 \in \partial(A \cap B)$. Consider the solid ellipsoid

$$C = \{\, z \in \mathbb{E}^m \;:\; f(z) \leqslant f(z_0) \,\}.$$

Note that C is compact, convex and has smooth boundary.

Suppose $z_0 \in \partial A \cap \operatorname{Int} B$. Then A and C touch at z_0 and we can set $\dot{A}$ to be the uniquely defined supporting half-space to A at z_0 and $\dot{B}$ to be any half-space containing B. The case $z_0 \in \partial B \cap \operatorname{Int} A$ is treated similarly.

Finally, suppose $z_0 \in \partial A \cap \partial B$. Then the set $\dot{A}$ (respectively, $\dot{B}$) is defined as the unique supporting half-space to A (respectively, B) at z_0 containing A (respectively, B).

Suppose $f(z) < f(z_0)$ for some $z \in \dot{A} \cap \dot{B}$. Since f is concave, $f(\bar{z}) < f(z_0)$ for any $\bar{z} \in$ $\in [zz_0[$. Since $[zz_0[\cap A \cap B \neq \varnothing$, the latter contradicts the fact that z_0 is minimum point of f on $A \cap B$.

8.13. Fix two open balls $B_1 = \mathrm{B}(0, r_1)$ and $B_2 = \mathrm{B}(0, r_2)$ such that

$$B_1 \subset A^i \subset B_2$$

for each wall A^i.

Suppose X is an intersections of the walls. Observe that

$$B_1 \subset X \subset B_2.$$

Therefore if $x \in X$, then X contains the convex hull $\operatorname{Conv}(B_1 \cup \{x\}$; therefore all intersections of the walls have ε-wide corners for $\varepsilon =$ $= 2 \cdot \arcsin \frac{r_1}{r_2}$.

8.14. Note that any centrally symmetric convex closed set in Euclidean space is a product of a compact centrally symmetric convex set and a subspace.

It follows that there is $R < \infty$ such that if X is an intersection of an arbitrary number of walls, then for any point $p \in X$ there is an isometry of X that moves p to a point in the ball $\mathrm{B}(0, R)$.

It remains to apply the argument in 8.13.

8.18. Note that we can assume that the balls have zero radiuses.

Observe that at each collision the balls exchange their velocities. Let us also change their labels at each collision. Note that after the relabeling, the coordinates functions $t \mapsto x_i(t)$ of the balls are linear functions in time.[1]

It remains to show n lines on the plane have at most $\frac{n\cdot(n-1)}{2}$ intersections. It follows since any pair of lines have at most one intersection.

Remarks. For nonidentical balls, the problem is a bit more interesting; Grant Sanderson [121] has couple of funny movies on a partial case of this problem.

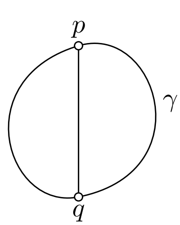
Recall that in the 3-dimensional case the number of collisions grows exponentially in n; the two-dimensional case is open [38].

9.5. Note that the existence of a null-homotopy is equivalent to the following. There are two one-parameter families of paths α_τ and β_τ, $\tau \in [0,1]$ such that:

- ◇ $\operatorname{length} \alpha_\tau$, $\operatorname{length} \beta_\tau < \pi$ for any τ.
- ◇ $\alpha_\tau(0) = \beta_\tau(0)$ and $\alpha_\tau(1) = \beta_\tau(1)$ for any τ.
- ◇ $\alpha_0(t) = \beta_0(t)$ for any t.
- ◇ $\alpha_1(t) = \alpha(t)$ and $\beta_1(t) = \beta(t)$ for any t.

By Corollary 9.3, the construction in Corollary 9.4 produces the same result for α_τ and β_τ. Hence the result.

9.10. The following proof works for compact locally simply connected metric spaces; it includes compact length, locally CAT(0) spaces.



By the globalization theorem there is a nontrivial homotopy class of closed curves.

Consider a shortest noncontractible closed curve γ in $\mathcal{X}$; note that such a curve exists.

[1] We use here that radiuses vanish; otherwise, $\tilde{x}_i = x_i - 2 \cdot k_i \cdot r$ are linear, where k_i is the number of i-th ball counted from left.

Indeed, let L be the infimum of lengths of all noncontractible closed curves in $\mathcal{X}$. Compactness and local contractibility of $\mathcal{X}$ imply that any two sufficiently close closed curves in $\mathcal{X}$ are homotopic. Then choosing a sequence of unit speed noncontractible curves whose lengths converge to L, an Arzelá–Ascoli type of argument shows that these curves subconverge to a noncontractible curve of length L.

Assume that γ is not a geodesic circle, that is, there are two points p and q on γ such that the distance $|p-q|$ is shorter then the lengths of the arcs, say α_1 and α_2, of γ from p to q. Consider the products, say γ_1 and γ_2, of $[qp]$ with α_1 and α_2. Then

- ⋄ γ_1 or γ_2 is noncontractible,
- ⋄ $\operatorname{length}\gamma_1, \operatorname{length}\gamma_2 < \operatorname{length}\gamma$,

a contradiction.

In the CAT(1) case we also have a geodesic circle. The proof is done nearly the same way, but we need to consider the homotopy classes of closed curves shorter than $2\cdot\pi$. One also need to apply 9.5, to show that curves γ_1 and γ_2 are not contractible in the class of curves shorter than $2\cdot\pi$.

Remark. The statement of the exercise fails if the requirement that $\mathcal{X}$ be compact is replaced by the assumption that it is proper. For example, the surface of revolution of the graph of $y = e^x$ around the x-axis is locally CAT(0) but has no closed geodesics.

9.11. Consider a closed ε-neighborhood A of the geodesic. Note that A_ε is convex. By the Reshetnyak gluing theorem, the double $\mathcal{W}_\varepsilon$ of $\mathcal{U}$ along A_ε is CAT(0).

Consider the space $\mathcal{W}'_\varepsilon$ obtained by doubly covering $\mathcal{U}\setminus A_\varepsilon$ and gluing back A_ε.

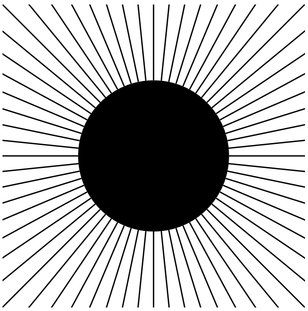
Observe that $\mathcal{W}'_\varepsilon$ is locally isometric to $\mathcal{W}_\varepsilon$. That is, for any point $p'\in\mathcal{W}'_\varepsilon$ there is a point $p\in\mathcal{W}_\varepsilon$ such that the δ-neighborhood of p' is isometric to the δ-neighborhood of p for all small $\delta>0$.

Further observe that $\mathcal{W}'_\varepsilon$ is simply connected since it admits a deformation retraction onto A_ε, which is contractible. By the globalization theorem, $\mathcal{W}'_\varepsilon$ is CAT(0).

It remains to note that $\widetilde{\mathcal{U}}$ can be obtained as a limit of $\mathcal{W}'_\varepsilon$ as $\varepsilon\to 0$, and apply Proposition 7.8.

10.1. Recall that by Proposition 8.1, any local geodesic shorter in $\mathcal{U}$ is a geodesic.

Consider a sequence of directions ξ_n at p of geodesics $[pq_n]$. Since the geodesics are extendable, we can assume that the distances $|p-q_n|_{\mathcal{U}} = 1$ for any n.



Since $\mathcal{U}$ is proper, we can pass to a converging subsequence of q_n; denote its limit by q. Since $q_n \to q$, the comparison implies that $\measuredangle[p\,{}^{q_n}_{q}]\to 0$ as $n\to\infty$. Therefore the direction ξ of $[pq]$ is the limit of directions ξ_n.

Note that the unit disc in the plane with attached half-line to each point is a complete CAT(0) length space with extendable geodesics. However, the space of geodesic directions on the boundary of the disc is not complete — there is no geodesic tangent to the boundary of the disc. This provides a counterexample to the statement of the exercise if $\mathcal{U}$ is not assumed to be proper.

10.2. Given a constant speed geodesic α starting at p, consider sequence of points $x_n = \alpha(\tfrac1n)$. Note that $n\cdot|p-x_n|$ is constant. Therefore if we consider x_n as a point in $n\cdot\mathcal{X}$, then this sequence has an ω-limit $\iota(\alpha)$ in T^ω_p.

Observe that ι defines a distance-preserving map $\mathrm{T}'_p\to\mathrm{T}^\omega_p$. Since T^ω_p is complete, this map can be extended to T_p. Whence the statement follows.

Since $\mathcal{X}$ is CAT(0), so is $n\cdot\mathcal{X}$, and by 7.8 so is $\mathrm{T}^\omega_p\mathcal{X}$. Since $\mathrm{T}_p\mathcal{X}$ is naturally isometric to a subspace of T^ω_p, we get that $\mathrm{T}_p\mathcal{X}$ is CAT(0) as well.

Remark. The ultratangent space might be larger than tangent space. For example, let Ш be a c o m b with a spine formed by a real line and a half-line (a tooth) attached to each point of the spine. Then for $p=0$ on the spine, T_pШ is formed by three half-lines meeting at one point, while T^ω_pШ is isometric to Ш.

10.3. Observe that it is sufficient to show that the space of directions Σ_p is a π-l e n g t h s p a c e;

the latter means that the defining condition of length space holds for pairs of points on distance less than π.

Since Σ_p is complete, the same argument as in 1.27*a*, shows that it sufficient to prove existence of almost midpoints for pairs of point on distance less than π; that is, if $\measuredangle(\xi,\zeta)<\pi$, then, given $\varepsilon>0$, there is $\mu\in\Sigma_p$ such that

$$❶ \qquad \measuredangle(\xi,\mu),\quad \measuredangle(\mu,\zeta)\leqslant \tfrac{1}{2}\cdot\measuredangle(\xi,\zeta)+\varepsilon.$$

Without loss of generality, we may assume that ζ and ξ are geodesic directions; so there are geodesics $[px]$ and $[pz]$ that start from p in these directions; in particular, $\measuredangle[p\,{}^x_z]=\measuredangle(\xi,\zeta)$. Fix small $r>0$ and choose points $\bar x=[px]$ and $\bar z=[pz]$ on the distance r from p. Since r is small, we can assume that

$$\measuredangle[p\,{}^x_z]+\varepsilon>\tilde\measuredangle(p\,{}^{\bar x}_{\bar z}).$$

Take a midpoint m of $[\bar x\bar y]$. By Alexandrov's lemma (7.14)

$$\tilde\measuredangle(p\,{}^{\bar x}_{m}),\quad \tilde\measuredangle(p\,{}^{m}_{\bar z})\leqslant\tfrac{1}{2}\cdot\tilde\measuredangle(p\,{}^{\bar x}_{\bar z}).$$

By comparison

$$\tilde\measuredangle(p\,{}^{\bar x}_{m})\geqslant\measuredangle[p\,{}^{\bar x}_{m}]\quad\text{and}\quad\tilde\measuredangle(p\,{}^{\bar z}_{m})\geqslant\measuredangle[p\,{}^{\bar z}_{m}].$$

Whence ❶ holds for the direction μ of $[pm]$.

10.8. Assume $\mathcal{P}$ is not CAT(0). Then by 10.7, a link Σ of some simplex contains a closed geodesic α with length $4\cdot\ell<2\cdot\pi$. We can assume that Σ has minimal possible dimension; so, by 10.7, Σ is locally CAT(1).

Divide α into two equal arcs α_1 and α_2.

Assume α_1 and α_2 are length minimizing; parameterize them by $[-\ell,\ell]$. Fix a small $\delta>0$ and consider two curves in $\operatorname{Cone}\Sigma$ written in polar coordinates as

$$\gamma_i(t)=(\alpha_i(\arctan\tfrac{t}{\delta}),\sqrt{\delta^2+t^2}).$$

Observe that both curves γ_1 and γ_2 are geodesics in $\operatorname{Cone}\Sigma$ and have common ends.

Observe that a small neighborhood of the tip of $\operatorname{Cone}\Sigma$ admits an isometric embedding into $\mathcal{P}$. Hence we can construct two geodesics γ_1 and γ_2 in $\mathcal{P}$ with common endpoints.

It remains to consider the case when α_1 (and therefore α_2) is not length minimizing.

Pass to its maximal length minimizing arc $\bar\alpha_1$ of α_1. Since Σ is locally CAT(1), 9.3 implies that there is another geodesic $\bar\alpha_2$ in Σ_p that shares endpoints with $\bar\alpha_1$. It remains to repeat the above construction for the pair $\bar\alpha_1$, $\bar\alpha_2$.

Remark. By 7.12, the given condition is a necessary and sufficient.

10.15. Use induction on the dimension to prove that if in a spherical simplex $\triangle$ every edge is at least $\frac{\pi}{2}$, then all dihedral angles of $\triangle$ are at least $\frac{\pi}{2}$.

The rest of the proof goes along the same lines as the proof of the flag condition (10.14). The only difference is that a geodesic may spend time *at least* π on each visit to Star_v.

Remark. Note that it is not sufficient to assume only that the all dihedral angles of the simplices are at least $\frac{\pi}{2}$. Indeed, the two-dimensional sphere with removed interior of a small rhombus is a spherical polyhedral space glued from four triangles with all the angles at least $\frac{\pi}{2}$. On the other hand the boundary of the rhombus is closed local geodesic in this space. Therefore the space cannot be CAT(1).

10.17. The space $\mathcal{T}_n$ has a natural cone structure with the vertex formed by the completely degenerate tree — all its edges have zero length.

Note that the space Σ over which the cone is taken comes naturally with a triangulation with all-right spherical simplicies. Each simplex corresponds to a combinatorics of a possibly degenerate tree.

Note that the link of any simplex of this triangulation satisfies the no-triangle condition (10.10). Indeed, fix a simplex $\triangle$ of the complex; suppose it is described by a possibly degenerate topological tree t. A triangle in the link of $\triangle$ can be described by three ways to resolve a degeneracy of t by adding one edge, such that (1) any pair of these resolutions can be done simultaneously, but (2) all three cannot be done simultaneously. Direct inspection shows that this is impossible.

Therefore, by Proposition 10.12 our complex is flag. It remains to apply the flag condition (10.14), and then 7.11.

11.2. If the complex $\mathcal{S}$ is flag, then its cubical analog $\square_{\mathcal{S}}$ is locally CAT(0) and therefore aspherical.

Assume now that the complex $\mathcal{S}$ is not flag. Extend it to a flag complex $\mathcal{T}$ by gluing a simplex in every clique (that is, a complete subgraph) of its one-skeleton.

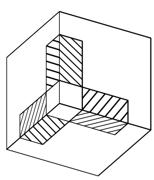
Note that the cubical analog $\square_{\mathcal{S}}$ is a proper subcomplex in $\square_{\mathcal{T}}$. Since $\mathcal{T}$ is flag, $\tilde\square_{\mathcal{T}}$, the universal cover of $\square_{\mathcal{T}}$, is CAT(0). Let $\tilde\square_{\mathcal{S}}$ be the inverse image of $\square_{\mathcal{S}}$ in $\tilde\square_{\mathcal{T}}$.

Choose a cube Q with minimal dimension in $\tilde\square_{\mathcal{T}}$ which is not present in $\tilde\square_{\mathcal{S}}$. By 7.24, Q is a convex set in $\tilde\square_{\mathcal{T}}$. The closest point projection $\tilde\square_{\mathcal{T}} \to Q$ is a retraction. It follows that the boundary ∂Q is not contractible in $\tilde\square_{\mathcal{T}} \setminus \operatorname{Int} Q$. Therefore the spheroid ∂Q is not contractible in $\tilde\square_{\mathcal{S}}$. That is, a covering of $\square_{\mathcal{S}}$ is not aspherical and therefore $\square_{\mathcal{S}}$ is not as well.

11.5. The solution goes along the same lines as the proof of Lemma 11.4, but few changes are needed.

The cycle γ is taken in the complement $\mathcal{S} \setminus \{v\}$ (or, alternatively, in the link of v in $\mathcal{S}$). Instead of a vertex, one has to take edge e in $\tilde Q$ that corresponds to v; so we show existence of large cycle in $\tilde Q$ that is not contractible in $\tilde Q \setminus e$. The last change is not principle: it is more visual to think that G is made from the squares parallel to the squares of the cubical complex which meet the edges of the complex orthogonally at their midpoints (in this case, formally speaking G, is not a subcomplex of the cubical analog).

11.9; *(b)⇒(a).* By 3.3c, Q is contractible. Therefore the globalization theorem and flag condition (9.6 and 10.14) imply that it is sufficient to show that each link in Q is flag. Further, by 10.12 it is sufficient to show that link of each cube in Q satisfies no-triangle condition.



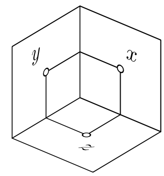
Arguing by contradiction, we can assume that no-triangle condition does not hold at a vertex v; that is, a zero-dimensional cube. In this case v is a vertex of there edges e_x, e_y, and e_z; each pair of edges belong to one of the squares s_x, s_y, and s_z with complementary index, but the squares s_x, s_y, s_z do not belong to one cube. For higher dimensional cubes we have a product of this configuration with a cube.

Let m_x, m_y and m_z be the midpoints of e_x, e_y, and e_z respectively. Consider 3 balls with centers m_x, m_y and m_z and radius $\frac{1}{4}$. Observe that each pair of balls have a common point; but all three together have no points of intersection. By 3.14c, the latter implies that (Q, ℓ^∞) is not an injective space — a contradiction.

(c)⇒(a). Observe that median point $m(x,y,z)$ of depends continuously on triple of points (x,y,z) and $m(x,x,y) = x$.

Given a loop $\gamma\colon [0,1] \to Q$ with base at $p = \gamma(0) = \gamma(1)$, consider the map $(a,b) \mapsto$ $\mapsto m(p, \gamma(a), \gamma(b))$ of the triangle $\triangle$ defined by $0 \leqslant a \leqslant b \leqslant 1$. Note that boundary of triangle runs along γ. It follows that γ is null homotopic and therefore Q is simply connected.

It remains to check that all links of Q satisfy no-triangle condition.



Assume that a link of Q does not satisfy the no-triangle condition. The same way as in the previous problem, we can assume that it is a link of a vertex; so we have a configuration of three squares s_x, s_y, and s_z, three edges e_x, e_y, and e_z, and one common vertex v as above. Observe that the centers x, y, and z of the squares s_x, s_y, and s_z. Observe that the geodesics $[xy]_{\ell^1}$, $[xz]_{\ell^1}$, and $[yz]_{\ell^1}$ are uniquely defined and they have no common point. It follows that the triple (x,y,z) does not have a median; that is, (Q, ℓ^1) is not a median space — a contradiction.

12.22. Let α be a closed curve in $\mathbb{S}^2$ of length $2{\cdot}\ell$.

Assume $\ell < \pi$. Let α_1 be a subarc of α of length ℓ, with endpoints p and q. Since $|p - q| \leqslant \ell < \pi$, there is a unique geodesic $[pq]$ in $\mathbb{S}^2$. Let z be the midpoint of $[pq]$.

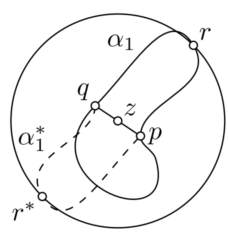
We claim that α lies in the open hemisphere H centered at z.

Assume the contrary; that is, α meets the equator ∂H at a point r. Without loss of generality, we may assume that $r \in \alpha_1$.

The arc α_1 together with its reflection α_1^* in z form a closed curve of length $2{\cdot}\ell$ which meets r and its antipodal point r'. Thus

$$\begin{aligned}\ell = \operatorname{length} \alpha_1 \geqslant \\ \geqslant |r - r'| = \pi\end{aligned}$$

— a contradiction.



Solution with the Crofton formula. Let α be a closed curve in $\mathbb{S}^2$ of length $\leqslant 2{\cdot}\pi$. We wish to prove that α is contained in a hemisphere in $\mathbb{S}^2$. By approximation it suffices to prove this for smooth curves α of length $< 2{\cdot}\pi$ with transverse self-intersections.

Given $v \in \mathbb{S}^2$, denote by $v^\perp$ the equator in $\mathbb{S}^2$ with the pole at v. Further, $\#X$ will denote the number of points in the set X.

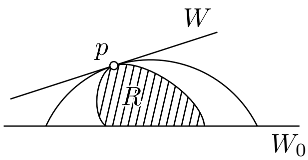
Obviously, if $\#(\alpha \cap v^\perp) = 0$, then α is contained in one of the hemispheres determined by $v^\perp$. Note that $\#(\alpha \cap v^\perp)$ is even for almost all v.

Therefore, if α does not lie in a hemisphere, then $\#(\alpha \cap v^\perp) \geqslant 2$ for almost all $v \in \mathbb{S}^2$.

By the Crofton formula we have that

$$\begin{aligned}\operatorname{length}(\alpha) &= \frac{1}{4}\cdot\int\limits_{\mathbb{S}^2}\#(\alpha\cap v^\perp)\cdot d_v\,\mathrm{area} \geqslant\\ &\geqslant 2\cdot\pi.\end{aligned}$$

12.23. Since Ω is not two-convex, we can choose a simple closed curve γ that lies in the intersection of a plane W_0 and Ω, and is contractible in Ω but not contractible in $\Omega \cap W_0$.



Let $\varphi\colon \mathbb{D} \to \Omega$ be a disc that shrinks γ. Applying the loop theorem (arguing as in the proof of Proposition 12.8), we can assume that φ is an embedding and $\varphi(\mathbb{D})$ lies on one side of W_0.

Let Q be the bounded closed domain cut from $\mathbb{E}^3$ by $\varphi(\mathbb{D})$ and W_0. By assumption it contains a point that is not in Ω. Changing W_0, γ and φ slightly, we can assume that such a point lies in the interior of Q.

Fix a circle Γ in W_0 that surrounds $Q \cap W_0$. Since Q lies in a half-space with boundary W_0, there is a smallest spherical dome with boundary Γ that includes the set $R = Q \setminus \Omega$.

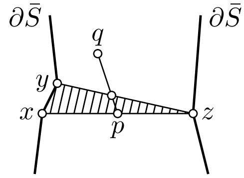
The dome has to touch R at some point p. The plane W tangent to the dome at p has the required property — the point p is an isolated point of the complement $W \setminus \Omega$. Further, by construction a small circle around p in W is contractible in Ω.

12.26. The proof is simple and visual, but it is hard to write it formally in a non-tedious way.

Consider the surface $\bar{S}$ formed by the closure of the remaining part S of the boundary. Note that the boundary ∂S of $\bar{S}$ is a collection of closed polygonal lines.

Assume $\bar{S}$ is not piecewise linear. Show that there is a line segment $[pq]$ in $\mathbb{E}^3$ that is tangent to $\bar{S}$ at some point p and has no common points with $\bar{S}$ except p.

Since $\bar{S}$ is locally concave, there is a local inner supporting plane Π at p that contains the segment $[pq]$.



Show that $\Pi \cap \bar{S}$ contains a segment $[xy] \ni\ni p$ with the ends in $\partial\bar{S}$. Denote by Π^+ the half-plane in Π that contains $[pq]$ and has $[xy]$ in its boundary.

Use the fact that $[pq]$ is tangent to S to show that there is a point $z \in \partial\bar{S}$ such that the line segment $[xz]$ or $[yz]$ lies in $\partial\bar{S} \cap \Pi^+$.

From the latter statement and local convexity of $\bar{S}$, it follows that the solid triangle $[xyz]$ lies in $\bar{S}$. In particular, all points on $[pq]$ sufficiently close to p lie in $\bar{S}$ — a contradiction.

13.1. Choose a function $r \mapsto \alpha(r)$ such that $\alpha'(r)\cdot r \to 0$ and $\alpha(r) \to \infty$ as $r \to 0$. Consider the reparametrization of the Euclidean plane given by $\iota\colon (r,\theta) \mapsto (r, \theta + \alpha(r))$ in the polar coordinates. Observe that ι is not differentiable at the origin, but the metric tensor g induced by ι is continuous.

For more on the subject read the paper of Eugenio Calabi and Philip Hartman [43].

13.5; *(a).* Suppose $p = f(x) = f(y)$ and the points $x, y \in \mathcal{M}$ are distinct. Since f is short, we get for any $r > 0$ the ball $\mathrm{B}(p,r)_{\mathcal{N}}$ contains the images of $\mathrm{B}(x,r)_{\mathcal{M}}$ and $\mathrm{B}(y,r)_{\mathcal{M}}$. Since f is volume-preserving, we get

❶ $\operatorname{vol}\mathrm{B}(x,r)_{\mathcal{M}} + \operatorname{vol}\mathrm{B}(y,r)_{\mathcal{M}} \leqslant \operatorname{vol}\mathrm{B}(p,r)_{\mathcal{N}}.$

By 13.2, for any $\varepsilon > 0$ and all sufficiently small $r > 0$ the volumes of the balls $\mathrm{B}(x,r)_{\mathcal{M}}$, $\mathrm{B}(y,r)_{\mathcal{M}}$ and $\mathrm{B}(p,r)_{\mathcal{N}}$, lie in the range $\omega_n\cdot e^{\mp 2\cdot n\cdot\varepsilon}\cdot r^n$, where ω_n denotes the volume of the unit ball in the n-dimensional Euclidean space. The latter contradicts ❶ for appropriate choice of ε and r.

(b). Denote by $\sigma(r,a)$ the volume of union of two r-balls in the n-dimensional Euclidean space such that the distance between their centers is a. Observe that the function $(a,r) \mapsto \sigma(r,a)$ is

continuous and increasing in a and r for $a \leqslant r$. Further, note that

$$\sigma(\lambda{\cdot}r, \lambda{\cdot}a) = \lambda^n{\cdot}\sigma(r, a)$$

for any $\lambda > 0$.

Choose a point $z \in \mathcal{M}$ and small $\varepsilon > 0$. By 13.2 there is $R > 0$ such that $\mathrm{B}(z, 10{\cdot}R)$ admits a $e^{\mp\varepsilon}$-bilipschitz map to the n-dimensional Euclidean space.

Choose $x, y \in \mathrm{B}(z, R)$. The argument used in part *(a)* implies that

$$\text{❷}\quad \begin{aligned} &e^{-n\cdot\varepsilon}{\cdot}\sigma(e^{-\varepsilon}{\cdot}r, e^{-\varepsilon}{\cdot}|x - y|_{\mathcal{M}}) \leqslant \\ &\leqslant e^{n\cdot\varepsilon}{\cdot}\sigma(e^{\varepsilon}{\cdot}r, e^{\varepsilon}{\cdot}|f(x) - f(y)|_{\mathcal{N}}). \end{aligned}$$

This inequality implies a lower bound on $|f(x) - f(y)|_{\mathcal{N}}$ in terms of $|x - y|_{\mathcal{M}}$.

Use the listed properties of the function $(a, r) \mapsto \sigma(r, a)$ to show that for any $c < 1$ there is $\varepsilon > 0$ such that ❷ implies that $b > c{\cdot}a$ for all sufficiently small a.

Finally, since $\mathcal{M}$ and $\mathcal{N}$ are length-metric spaces, part *(b)* implies that f is locally distance preserving. (An inclusion map from a nonconvex open subset to the plane gives an example of volume preserving short map that is not distance preserving.)

A more general result is discussed by Paul Creutz and Elefterios Soultanis [48].

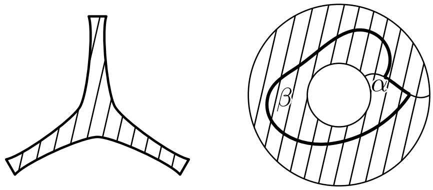
13.10. Denote by $\mathcal{M}$ and $\mathcal{M}^\circ$ the space of (M, g) and (M°, g); further denote by $\bar{\mathcal{M}}^\circ$ the completion of $\mathcal{M}^\circ$. Observe that the inclusion $M^\circ \hookrightarrow M$ induces a short onto map $\iota\colon \bar{\mathcal{M}}^\circ \to \mathcal{M}$.

Recall that M is bounded by hypersurface that is locally a graph. Use it to show that any sufficiently short curve γ in (M, g) can be approximated by a curve in $\mathcal{M}^\circ$ with g-length arbitrary close to $\mathrm{length}_g\,\gamma$. Conclude that ι is an isometry.

13.13. From the proof of Besicovitch inequality, one can see that the restriction of $\boldsymbol{f}$ to the interior of $\mathcal{M}$ is (1) volume-preserving, and (2) its differential $d_p\boldsymbol{f}\colon \mathrm{T}_p \to \mathrm{T}_{\boldsymbol{f}(p)}$ is an isometry for almost all p.

Since $\boldsymbol{f}$ is Lipschitz, (2) can be used to show that $\boldsymbol{f}$ is short. It remains to apply 13.5 and 13.10.

13.14. Consider the hexagon with flat metric and curved sides shown on the diagram. Observe that its area can be made arbitrarily small while keeping the distances from the opposite sides at least 1.



13.15; *(a).* Let α be a shortest curve that runs between the boundary components of the cylinder. Cut the cylinder along α. We get a square with Riemannian metric on it $(\square, g)$.

Two opposite sides of $\square$ correspond to the boundary components of the cylinder. The other pair corresponds to the sides of the cut. By assumption, the g-distance between the first pair of sides is at least 1.

Consider a shortest curve β that connects this pair of sides; let us keep the same notation for the projection of β in the cylinder.

Note that a cyclic concatenation γ of β with an arc of α is homotopic to a boundary circle. Therefore $\mathrm{length}_g\,\gamma \geqslant 1$. Since α is a shortest path, its arc cannot be longer than any curve connecting its ends; therefore

$$\mathrm{length}_g\,\beta \geqslant \tfrac{1}{2}\cdot \mathrm{length}_g\,\gamma \geqslant \tfrac{1}{2}.$$

That is, the other pair of sides of $\square$ lies on g-distance at least $\frac{1}{2}$ from each other. By 13.12, $\mathrm{area}(\square, g) \geqslant \frac{1}{2}$, hence the result.

(b). Note that any curve in the cylinder that is bordant to a boundary component has length at least 1. Therefore if $0 \leqslant t \leqslant 1$, then the level sets

$$L_t = \left\{\, x \in \mathbb{S}^1 \times [0, 1] \;:\; \mathrm{dist}_{\mathbb{S}^1\times\{0\}}(x)_g = t \,\right\}$$

have length at least 1. Applying the coarea inequality, we get that

$$\mathrm{area}(\mathbb{S}^1 \times [0, 1], g) \geqslant 1.$$

13.16; *(a).* Argue the same way as in 13.11, but observe in addition that $\mathrm{vol}\,\Sigma = \mathrm{vol}\,\boldsymbol{f}(\Sigma) = 0$ and use it time to time.

(b). Without loss of generality, we may assume that V lies in a unit cube $\square$. Consider a noncontinuous metric tensor $\bar{g}$ on $\square$ that coincides with g inside V and with the canonical flat metric tensor outside of V.

Observe that the $\bar{g}$-distances between opposite faces of $\square$ are at least 1. Indeed this is true for the Euclidean metric and the assumption $|p - q|_g \geqslant |p - q|_{\mathbb{E}^d}$ guarantees that one cannot make a shortcut in V. Therefore, the $\bar g$-distances between every pair of opposite faces is at least as large as 1 which is the Euclidean distance.

Applying part *(a)*, we get that $\operatorname{vol}(\square,\bar g)\geqslant$ $\geqslant\operatorname{vol}\square$. Whence the statement follows.

13.17. Let $x\in\mathbb{S}^2$ be a point that minimize the distance $|x-x'|_g$. Consider a shortest path γ from x to x'. We can assume that

$$|x-x'|_g=\operatorname{length}\gamma=1.$$

Let γ' be the antipodal arc to γ. Note that γ' intersects γ only at the common endpoints x and x'. Indeed, if $p'=q$ for some $p,q\in\gamma$, then $|p-q|\geqslant 1$. Since $\operatorname{length}\gamma=1$, the points p and q must be the ends of γ.

It follows that γ together with γ' forms a closed simple curve in $\mathbb{S}^2$; it divides the sphere into two disks D and D'.

Let us divide γ into two equal arcs γ_1 and γ_2; each of length $\frac12$. Suppose that $p,q\in\gamma_1$, then

$$\begin{aligned}|p-q'|_g&\geqslant|q-q'|_g-|p-q|_g\geqslant\\&\geqslant 1-\tfrac12=\tfrac12.\end{aligned}$$

That is, the minimal distance from γ_1 to γ_1' is at least $\frac12$. The same way we get that the minimal distance from γ_2 to γ_2' is at least $\frac12$. By Besicovitch inequality, we get that

$$\operatorname{area}(D,g)\geqslant\tfrac14\quad\text{and}\quad\operatorname{area}(D',g)\geqslant\tfrac14.$$

Therefore

$$\operatorname{area}(\mathbb{S}^2,g)\geqslant\tfrac12.$$

A better estimate. Let us indicate how to improve the obtained bound to

$$\operatorname{area}(\mathbb{S}^2,g)\geqslant 1.$$

Suppose x, x', γ and γ' are as above. Consider the function

$$f(z)=\min_t\{\,|\gamma'(t)-z|_g+t\,\}.$$

Observe that f is 1-Lipschitz.

Show that two points $\gamma'(c)$ and $\gamma(1-c)$ lie on one connected component of the level set $L_c=\{\,z\in\mathbb{S}^2\;:\;f(z)=c\,\}$; in particular

$$\operatorname{length}L_c\geqslant 2\cdot|\gamma'(c)-\gamma(1-c)|_g.$$

By the triangle inequality, we have that

$$\begin{aligned}|\gamma'(c)-\gamma(1-c)|_g&\geqslant 1-|\gamma(c)-\gamma(1-c)|_g=\\&=1-|1-2\cdot c|.\end{aligned}$$

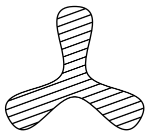
The coarea inequality (13.9)

$$\operatorname{area}(\mathbb{S}^2,g)\geqslant\int\limits_0^1\operatorname{length}L_c\cdot dc$$

finishes the proof.

The bound $\frac12$ was proved by Marcel Berger [18]. Christopher Croke conjectured that the optimal bound is $\frac4\pi$ and the round sphere is the only space that achieves this [Conjecture 0.3 in 49] — if you solved the last part of the problem, then publish the result.

13.18. Given $\varepsilon>0$, construct a disk Δ in the plane with



$$\operatorname{length}\partial\Delta<10$$

and

$$\operatorname{area}\Delta<\varepsilon$$

that admits an continuous involution ι such that

$$|\iota(x)-x|\geqslant 1$$

for any $x\in\partial\Delta$.

An example of Δ can be guessed from the picture; the involution ι makes a length preserving half turn of its boundary $\partial\Delta$.

Take the product $\Delta\times\Delta\subset\mathbb{E}^4$; it is homeomorphic to the 4-ball. Note that

$$\operatorname{vol}_3[\partial(\Delta\times\Delta)]=2\cdot\operatorname{area}\Delta\cdot\operatorname{length}\partial\Delta<20\cdot\varepsilon.$$

The boundary $\partial(\Delta\times\Delta)$ is homeomorphic to $\mathbb{S}^3$ and the restriction of the involution $(x,y)\mapsto$ $\mapsto(\iota(x),\iota(y))$ has the needed property.

It remains to smooth $\partial(\Delta\times\Delta)$ a bit.

Remark. This example is given by Christopher Croke [49]. Note that according to 14.24, the involution ι cannot be made isometric.

13.19. Note that if (M,g_∞) is $e^{\mp\varepsilon}$-bilipschitz to a cube, then applying Besicovitch inequality, we get that

$$\varliminf_{n\to\infty}\operatorname{vol}(M,g_n)\geqslant e^{-n\cdot\varepsilon}\cdot\operatorname{vol}(M,g_\infty).$$

By the Vitali covering theorem, given $\varepsilon>0$, we can cover the whole volume of (M,g_∞) by $e^{\pm\varepsilon}$-bilipschitz cubes. Applying the above observation and summing up the results, we get that

$$\varliminf_{n\to\infty}\operatorname{vol}(M,g_n)\geqslant e^{-n\cdot\varepsilon}\cdot\operatorname{vol}(M,g_\infty).$$

The statement follows since ε is an arbitrary positive number.

To solve the second part of the exercise, start with g_∞ and construct g_n by adding many tiny bubbles. The volume can be increased arbitrarily with an arbitrarily small change of metric.

Remark. A more general result was obtained by Sergei Ivanov [77]. Note that the statement does not hold true for Gromov–Hausdorff convergence. In fact any compact metric space $\mathcal{X}$ can be GH-approximated by a Riemannian surface with an arbitrarily small area. To show the latter statement, approximate $\mathcal{X}$ by a finite graph Γ, embed Γ isometrically to the Euclidean space, and pass to the surface of its neighborhood.

13.20. Set $s=\operatorname{sys}(\mathbb{T}^2,g)$.

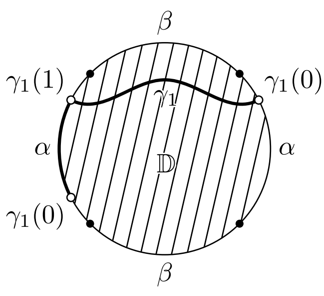
Cut $\mathbb{T}^2$ along a shortest closed noncontractible curve γ. We get a cylinder $(\mathbb{S}^1,g)$ with a Riemannian metric on it.

Applying the argument in 13.15a, we get that the g-distance between the boundary components is at least $\frac{s}{2}$. Then 13.15a implies that the area of torus is at least $\frac{s^2}{2}$.

Remark. The optimal bound is $\frac{\sqrt3}{2}\cdot s^2$; see 13G.

13.21. Set $s=\operatorname{sys}(\mathbb{RP}^2,g)$. Cut $(\mathbb{RP}^2,g)$ along a shortest noncontractible curve γ. We obtain $(\mathbb{D}^2,g)$ — a disc with metric tensor which we still denote by g.

Divide γ into two equal arcs α and β. Denote by A and A' the two connected components of the inverse image of α. Similarly denote by B and B' the two connected components of the inverse image of β.



Let γ_1 be a path from A to A'; map it to $\mathbb{RP}^2$ and keep the same notation for it. Note that γ_1 together with a subarc of α forms a closed noncontractible curve in $\mathbb{RP}^2$. Since length $\alpha=\frac{s}{2}$, we have that length $\gamma_1\geqslant\frac{s}{2}$. It follows that the distance between A and A' in $(\mathbb{D}^2,g)$ is at least $\frac{s}{2}$. The same way we show that the distance between B and B' in $(\mathbb{D}^2,g)$ is at least $\frac{s}{2}$.

Note that $(\mathbb{D}^2,g)$ can be parameterized by a square with sides A, B, A' and B' and apply 13.11 to show that

$$\operatorname{area}(\mathbb{RP}^2,g)=\operatorname{area}(\mathbb{D}^2,g)\geqslant\tfrac14\cdot s^2.$$

Remark. The optimal bound is $\frac{2}{\pi}\cdot s^2$; see 13G. In fact any Riemannian metric on the disc with the boundary globally isometric to a unit circle with angle metric has the area at least as large as the unit hemisphere. It is expected that the same inequality holds for any compact surface with connected boundary (not necessarily a disc); this is the so-called filling area conjecture [it is mentioned Mikhael Gromov in 5.5.B′(e′) of 62].

13.22. Cut the surface along a shortest noncontractible curve γ. We might get a surface with one or two components of the boundary. In these two cases repeat the arguments in 13.21 or 13.20 using 13.12 instead of 13.11.

13.23. Consider the product of a small 2-sphere with the unit circle.

13.25. Apply the same construction as in the original Besicovitch inequality, assuming that the target rectangle $[0,d_1]\times\dots\times[0,d_n]$ equipped with the metric induced by the ℓ^∞ norm; apply 13.24 where it is appropriate.

13.26. Suppose that $\Delta_1\ne\Delta_2$. Consider the map $f\colon\mathbb{S}^n\to\mathcal{X}$ such that the restriction to north and south hemispheres describe Δ_1 and Δ_2 respectively. Show that if $\Delta_1\ne\Delta_2$, then $\mathbb{S}^n$ can be parameterized by the boundary of the unit cube $\square$ in such a way that for any pair A, A' of opposite faces their images $f(A)$, $f(A')$ do not overlap.

Since $\mathcal{X}$ is contractible, the map f can be extended to a map of the whole cube. By 13.25

$$\operatorname{haus}_{n+1}[f(\square)]>0,$$

a contradiction.

14.4. The following claim resembles Besicovitch inequality; it is key to the proof:

(∗) Let a be a positive real number. Assume that a closed curve γ in a metric space $\mathcal{X}$ can be subdivided into 4 arcs α, β, α', and β' in such a way that

- $|x - x'| > a$ for any $x \in \alpha$ and $x' \in \alpha'$ and
- $|y - y'| > a$ for any $y \in \beta$ and $y' \in \beta'$.

Then γ is not contractible in its $\frac{a}{2}$-neighborhood.

To prove $(*)$, consider two functions defined on $\mathcal{X}$ as follows:

$$w_1(x) = \min\{\, a, \mathrm{dist}_\alpha(x)\, \}$$
$$w_2(x) = \min\{\, a, \mathrm{dist}_\beta(x)\, \}$$

and the map $\boldsymbol{w}\colon \mathcal{X} \to [0,a] \times [0,a]$, defined by

$$\boldsymbol{w}\colon x \mapsto (w_1(x), w_2(x)).$$

Note that

$$\boldsymbol{w}(\alpha) = 0 \times [0,a], \qquad \boldsymbol{w}(\beta) = [0,a] \times 0,$$
$$\boldsymbol{w}(\alpha') = a \times [0,a], \qquad \boldsymbol{w}(\beta') = [0,a] \times a.$$

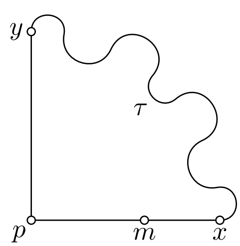
Therefore, the composition $\boldsymbol{w} \circ \gamma$ is a degree 1 map

$$\mathbb{S}^1 \to \partial([0,a] \times [0,a]).$$

It follows that if $h\colon \mathbb{D} \to \mathcal{X}$ shrinks γ, then there is a point $z \in \mathbb{D}$ such that $\boldsymbol{w} \circ h(z) = (\frac{a}{2}, \frac{a}{2})$. Therefore, $h(z)$ lies at distance at least $\frac{a}{2}$ from α, β, α', β' and therefore from γ. It proves the claim.

Coming back to the problem, let $\{W_i\}$ be an open covering of the real line with multiplicity 2 and $\operatorname{rad} W_i < R$ for each i; for example, take the covering by the intervals $((i - \frac{2}{3})\cdot R, (i + \frac{2}{3})\cdot R)$.

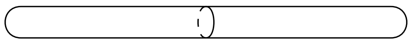
Choose a point $p \in \mathcal{X}$. Denote by $\{V_j\}$ the connected components of $\mathrm{dist}_p^{-1}(W_i)$ for all i. Note that $\{V_j\}$ is an open finite cover of $\mathcal{X}$ with multiplicity at most 2. It remains to show that $\operatorname{rad} V_j < 100\cdot R$ for each j.



Arguing by contradiction assume there is a pair of points $x, y \in V_i$ such that $|x - y|_{\mathcal{X}} \geqslant 100\cdot R$. Connect x to y with a curve τ in V_j. Consider the closed curve σ formed by τ and two shortest paths $[px]$, $[py]$.

Note that $|p - x| > 40$. Therefore, there is a point m on $[px]$ such that $|m - x| = 20$.

By the triangle inequality, the subdivision of σ into the arcs $[pm]$, $[mx]$, τ and $[yp]$ satisfy the conditions of the claim $(*)$ for $a = 10\cdot R$, hence the statement.



The quasiconverse does not hold. As an example take a surface that looks like a long cylinder with closed ends; it is a smooth surface diffeomorphic to a sphere. Assuming the cylinder is thin, it has macroscopic dimension 1 at a given scale. However, a circle formed by a section of cylinder around its midpoint by a plane parallel to the base is a circle that cannot be contracted in its small neighborhood.

Source: [62, Appendix 1(E$_2$)].

14.5; *only-if part.* Suppose $\mathrm{width}_n\,\mathcal{X} < R$. Consider a covering $\{V_1, \dots, V_k\}$ of $\mathcal{X}$ guaranteed by the definition of width. Let $\mathcal{N}$ be its nerve and $\boldsymbol{\psi}\colon \mathcal{X} \to \mathcal{N}$ be the map provided by 14.2.

Since the multiplicity of the covering is at most $n+1$, we have $\dim \mathcal{N} \leqslant n$.

Note that if $x \in \mathcal{N}$ lies in a star of a vertex v_i, then $\boldsymbol{\psi}^{-1}\{x\} \subset V_i$; in particular, we have $\operatorname{rad}[\boldsymbol{\psi}^{-1}\{x\}] < R$.

If part. Choose $x \in \mathcal{N}$. Since the inverse image $\boldsymbol{\psi}^{-1}\{x\}$ is compact, $\boldsymbol{\psi}$ is continuous, and $\operatorname{rad}[\boldsymbol{\psi}^{-1}\{x\}] < R$, there is a neighborhood $U \ni x$ such that the $\operatorname{rad}[\boldsymbol{\psi}^{-1}(U)] < R$.

Since $\mathcal{X}$ is compact, there is a finite cover $\{U_i\}$ of $\mathcal{N}$ such that $\boldsymbol{\psi}^{-1}(U_i) \subset \mathcal{X}$ has a radius smaller than R for each i. Since $\mathcal{N}$ has dimension n, we can inscribe[1] in $\{U_i\}$ a finite open cover $\{W_i\}$ with multiplicity at most $n+1$. It remains to observe that $V_i = \boldsymbol{\psi}^{-1}(W_i)$ defines a finite open cover of $\mathcal{X}$ with multiplicity at most $n+1$ and $\operatorname{rad} V_i < R$ for any i.

14.6. Assume that $\mathcal{P}$ is connected.

Let us show that $\operatorname{diam} \mathcal{P} < R$. If this is not the case, then there are points $p, q \in \mathcal{P}$ on distance R from each other. Let γ be a shortest path from p to q. Clearly $\operatorname{length}\gamma \geqslant R$ and γ lies in $\mathrm{B}(p,R)$ except for the endpoint q. Therefore, $\operatorname{length}[\mathrm{B}(p,R)_{\mathcal{P}}] \geqslant R$. Since

[1]Recall that a covering $\{W_i\}$ is inscribed in the covering $\{U_i\}$ if for every W_i is a subset of some U_j.

$\operatorname{VolPro}_{\mathcal{P}}(R) \geqslant \operatorname{length}[\mathrm{B}(p,R)_{\mathcal{P}}]$, the latter contradicts $\operatorname{VolPro}_{\mathcal{P}}(R) < R$.

In general case, we get that each connected component of $\mathcal{P}$ has a radius smaller than R. Whence the width of $\mathcal{P}$ is smaller than R.

Second part. Again, we can assume that $\mathcal{P}$ is connected.

The examples of line segment or a circle show that the constant $c = \frac{1}{2}$ cannot be improved. It remains to show that the inequality holds with $c = \frac{1}{2}$.

Choose $p \in \mathcal{P}$ such that the value

$$\rho(p) = \max\{\, |p-q|_{\mathcal{P}} \;:\; q \in \mathcal{P}\,\}$$

is minimal. Suppose $\rho(p) \geqslant \frac{1}{2}\cdot R$. Observe that there is a point $x \in \mathcal{P}\backslash\{p\}$ that lies on any shortest path starting from p and length $\geqslant \frac{1}{2}\cdot R$. Otherwise, for any $r \in (0, \frac{1}{2}\cdot R)$ there would be at least two points on distance r from p; by coarea inequality we get that the total length of $\mathcal{P}\cap\mathrm{B}(p,\frac{1}{2}\cdot R)$ is at least R — a contradiction.

Moving p toward x reduces $\rho(p)$ which contradicts the choice of p.

14.18. The inequality $6\cdot R < s$ used twice:

- ⋄ to shrink the triangle $[p_ip_jp_k]$ to a point;
- ⋄ to extend the constructed homotopy on $\mathcal{M}^0$ to $\mathcal{M}^1$.

The first issue can be resolved by passing to a barycentric subdivision of $\mathcal{N}^2$; denote by v_{ij} and v_{ijk} the new vertices in the subdivision that correspond to edge $[v_iv_j]$ and triangle $[v_iv_jv_k]$ respectively.

Further for each vertex v_{ij} choose a point $p_{ij} \in V_i \cap V_j$ and set $f(v_{ij}) = p_{ij}$. Similarly for each vertex v_{ijk} choose a point $p_{ijk} \in V_i\cap V_j\cap V_k$ and set $f(v_{ijk}) = p_{ijk}$.

Note that

$$|p_i - p_{ij}| < R,$$
$$|p_i - p_{ijk}| < R,$$
$$|p_{ij} - p_{ijk}| < 2\cdot R.$$

Therefore, perimeter of the triangle $[p_ip_{ij}p_{ijk}]$ in the subdivision is less that $4\cdot R$. It resolves the first issue.

The second issue disappears if one estimates the distances a bit more carefully.

14.19. Choose a fine covering of $\mathcal{M}$ with multiplicity at most n. Choose $\boldsymbol{\psi}$ from $\mathcal{M}$ to the nerve $\mathcal{N}$ of the covering the same way as in the proof of 14.15.

It remains to construct $f\colon \mathcal{N} \to \mathcal{M}$ and show that $f\circ\boldsymbol{\psi}$ is homotopic to the identity map. To do this, apply the same strategy as in the proof of 14.15 together with the so-called geodesic cone construction described below.

Let $\triangle$ be a simplex in a barycentric subdivision of $\mathcal{N}$. Suppose that a map f is defined on one facet $\triangle'$ of $\triangle$ to $\mathcal{M}$ and $\mathrm{B}(p,r) \supset f(\triangle')$. Then one can extend f to whole $\triangle$ such that the remaining vertex v maps to p. Namely connect each point $f(x)$ to p by minimizing geodesic path γ_x (by assumption it is uniquely defined) and set

$$f\colon t\cdot x + (1-t)\cdot v \mapsto \gamma_x(t).$$

14.21. Suppose M is an essential manifold and N is an arbitrary closed manifold. Observe that shrinking N to a point produces a map $N\#M \to M$ of degree 1. In particular, there is a map $f\colon N\#M \to M$ that sends the fundamental class of $N\#M$ to the fundamental class of M.

Since M is essential, there is an aspherical space K and a map $\iota\colon M \to K$ that sends the fundamental class of M to a nonzero homology class in K. From above, the composition $\iota\circ f\colon N\#M \to K$ sends the fundamental class of $N\#M$ to the same homology class in K.

14.22. Suppose M_1 and M_2 are essential. Let $\iota_1\colon M_1 \to K_1$ and $\iota_2\colon M_2 \to K_2$ are the maps to aspherical spaces as in the definition (14.20). Show that the map $(\iota_1,\iota_2)\colon M_1\times M_2 \to K_1\times K_2$ meets the definition.

Remarks. Choose a group G. Note that there is an aspherical connected space CW-complex K with fundamental group G. The space K is called an Eilenberg–MacLane space of type $K(G,1)$, or briefly a $K(G,1)$ space. Moreover it is not hard to check that

- ⋄ K is uniquely defined up to a weak homotopy equivalence;
- ⋄ if $\mathcal{W}$ is a connected finite CW-complex. Then any homomorphism $\pi_1(\mathcal{W},w) \to \pi_1(K,k)$ is induced by a continuous map $\varphi\colon(\mathcal{W},w)\to(K,k)$. Moreover, φ is uniquely defined up to homotopy equivalence.
- ⋄ Suppose that M is a closed manifold, K is a $K(\pi_1(M),1)$ space and a map $\iota\colon M \to K$ induces an isomorphism of

fundamental groups. Then M is essential if and only if ι sends the fundamental class of M to a nonzero homology class of K.

The property described in the last statement is the original definition of essential manifold. It can be used to prove a converse to the exercise; namely *the product of a nonessential closed manifold with any closed manifold is not essential.*

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