# Alexandrov's spaces with curvatures bounded from below II

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This paper contains the proofs of the results announced in [I, § 17]. The reader is supposed to be familiar with the definitions of Alexandrov's spaces (this name is used for finitedimensional spaces with curvatures bounded from below, FSCBB in [I]) (see [I, § 2]), the basic examples - cones and spherical suspensions ([I, 3.6, 3.7]), the generalised Toponogov's theorem ([I, § 4]), the notions and basic properties of strained points ([I, § 5]), rough volume ([I, §§ 6, 9]), spaces of directions and tangent cones ([I, § 7]), boundary ([I, 7.19]), and (directionally) differentiable functions ([I, 12.2-12.6]). The topological tools from [S] concerning deformation of homeomorphisms are used as well. Our principal results can be expressed as follows.

0.1. The Theorem on spherical neighborhood.

A sufficiently small spherical neighborhood of a point in Alexandrov's space is homeomorphic to the tangent cone at this point.

0.2. Corollary. An Alexandrov's space has a natural stratification into topological manifolds.

0.3. The Stability Theorem.

A compact Alexandrov's space $M^n$ has a neighborhood in Gromov-Hausdorff metric, such that any complete Alexandrov's space $\tilde{M}^n$ in this neighborhood, with the same lower bound of curvatures and the same dimension, is homeomorphic to $M^n$.

§ 1 contains a topological construction showing that a point in an Alexandrov's space has a conical neighborhood - a Morse-theoretic argument, based on the deformation theorems from [S] and the properties of non-critical maps, defined in §3. The same argument proves that a proper non-critical map, (from Alexandrov's space to euclidean space) is a (locally trivial) bundle projection (Theorem 1.4.1). § 3 contains the definition of non-critical maps and proofs of their properties, used in § 1. This definition is (for technical reasons) rather complicated and by no means canonical. The admissible maps from [I, 17.1] are particular cases of these non-critical maps. § 2 contains preliminary lemmas, which are used exstensively in § 3. The arguments of §§ 2, 3 are purely geometrical, based on the comparison inequalities. § 4 contains the proof of the theorem 4.3, that generalize the stability Theorem 0.3. This proof is essencially topological and based on the results of §§ 1, 3. The theorems 1.4.1 and 4.3 imply (by simple arguments) the Theorem 0.1 (see 4.4), a topological characterisation of boundary points of Alexandrov's space (4.6) and a natural generalisation of the Diameter sphere theorem of Grove and Shiohama [GSh] (4.5). § 5 contains the proof of the Doubling Theorem, stating that the (naturally defined) doubling of an Alexandrov's space with boundary is also an Alexandrov's space (with the same lower bound of curvatures). § 6 shows how to generalize the Soul Theorem of Cheeger and Gromoll [CG], and the Sharafutdinov's retraction [Sh] to the case of nonnegatively curved Alexandrov's spaces.

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Notations and conventions.

$A \approx B$ means that $A$ is homeomorphic to $B$,

$(A,B,C) \approx (X,Y,Z)$ means that there exists a homeomorphism $\theta: A \to X$ such that $\theta(B)=Y$, $\theta(C)=Z$ $(B, C \subset A; Z, Y \subset X)$.

Two maps $\varphi, \psi: X \to Y$ are $\nu$-close iff $\forall x \in X$ $|\varphi(x)\,\psi(x)| \leq \nu$

A map $\theta: X \to Y$ is a $\nu$-approximation iff $\forall x_1, x_2 \in X$ $||\theta(x_1)\,\theta(x_2)| - |x_1 x_2|| \leq \nu$ and $\forall y \in Y\ \exists x \in X: |y\,\theta(x)| < \nu$

A map $\theta: (A,B) \to (X,Y)$ is a $\nu$-approximation iff $\theta(B) \subset Y$ and $\theta|_B: B \to Y$ is a $\nu$-approximation as well as $\theta: A \to X$.

$(M_i, N_i, p_i)$ converge to $(M, N, p)$ in Gromov-

Hausdorff sense iff for any $R>0$, $\nu>0$ there exists $\bar{N}>0$ such that for any $i>\bar{N}$ there exists a $\nu$-approximation

$$\theta_i : (M_i \cap B_{p_i}(R), N_i \cap B_{p_i}(R), p_i) \to (M \cap B_p(R), N \cap B_p(R), p)$$

$K(M)$ may denote the topological open cone on $M$ or the metric cone on $M$, in case $M$ is an Alexandrov's space with curvatures $\geqslant 1$. In this case $S(M)$ denotes the spherical suspension on $M$. $\bar{K}(M)$ denotes the topological closed cone on $M$, that is a join of $M$ and a point. $K_p(M)$ means the cone on $M$ with apex $p$.

$B_p(R)$ denotes the open metric ball of radius $R$, centered at $p$.

$R \cdot M^n$ denotes the space $M^n$ with metric multiplied by $R$.

$\beta_\varepsilon(X)$ denotes the maximal mnumber of points $x_i \in X$ such that $|x_i x_j| \geqslant \varepsilon$ $(i \neq j)$.

$V_{r_k}$ denotes the $k$-dimensional rough volume.

$\Sigma_p$ denotes the space of directions at $p$.

$f'_{(p)}(\xi)$ denotes the derivative of $f$ at $p$ in the direction $\xi \in \Sigma_p$.

$Q' \subset \Sigma_p$ denotes the set of directions of all shortest lines $pQ$ (a shortest line $pQ$ is a shortest line $pq$ such that $q \in Q$ and $|pq| = |pQ|$).

$Q' \in \Sigma_p$ denotes the direction of some shortest line $pQ$.

$\tilde{\angle} ApB$ denotes the angle at $p$ in the comparison triangle with sidelengths $|Ap|, |Bp|, |AB|$ ; if $|AB| \leq ||Ap| - |Bp||$ then $\tilde{\angle} ApB = 0$. Clearly $\tilde{\angle} ApB$ satisfies the comparison inequality $\tilde{\angle} ApB \leq |A'B'|$, $|A', B'| \subset \Sigma_p$.

$I^k$ denotes a $k$-dimensional closed cube in euclidean space, with edges parallel to (some) coordinate axes. $\mathring{I}^k$ denotes the corresponding open cube. $I^k_p(R)$ means the cube $\{x \in \mathbb{R}^k : |x_i - p_i| \leq R, 1 \leq i \leq k\}$. $I^m \subset I^\ell$ means in particular that the edges of $I^m$ are parallel to some edges of $I^\ell$.

The distance in euclidean space $\mathbb{R}^k$, denoted by $|\cdot,\cdot|$ is induced by the norm $|x| = \max_i |x_i|$.

Positive constants are denoted by $c$. We ignore in notation the dependence of such constants on the lower bound of curvatures and the dimension-like parameters. $C(\varepsilon)$ denotes a constant depending on a parameter $\varepsilon$. We denote by $\varkappa$ positive increasing continuous functions defined for sufficiently small positive arguments, and tending to zero when their arguments tend to zero. The dependence of these functions on dimension-like parameters and the lower bound of curvature is ignored as well. The function $\varkappa$ may depend on additional parameters that are indicated explicitly. Any emergence of $C$ or $\varkappa$ means the statement of existence of such a constant or function, and the assertions, which contain $C$ or $\varkappa$, are supposed to hold only for suitably chosen $C$ and $\varkappa$.

## 1. The topological construction.

### 1.1. Spaces with multiple conical singularities (MCS-spaces).

<u>Definition</u>. A metrizable space $X$ is an MCS-space of dimension $n$ $(n \geq 0)$ iff each point $x \in X$ has a neighborhood pointed homeomorphic to an open cone on a compact $(n-1)$-dimensional MCS-space. (We assume the empty set to be the unique compact $(-1)$-dimensional MCS-space).

<u>Remark</u>. An open conical neighborhood is unique up to a pointed homeomorphism, see [K].

It is clear that a join of two compact MCS-spaces as well as a product of any two MCS-spaces is an MCS-space.

There is a natural stratification of an MCS-space; the $\ell$-dimensional strata consists of such points $x$ that the conical neighborhood of $X$ admits a splitting $\mathbb{R}^m \times K(S_m)$, where $S_m$ being a compact MCS-space, iff $m \leq \ell$. It is clear that the $\ell$-dimensional strata is an $\ell$-dimensional topological manifold, and an MCS-space is a WCS set in the sense of [S, def.5.1].

### 1.2. Background from topology.

Theorem A. Let $X$ be a metric space, $f: X \to \mathbb{R}^k$ be a continuous proper open map, such that for each point $x \in X$

1) There is an open product neighborhood $\mathcal{U}_x \ni x$ and a homeomorphism $f_x : \mathcal{U}_x \to (\mathcal{U}_x \cap f^{-1}(f(x))) \times f(\mathcal{U}_x)$ respecting $f$ (that is $pr \circ f_x \equiv f$, where $pr : (\mathcal{U}_x \cap f^{-1}(f(x))) \times f(\mathcal{U}_x) \to f(\mathcal{U}_x)$ denotes the projection);

2) $f^{-1}(f(x))$ is a compact MCS-space.

Then $f$ is a (locally trivial) bundle map.

Complement to theorem A. Assume in addition that a product neighborhood $\mathcal{U}_x$ satisfies $f(\mathcal{U}_x) = f(X) = I^k$, and fix a compact subset $K \subset \mathcal{U}_x$. Then there exists a homeomorphism $\varphi : X \to f^{-1}(f(x)) \times I^k$ such that $\varphi|_K \equiv f_x|_K$.

Theorem B. Let $X$ be a compact metric MCS space, $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$ be a finite open covering of $X$. Given a function $\varkappa_0$ there exists a function $\varkappa$, depending on $X$, $\{\mathcal{U}_\alpha\}$ and $\varkappa_0$, with the following property.

If $\widetilde{X}$ is a metric space, such that any two close points $\widetilde{x}_1$, $\widetilde{x}_2 \in \widetilde{X}$ can be connected by a curve in $\widetilde{X}$ of diameter $< \varkappa_0(|\widetilde{x}_1 \widetilde{x}_2|)$, $\{\widetilde{\mathcal{U}}_\alpha\}_{\alpha \in \mathcal{A}}$ is an open covering of $\widetilde{X}$, $\varphi : X \to \widetilde{X}$ is a $\delta$-approximation, $\varphi_\alpha : \mathcal{U}_\alpha \to \widetilde{\mathcal{U}}_\alpha$, $\alpha \in \mathcal{A}$, are homeomorphisms, $\delta$-close to $\varphi$, then there exists a homeomorphism $\eta : X \to \widetilde{X}$, $\varkappa(\delta)$-close to $\varphi$.

Complement to theorem B. Given in addition continuous maps $f : X \to \mathbb{R}^k$, $\widetilde{f} : \widetilde{X} \to \mathbb{R}^k$, $h : X \to \mathbb{R}$, $\widetilde{h} : \widetilde{X} \to \mathbb{R}$ and a compact subset $K \subset X$, suppose that for $\widetilde{\mathcal{U}}_\alpha$ intersecting $K$ (respectively, non-intersecting $K$) we have $(\widetilde{f}, \widetilde{h}) \circ \varphi_\alpha \equiv (f, h)$ on $\mathcal{U}_\alpha$ ($\widetilde{f} \circ \varphi_\alpha \equiv f$ on $\mathcal{U}_\alpha$), and each such $\widetilde{\mathcal{U}}_\alpha$ is contained in a product neighborhood $V_\alpha$ w.r.t. $(f, h)$ (w.r.t. $f$) (we say that $V$ is a product neighborhood w.r.t. $g : V \to \mathbb{R}^\ell$ if there exist a point $v \in g(V)$ and a homeomorphism $g' : V \to g^{-1}(v) \times I^\ell$, such that $g \equiv pr \circ g'$, $pr$ being the projection onto $I^\ell$, and $g^{-1}(v)$ is an MCS-space).

Then the homeomorphism $\eta : X \to \widetilde{X}$ in the conclusion of theorem B can be chosen to satisfy $f \equiv \widetilde{f} \circ \eta$ on $X$ and

$(f,h)\equiv(\tilde f,\tilde h)\circ\eta$ on $K$. (The function $\varkappa$ may now depend on $X,\{U_\alpha\},\varkappa_0,K,f,h$.

Theorem A was proved by L.C.Siebenmann [S, cor.6.14, th.5.4], the complement follows from [S, 6.9]. The following proof of Theorem B exploits the same arguments.

$\overline{W}\subset V$

$\overline{V}\subset U$

Assertion 1. Let $X$ be a compact metric MCS-space, $W \Subset V \Subset U \subset X$ be open subsets. Then for any open embedding $\varphi: U\to X$, $\delta$-close to the inclusion $i$, there exists an open embedding $\psi: U\to X$, $\varkappa(\delta)$-close to $i$, such that $\psi\equiv\varphi$ on $W$ and $\psi\equiv i$ on $U\setminus V$. ($\varkappa$ depends on $W,V,U,X$).

$\delta$ small enough

Complement. If $X\approx X_1\times I^k$, where $X_1$ is a compact MCS-space, and $\varphi$ respects the projection onto $I^k$, then $\psi$ can be chosen to respect this projection.

Proof. We can apply the deformation theorem [S, th.5.4] to the open embedding $\varphi|_{U\setminus\overline{W}}$ and obtain an open embedding $\varphi_1: U\setminus\overline{W}\to X$, $\varkappa(\delta)$-close to the inclusion, which coincides with $i$ in some neighborhood of $\neg V$ and is equal to $\varphi$ outside some compact subset of $U\setminus\overline{W}$. Now let

$$\psi(x)=\begin{cases}\varphi(x), & x\in W\\ \varphi_1(x), & x\in V\setminus W\\ x, & x\in U\setminus V\end{cases}$$

. To prove the complement use [S, th.6.1.] in addition to [S, th.5.4]. ∎

Assertion 2. In conditions of Theorem B, if $x\in X,\tilde x\in\tilde X$ satisfy $|\varphi(x),\tilde x|<\delta$, $V\supset B_x(\varkappa_0(\delta)+10\delta)$ is an open subset of $X$, $\psi: V\to\tilde X$ is an open embedding, $\delta$-close to $\varphi$, then $\tilde x\in\psi(V)$.

This is clear. ∎

Now assume the conditions of Theorem B, and suppose $U_{\alpha_1}\cap U_{\alpha_2}\neq\emptyset$. Let $U_1^4\Subset U_1^3\Subset U_1^2\Subset U_1^1\Subset U_{\alpha_1}$, $U_2^4\Subset U_2^3\Subset U_2^2\Subset U_2^1\Subset U_{\alpha_2}$ be open subsets such that $X\setminus\bigcup_{\alpha\in\mathcal{A}\setminus\{\alpha_1,\alpha_2\}}U_\alpha\subset U_1^4\cup U_2^4$. Assertion 2 implies

$\varphi_{\alpha_1}(U_1^1\cap U_2^1)\subset\varphi_{\alpha_2}(U_{\alpha_2})$ provided $\delta$ is small. Thus we may consider the open embedding $\varphi_{\alpha_2}^{-1}\circ\varphi_{\alpha_1}: U_1^1\cap U_2^1\to U_{\alpha_2}$, which is $2\delta$ -

close to the inclusion $i$ . By Assertion 1 there is an open embedding $\psi: U_1^1 \cap U_2^1 \to U_{\alpha_2}$ , $\varkappa(\delta)$-close to $i$ , such that $\psi \equiv \varphi_{\alpha_2}^{-1} \circ \varphi_{\alpha_1}$ on $U_1^3 \cap U_2^3$ and $\psi \equiv i$ on $U_1^1 \cap U_2^1 \setminus U_1^2 \cap U_2^2$. Extend $\psi$ onto $U_2^1$ letting $\psi \equiv i$ on $U_2^1 \setminus U_1^2 \cap U_2^2$ , and define $\varphi'_{\alpha_2} = \varphi_{\alpha_2} \circ \psi$ . Now we can define an immersion $\varphi' : U_1^4 \cup U_2^4 \to \tilde{X}$ letting $\varphi'(x) = \begin{cases} \varphi_{\alpha_1}(x), & x \in U_1^4 \\ \varphi'_{\alpha_2}(x), & x \in U_2^4 \end{cases}$

In fact $\varphi'$ is clearly an open embedding provided $\delta$ is small. Moreover, Assetrion 2 implies $\tilde{X} \setminus \bigcup_{\alpha \in \mathfrak{A} \setminus \{\alpha_1 \alpha_2\}} \bar{U}_\alpha \subset \varphi'(U_1^4 \cup U_2^4)$ provided $\delta$ is small. Now the proof of Theorem B can be completed by induction. The proof can be generalized trivially to handle the complement. ■

def ?? compl?

1.3. Properties of non-critical maps.

Let $U \subset M^n$ be a domain in Alexandrov's space, $f: U \to \mathbb{R}^k$ $(k \le n)$ be a continuous map, $p \in U$ . We say that $f$ is non-critical at $p$ iff it satisfies some conditions, listed in 3.1, 3.7. Now we need only the following properties of non-critical maps, that will be established in §3.

1.3.1. A set of non-critical points of a map is open, and a map is open near its non-critical point.

1.3.2. If $f: U \subset M^n \to \mathbb{R}^n$ is non-critical at $p$, then $f$ maps homeomorphically some neighborhood of $p$ onto a cube $I^n_{f(p)}$, in $\mathbb{R}^n$.

1.3.3. Let $f: U \subset M^n \to \mathbb{R}^k$ be non-critical and incomplementable at $p$ , that is for any function $f_1$ in a neighborhood of $p$ the map $(f, f_1)$ to $\mathbb{R}^k \times \mathbb{R} = \mathbb{R}^{k+1}$ is critical at $p$ . Then there exists a function $\varkappa_1$, and for sufficiently small $R>0$ and $R'>0$ , such that $\varkappa_1(2R') < R$ , there exists a continuous function $h: U_1 = \bar{B}_p(R) \cap f^{-1}(I^k_{f(p)}(R')) \to [0,R]$ with the following properties

a) $h(x) = |px|$ if $|px| > R/2$

b) $f$ is injective on $S = h^{-1}(0)$

c) $f$ is complementable at any point of $U_1 \setminus S$

d) If $x \in U_1$ satisfies $\varkappa_1(|f(x) f(S)|) < h(x)$ then $x$

$S = h^{-1}(0)$

is non-critical for the map $(f,h):U_1\to\mathbb{R}^{k+1}$.

Moreover, for each $v\in I^k_{f(p)}(R')$ there exist a continuous function $h_v:U_1\to[0,R]$ and a point $O_v\in f^{-1}(v)\cap U_1$ such that

e) $h_v(x)=0 \Leftrightarrow x=O_v$, $h_v(x)=R \Leftrightarrow h(x)=R$. $(x\in f^{-1}(v)\cap U_1)$

f) Each point $x\in f^{-1}(v)\setminus\{O_v\}$ is non-critical for $(f,h_v):U_1\to\mathbb{R}^{k+1}$.

Remark. It is clear that $p\in S$, and we may take $h_v\equiv h$ for $v\in f(S)$.

1.4. Formulations and reductions.

Our aim in this section is to prove the following assertion.

Theorem 1.4.1. A proper map $f:U\subset M^n\to\mathbb{R}^k$ (open) without critical points is a (locally trivial) bundle map.

In order to prove this theorem we need also the two following assertions.

Proposition 1.4.2. Let $f:U\subset M^n\to\mathbb{R}^k$ be non-critical and incomplementable at $p$. Then

a) for $R>0$ sufficiently small

$$(\overline{B}_p(R)\cap f^{-1}(f(p)),\partial B_p(R)\cap f^{-1}(f(p)),p)\approx$$
$$\approx(\overline{K}_p(\partial B_p(R)\cap f^{-1}(f(p))),\partial B_p(R)\cap f^{-1}(f(p)),p)$$

b) for $R'>0$ small enough comparing to $R$, there is a homeomorphism

$$\varphi:(\overline{B}_p(R)\cap f^{-1}(I^k_{f(p)}(R')),\partial B_p(R)\cap f^{-1}(I^k_{f(p)}(R'))\to$$
$$(\overline{B}_p(R)\cap f^{-1}(f(p)))\times I^k_{f(p)}(R'),\partial B_p(R)\cap f^{-1}(f(p)))\times I^k_{f(p)}(R'),$$

which respects $f$, that is $f\equiv pr\circ\varphi$.

c) The map $(f,|p,\cdot|)$ is non-critical at points of $\partial B_p(R)\cap f^{-1}(f(p))$.

Proposition 1.4.3. A level set $f^{-1}(v)$ of a map $f: U\subset M^n\to\mathbb{R}^k$ is homeomorphic to an MCS-space provided it does not contain critical points.

The case $k=n$ of 1.4.1, 1.4.2, 1.4.3 follows immediately from 1.3.2. Theorem 1.4.1 for $k=\ell$ follows from 1.4.2, 1.4.3 for $k\geqslant\ell$ and Theorem A. Proposition 1.4.3 for $k=\ell$ follows from 1.4.2 for $k\geqslant\ell$ and 1.4.3 for $k>\ell$. It remains to prove that 1.4.1, 1.4.2, 1.4.3 for $k>\ell$ imply 1.4.2 for $k=\ell$.

1.5. Proof of 1.4.1, 1.4.2, 1.4.3.

Assume 1.4.1, 1.4.2, 1.4.3 to be true for $k>\ell$ and let $f: U\subset M^n\to R^\ell$ be non-critical and incomplementable at $p$; take $R, R', \varkappa_1, h, h_v, U_1, S$ as in 1.3.3. Then 1.4.2.c is clear. We prove first an assertion slightly generalizing 1.4.2.a. Let $\Sigma = f^{-1}(f(p))\cap\partial B_p(R)$.

Assertion 3. Let $v\in \underbrace{I^\ell_{f(p)}(R')}$ satisfy $\varkappa_1(|v, f(S)|)<R_0\leq R$. Then $(f^{-1}(v)\cap h^{-1}[0,R_0],\ f^{-1}(v)\cap h^{-1}(R_0))\approx(\overline{K}(\Sigma),\Sigma)$. If $R_0=R$ then the homeomorphism above maps $O_v$ to the apex of the cone.

Proof. 1.3.3.a, d imply that $(f,h)$ has no critical points in $\partial B_p(R)\cap U_1$, hence $\underline{f^{-1}(v)\cap\partial B_p(R)\approx\Sigma}$ by 1.4.1 for $k=\ell+1$. Furthermore 1.3.3.e, f imply that $(f,h_v)$ has no critical points in $f^{-1}(v)\cap h_v^{-1}(0,R)$, hence for any $0<R_1<R_2\leqslant R$ we have

$$(f^{-1}(v)\cap h_v^{-1}[R_1,R_2],\ f^{-1}(v)\cap h_v^{-1}(R_1),\ f^{-1}(v)\cap h_v^{-1}(R_2))\approx(\Sigma\times I, \Sigma\times\{0\}, \Sigma\times\{1\}).$$

and therefore $(f^{-1}(v)\cap\overline{B}_p(R),\ f^{-1}(v)\cap\partial B_p(R), O_v)\approx(\overline{K}_p(\Sigma),\Sigma,p)$. At last, choose $R_1$ such that $\varkappa_1(|v,f(S)|)<R_1<R_0$, and observe that $(f,h)$ has no critical points in $h^{-1}[R_1,R]\cap f^{-1}(v)$, hence

$$(f^{-1}(v)\cap h^{-1}[R_1,R],\ f^{-1}(v)\cap h^{-1}(R_1),\ f^{-1}(v)\cap h^{-1}(R))\approx(\Sigma\times I,\Sigma\times\{0\},\Sigma\times\{1\})\approx$$
$$\approx(f^{-1}(v)\cap h^{-1}[R_1,R_0],\ f^{-1}(v)\cap h^{-1}(R_1),\ f^{-1}(v)\cap h^{-1}(R_0))$$

and

$$(f^{-1}(v)\cap h^{-1}[0,R_0],\ f^{-1}(v)\cap h^{-1}(R_0))\approx(f^{-1}(v)\cap h^{-1}[0,R],\ f^{-1}(v)\cap h^{-1}(R))\approx(\overline{K}(\Sigma),\Sigma).$$

∎

From now on we may assume $\Sigma\neq\emptyset$ since $\ell=n$ otherwise.

choose $v=f(p)$

$\partial B_p(R)=\{x \mid |px|=R\}$.

$\{x: f(x)=v \wedge |px|=R\}$

In order to prove 1.4.2.b we construct a special cell decomposition of $\mathcal{U}_1 \setminus S$ with cells homeomorphic to $\Sigma \times \mathring{I}^m$ or $K(\Sigma) \times \mathring{I}^m$, $0 \leq m \leq \ell$. We use cells of 3 types.

The cells of type I are of the form $C_\alpha = h^{-1}(R_\alpha) \cap f^{-1}(\mathring{I}^{m_\alpha}_\alpha)$, $0 < R_\alpha \leq R$, $I^{m_\alpha}_\alpha \subset I^\ell_{f(p)}(R')$. We assume that $\varkappa_1(|v\, f(S)|) < R_\alpha$ for $v \in I^{m_\alpha}_\alpha$, and let $\overline{C}_\alpha = h^{-1}(R_\alpha) \cap f^{-1}(I^{m_\alpha}_\alpha)$.

The cells of type II are of the form $C_\beta = h^{-1}(R^1_\beta, R^2_\beta) \cap f^{-1}(\mathring{I}^{m_\beta}_\beta)$, $0 < R^1_\beta < R^2_\beta \leq R$, $I^{m_\beta}_\beta \subset I^\ell_{f(p)}(R')$. We assume that $\varkappa_1(|v\, f(S)|) < R^1_\beta$ for $v \in I^{m_\beta}_\beta$ and let $\overline{C}_\beta = h^{-1}[R^1_\beta, R^2_\beta] \cap f^{-1}(I^{m_\beta}_\beta)$.

The cells of type III are of the form $C_\gamma = h^{-1}([0, R_\gamma)) \cap f^{-1}(\mathring{I}^{m_\gamma}_\gamma)$, $0 < R_\gamma \leq R$, $I^{m_\gamma}_\gamma \subset I^\ell_{f(p)}(R')$. We let $\overline{C}_\gamma = h^{-1}([0, R_\gamma]) \cap$ $\cap f^{-1}(I^{m_\gamma}_\gamma)$ and assume that $\varkappa_1(|v\, f(S)|) < R_\gamma$ for $v \in I^{m_\gamma}_\gamma$ and that for any cell $C_{\gamma'}$ such that $C_\gamma \subset \overline{C}_{\gamma'}$, we have $\overline{C}_{\gamma'} \cap S = \varnothing$.

It follows from 1.4.1 for $k = \ell + 1$ and 1.3.3.d that a closed cell $\overline{C}_\alpha$ of type I is homeomorphic to $\Sigma \times I^{m_\alpha}_\alpha$ respecting $f$, and a closed cell $\overline{C}_\beta$ of type II is homeomorphic to $\Sigma \times I^{m_\beta}_\beta \times I$ respecting $(f, h)$. At last 1.3.3.c, 1.4.2 for $k > \ell$, Assertion 3, 1.4.3 for $k = \ell + 1$ and the complement to Theorem A imply that a cell $C_\gamma$ of type III satisfies $(\overline{C}_\gamma, C_\gamma) \approx (\overline{K}(\Sigma) \times I^{m_\gamma}_\gamma, K(\Sigma) \times \mathring{I}^{m_\gamma}_\gamma)$ respecting $f$.*)

For preliminary constructions we need also cells of type IV; their only distinction from the cells of type III is that the very last assumption is replaced by the opposite one: there exists a cell $C_{\gamma'}$ such that $\overline{C}_{\gamma'} \supset C_\gamma$ and $\overline{C}_{\gamma'} \cap S \neq \varnothing$.

($\Omega$) We proceed by an infinite sequence of steps. Before the $i$-th step we have a decomposition of $\mathcal{U}_1$ into finite set of cells of types I–IV, such that

the boundary $\overline{C} \setminus C$ of any cell consists of whole cells, and all cells of type IV have $R_\gamma = 2^{1-i} R$ and $\operatorname{diam} I^{m_\gamma}_\gamma = 2^{-n_i} R'$ (for $m_\gamma > 0$), where $n_i$ are integers satisfying

*) Indeed, Assertion 3 implies that $(f^{-1}(v) \cap h^{-1}[0, R_\gamma]) \approx \overline{K}(\Sigma)$. $\overline{K}(\Sigma)$ is a compact MCS space since $\Sigma$ is a compact MCS space by 1.4.3. for $k = \ell + 1$. 1.3.3.c and 1.4.2.b for $k > \ell$. imply that each point of $\overline{C}_\gamma$ has a product neighborhood w.r.t. $f$. At last, fix $R'_\gamma$ such that $\varkappa(|v, f(S)|) < R'_\gamma < R_\gamma$ for all $v \in I^{m_\gamma}_\gamma$ and observe that $f^{-1}(\mathring{I}^{m_\gamma}_\gamma) \cap$

$n_1=1, n_{i+1}>n_i$, $\varkappa_1(2^{-n_i}R')<2^{1-i}R$ . It follows from our definitions that the boundary of a cell of type $\overline{IV}$ contains only cells of types $I, \overline{IV}$ and any cell of type $\overline{IV}$ is contained in some closed top-dimensional cell of type $\overline{IV}$ . To perform the $i$ -th step we first subdivide each cell $h^{-1}[0,2^{1-i}R)\cap f^{-1}(\mathring{I}_\gamma^{m_\gamma})$ of type $\overline{IV}$ into $2^{m(n_{i+1}-n_i)}$ cells $h^{-1}[0,2^{1-i}R)\cap f^{-1}(\mathring{I}_{\gamma,j}^{m_\gamma})$ in a regular way, and obtain several cells of types $\overline{III}, \overline{IV}$ . Second, we subdivide some cells of type $I$ in $h^{-1}(2^{1-i}R)$ to ensure $(\Omega)$ . At last, we subdivide each new cell $h^{-1}[0,2^{1-i}R)\cap f^{-1}(\mathring{I}_{\gamma,j}^{m_{\gamma,j}})$ $(0\le m_{\gamma,j}\le m)$ of type $\overline{IV}$ into 3 cells $h^{-1}[0,2^{-i}R)\cap f^{-1}(\mathring{I}_{\gamma,j}^{m_{\gamma,j}})$, $h^{-1}(2^{-i}R)\cap f^{-1}(\mathring{I}_{\gamma,j}^{m_{\gamma,j}})$, $h^{-1}(2^{-i}R,2^{1-i}R)\cap f^{-1}(\mathring{I}_{\gamma,j}^{m_{\gamma,j}})$ of types $\overline{IV}$ , $I$ , $II$ respectively. The result of the infinite sequence of such steps is a locally finite decomposition of $U_1\setminus S$ into cells of types $I, II, III$ , satisfying $(\Omega)$ .

Now we are going to define the required homeomorphism $\varphi: U_1\to\overline{K}(\Sigma)\times I^\ell_{f(p)}(R')$ . We may view $\overline{K}(\Sigma)$ as a quotient $\{(x,y): x\in\Sigma, y\in[0,R]\}/\sim$ and define $\tilde{h}(z)=y$ for $z=(x,y)\in$ $\in\overline{K}(\Sigma)$ . Thus we have naturally defined functions $\tilde{h}, \tilde{f}_1,\dots,\tilde{f}_\ell$ on $\overline{K}(\Sigma)\cap I^\ell_{f(p)}(R')$, $\tilde{f}_1,\dots,\tilde{f}_\ell$ being the coordinate functions on $I^\ell_{f(p)}(R')$ . Define the corresponding cells in $\overline{K}(\Sigma)\times I^\ell_{f(p)}(R')$ by the same inequalities as in $U_1$ , with $\tilde{f},\tilde{h}$ instead of $f,h$ . We obtain the corresponding cell decomposition of $\overline{K}(\Sigma)\times I^\ell_{f(p)}(R')\setminus\{\tilde{p}\}\times f(S)$ , where $\tilde{p}$ denotes the apex of $\overline{K}(\Sigma)$ . Now we define $\varphi$ to map a cell in $U_1\setminus S$ onto the corresponding cell. First we define $\varphi$ on the cells of type $I$ in $h^{-1}(R)$ , then extend it to the closed cells of type $II$ in $h^{-1}[R/2,R]$ , starting from low-dimensional ones, next - extend it to the closed cells of type $II$ in $h^{-1}[R/4,R/2]$ , e.t.c. It is clear that $\varphi$ can be defined on the cells of types $I, II$ to respect $(f,h)$. At last we extend $\varphi$ respecting $f$ to the cells of type $III$ starting from the low-dimensional ones. It remains only to use 1.3.3.b and define $\varphi: S\to\{\tilde{p}\}\times f(S)$ respecting $f$. The bijectivity and continuity of $\varphi$ are obvious. ∎

## 2. Preliminary lemmas

All functions $\varkappa$ in this section may depend on the parameter, denoted by $\varepsilon$ .

2.1. Consecutive approximations.

2.1.1. Let $f: \mathcal{U} \subset M^n \to \mathbb{R}^k$ be a differentiable map from a domain in Alexandrov's space, and let $\|\cdot\|$ denote a norm on $\mathbb{R}^k$. Suppose that for any $x \in \mathcal{U}$ and $v \in \mathbb{R}^k$ , such that $f(x) \neq v$ , there exists a direction $\xi \in \Sigma_x$ such that $\|f(\cdot)-v\|'_{(x)}(\xi) < -\varepsilon$ . Then $f$ is clearly $\varepsilon$-open w.r.t. $\|\cdot\|$ (that is, given $x \in \mathcal{U}$ and $v \in \mathbb{R}^k$ such that $B_x(\|f(x)-v\| \cdot \varepsilon^{-1}) \subset \mathcal{U}$ , there exists a point $y \in \mathcal{U}$ such that $f(y)=v$ and $|xy| < \varepsilon^{-1}\|f(x)-v\|$ , c.f. [I.5.3])

2.1.2. In particular suppose that $f=(f_1,\dots,f_k): \mathcal{U} \subset M^n \to \mathbb{R}^k$ satisfies the following condition:

For any $p \in \mathcal{U}$ there are such directions $\xi_i^+$, $1 \le i \le k$, $\xi^-$ in $\Sigma_p$ that $|f'_{j(p)}(\xi_i^+)| < \delta$ for $i \neq j$ , $f'_{i(p)}(\xi_i^+) > \varepsilon$ , $-\varepsilon^{-1} < f'_{i(p)}(\xi^-) < -\varepsilon$ for all $i$ .

Then $f$ is $c(\varepsilon)$ -open w.r.t. euclidean norm in $\mathbb{R}^k$ , $(\delta < c(\varepsilon))$.

2.1.3. Let $f: \mathcal{U} \subset M^n \to \mathbb{R}^k$ be a differentiable $\varepsilon$-open map, let $p \in \mathcal{U}$ , $\xi \in \Sigma_p$ be such that $f'_{(p)}(\xi) = 0$ . Then given neighborhoods $V'$ of $\xi$ and $\mathcal{U}_1$ of $p$ there exists a point $q \in \mathcal{U}_1 \cap f^{-1}(f(p))$ such that $q' \subset V'$ . In particular, given a finite set of differentiable functions $g_i: \mathcal{U} \to R$ we can choose $q \in \mathcal{U}_1 \cap f^{-1}(f(p))$ to satisfy the inequalities $g_i(q) < g_i(p)$ if $g'_{i(p)}(\xi) < 0$ and $g_i(q) > g_i(p)$ if $g'_{i(p)}(\xi) > 0$ (c.f. [I.12.6]).

2.2. <u>Lemma</u>. A complete $n$-dimensional Alexandrov's space with curvatures $\geq 1$ can not contain $n+3$ compact subsets $A_i$ such that $|A_i A_j| > \pi/2 - \delta$ for $i \neq j$ , $|A_i A_1| > \pi/2 + \varepsilon$ for $i \geq 3$, $(\delta < c(\varepsilon))$. $\exists \delta(n, \varepsilon)$

<u>Proof</u>. We use induction on $n$ , the case $n=1$ being obvious. We may assume that $A_{n+3}$ is a point $p$ . Consider the sets of directions $A_i' \subset \Sigma_p$ , $1 \le i \le n+2$ . We have
$|pA_j| \leq 2\pi - |pA_1| - |A_1 A_j| < \pi - c(\varepsilon)$ $(j \neq 1)$ , $|A_1 p| \leq 2\pi - |pA_{n+2}| - |A_1 A_{n+2}| < \pi - c(\varepsilon)$.

Hence the comparison theorem implies $|A_i'A_j'| > \frac{\pi}{2} - \varkappa(\delta)$ $(i \neq j)$, $|A_i'A_1'| >$ $> \frac{\pi}{2} + c(\varepsilon)$, $i \geq 3$ , and this is a contradiction with the inductional assumption. ∎

2.3. Lemma. a) Let $M^n$ be a complete Alexandrov's space with curvatures $\geq 1$ , $\{A_i\}$, $1 \leq i \leq k+2$ $(0 \leq k \leq n)$ be compact subsets of $M^n$ such that $|A_i^* A_j^*| > \pi/2 - \delta$ $(i \neq j)$, $|A_1 A_i| > \frac{\pi}{2} + \varepsilon$ $(i \neq 1)$ . Then there is a point $x \in M^n$ such that $|xA_i| = \pi/2$ $(i \geq 3)$, $|xA_1| > \pi/2 + c(\varepsilon)$ , $|xA_2| < \pi/2 - c(\varepsilon)$. $(\delta < c(\varepsilon))$.

b) The assertion holds true if we replace the assumption $|A_1 A_2| > \pi/2 + \varepsilon$ by $|A_1 A_2| > \pi/2 - \delta$ and the conclusion $|xA_1| > \pi/2 + c(\varepsilon)$ by $|xA_1| > \pi/2 - \varkappa(\delta)$.

Proof of a). We use induction on $n$ , the case $n=1$ being obvious. First we move a point of $A_2$ towards $A_1$ to get a point $x_0$ such that $|x_0 A_i| \geq \pi/2$ $(i > 2)$, $|x_0 A_1| \geq \pi/2 + \mu$; $\mu = c(\varepsilon)$ . Next we construct inductively a sequence of points $x_\ell \in M^n$ and subsets $I_\ell \subset \{3, \ldots, k+2\}$ $(0 \leq \ell \leq k)$ such that $\# I_\ell = \ell$ , $I_{\ell+1} \supset I_\ell$ , and the following set of inequalities is satisfied with $x_\ell$ as $x$ :

(1) $|xA_i| = \pi/2$ for $i \in I_\ell$ , $|xA_i| \geq \pi/2$ for $i > 2, i \notin I_\ell$, $|xA_1| \geq \frac{\pi}{2} + \mu + \frac{\mu_1}{2}|xx_0|$,
where $\mu_1 = c(\varepsilon)$ is from (2) below.

Assume that $x_m$, $I_m$ are already constructed for $m \leq \ell$ $(\ell < k)$ and let $\mathcal{X}_\ell = \{x \in M^n :\ x$ satisfies (1) $\}$ . Choose any $j_0 > 2$ , $j_0 \notin I_\ell$ and let $x_{\ell+1}$ be the closest to $A_{j_0}$ point of $\mathcal{X}_\ell$ . Then $x_{\ell+1}$ satisfies (1) with $I_{\ell+1} = I_\ell \cup \{j_0\}$ instead of $I_\ell$ . Indeed, we have $|x_{\ell+1} A_i| <$ $< \pi - c(\varepsilon)$ for all $i$ and therefore for any $y$ in some neighborhood of $x_{\ell+1}$ the comparison theorem implies $|A_i' A_j'| > \pi/2 - \varkappa(\delta)$ $(i \neq j, i,j \neq 2)$, $|A_i' A_1'| > \pi/2 + c(\varepsilon)$ $(i > 2)$ in $\Sigma_y$ . Hence the inductional assumption allows us to apply 2.1.2 and conclude that the map $f(\cdot) = (|A_3, \cdot|, \ldots, |A_{k+2}, \cdot|)$ is $c(\varepsilon)$-open in some neihgborhood of $x_{\ell+1}$ . Again by the inductional assumption we can find a direction $\xi \in \Sigma_{x_{\ell+1}}$ , such that

(2) $|A_i' \xi| = \pi/2$ $(i \neq 1, 2, j_0)$, $|A_1' \xi| > \pi/2 + \mu_1$, $|A_{j_0}' \xi| < \pi/2$ . Hence either $|x_{\ell+1} A_{j_0}| = \pi/2$ or, by 2.1.3, there is a point near $x_{\ell+1}$ which satisfies (1) and is closer to $A_{j_0}$ than $x_{\ell+1}$ -

a contradition.

Now we have a point $x_k$ that satisfies all the requirements of our assertion except the last one. Let $\mathcal{X}$ be the set of all points $x \in M^n$ such that $|xA_i| = \pi/2$ $(i \geqslant 3)$, $|xA_1| \geqslant \pi/2 + \mu$, and let $\bar{x}$ be the closest to $A_2$ point of $\mathcal{X}$. To prove that $|\bar{x}A_2| < \frac{\pi}{2} - c(\varepsilon)$ it suffices to show that the assumption $|\bar{x}A_2| > \pi/2 - \varkappa(\delta)$ leads to a contradiction. Indeed, this assumption allows us to get a contradiction using the argument above, with $\bar{x}, A_2$ instead of $x_{l+1}, A_{j_0}$ and $(|A_2, \cdot|, \ldots, |A_{k+2}, \cdot|)$ instead of $f(\cdot)$. (In case $k = n$ the reference to the inductional assumption in this argument must be replaced by the reference to 2.2.) ▬

Proof of b). We use induction on $n$ and reverse induction on $k$ while $n$ is fixed. Repeat the first part of the proof of a) with $\mu = -\varkappa(\delta)$ instead of $\mu = c(\varepsilon)$, to get a point $x_k$, such that $|x_k A_i| = \frac{\pi}{2}$ $(i \geqslant 3)$, $|x_k A_1| \geqslant \frac{\pi}{2} + \mu + \mu_1/2 \, |x_k x_0|$, $\mu = -\varkappa(\delta)$, $\mu_1 = c(\varepsilon)$. If $|x_k A_2| < \frac{\pi}{2} - c(\varepsilon)$ we are done. Otherwise we have $|x_k A_2| > \frac{\pi}{2} - \varkappa(\delta)$, hence $|x_k A_1| > \pi/2 + c(\varepsilon)$. Therefore in case $k < n$ we can take $x_k$ as $A_{k+3}$ and apply the assumption of the reverse induction, and in case $k = n$ we get a contradiction to 2.2. ▬

2.4. Corollary. Under assumptions of 2.3.a) there is a point $x \in M^n$ such that $|xA_i| = \pi/2$ $(i > 2)$, $|xA_2| > \frac{\pi}{2} + c(\varepsilon)$.

Indeed, consider the cone $K(M^n)$ with apex $p$ and unit sphere identified with $M^n$. It follows from 2.3.a) and 2.1.2 that $f(\cdot) = (|A_2, \cdot|, \ldots, |A_{k+2}, \cdot|)$ is a differentiable $c(\varepsilon)$-open map near $p$. Take a sequence $\{v^i\} \subset \mathbb{R}^{k+1}$, $v^i \to f(p)$ such that $v^i_j = |A_{j+1} p|$ $(j \geqslant 2)$, $v^i_1 > |A_2 p|$, and let $p^i \in K(M^n)$ be such that $f(p^i) = v^i$, $c(\varepsilon)|p^i p| < |v^i f(p)|$. Then any limit point of $(p^i)'$ in $\Sigma_p = M^n$ satisfies our conditions. ▬

2.5. Volume estimates.

2.5.1. Let $M^n$ be a complete Alexandrov's space with curvatures $\geqslant 1$, $A \subset M$, $A[a_1, a_2] = \{x \in M : a_1 \leqslant |Ax| \leqslant a_2\}$. Let $0 \leqslant a_1 < a_2 < b_1 < b_2$, $0 < w \leqslant \min\{a_2 - a_1, b_2 - b_1\}$.

Then

$$\beta_{\omega\cdot\frac{a_2}{b_2}}\left(A[a_1,a_2]\right) \geqslant c\cdot \frac{a_2-a_1}{b_2-b_1}\,\beta_\omega\left(A[b_1,b_2]\right)$$

Indeed, the general case follows easily from the case $a_2-a_1=$ $=b_2-b_1=\omega$ . Let $\varphi.\ A[b_2-\omega,b_2]\to A[a_2-\omega,a_2]$ sends a point $x$ to a point $\varphi(x)$ on a shortest line $xA$ , such that $|A\varphi(x)|=a_2/b_2\,|Ax|$ . It follows easily from the comparison inequalities that $|\varphi(x)\varphi(y)|\geqslant a_2/b_2\,|xy|$ for any $x,y\in A[b_2-\omega,b_2]$ , and this is enough for our estimate.

2.5.2. It follows from 2.5.1 and [I.9.3] that there exists a constant $C_n>0$ , such that $\beta_\omega(A[\pi/2-\delta,\pi/2+\delta])\leqslant$ $\leqslant C_n\delta\cdot\omega^{-n}$ provided $0<\omega\leq\delta$ .

## 3. The definition and properties of noncritical maps.

All functions $\mathscr{X}$ in this section may depend on the parameter denoted by $\varepsilon$.

3.1. <u>Definition</u>. A map $f=(f_1,\dots,f_k):U\subset M^n\to\mathbb{R}^k$ $(k\geq 0)$ is called $(\varepsilon,\delta)$-noncritical at $p\in U$ if it satisfies the following set of conditions:

$$1.\ f_i=\inf_\gamma f_{i\gamma},\quad f_{i\gamma}(\cdot)=\varphi_{i\gamma}(|A_{i\gamma},\cdot|)+\sum_{l=1}^{i-1}\varphi_{i\gamma}^{l}(f_l(\cdot))+c_{i\gamma},$$

where $c_{i\gamma}\in\mathbb{R}$ , $A_{i\gamma}$ are compact subsets of $M^n$, $\varphi_{i\gamma}$, $\varphi_{i\gamma}^{l}$ have right and left derifatives, $\varphi_{i\gamma}^{l}$ are lipschitz functions with lipschitz constants $\leq\varepsilon^{-1}$ , $\varphi_{i\gamma}$ are increasing functions, satisfying $\varphi_{i\gamma}(0)=0$, $\varepsilon|x-y|\leq|\varphi_{i\gamma}(x)-\varphi_{i\gamma}(y)|\leq\varepsilon^{-1}|x-y|$.

2. The sets of indices $\Gamma_i(p)=\{\gamma:\ f_i(p)=f_{i\gamma}(p)\}$ satisfy $\#\,\Gamma_i(p)\leq\varepsilon^{-1}$ and there exists $\rho=\rho(p)>0$ such that for all $i$ $f_i(x)<f_{i\gamma}(x)-\rho$ for $x\in B_p(\rho)$, $\gamma\notin\Gamma_i(p)$.

3. $\measuredangle A_{i\alpha}pA_{j\beta}>\pi/2-\delta$ for $i\neq j$, $\alpha\in\Gamma_i(p)$, $\beta\in\Gamma_j(p)$.

4. There is a point $W=W(p)\in M^n$ such that $\measuredangle A_{i\gamma}pW>\pi/2+\varepsilon$ for $\gamma\in\Gamma_i(p)$ .

It is clear that the set of $(\varepsilon,\delta)$-noncritical points of $f'$ is open and $f$ is differentiable at any such point.

3.2. <u>Proposition</u>. Suppose that $f: \mathcal{U} \subset M^n \to \mathbb{R}^k$ has no $(\varepsilon,\delta)$-critical points in $\mathcal{U}$. Then $k \le n$ and $f$ is $C(\varepsilon)$-open. Furthermore, if $k=n$ then $f$ is a local (bilipschitz) homeomorphism.

<u>Proof</u>. Conditions 3.1.3, 4 imply that assumption $k>n$ contradicts to 2.2. It follows from 2.3.a, 2.4 that for any $p \in \mathcal{U}$ there are such directions $\xi_i^+, \xi_i^- \subset \Sigma_p$ $(1 \le i \le k)$ that $|A'_{j(p)}\xi_i^{\pm}| = \pi/2$ $(i \ne j)$, $|A'_{i(p)}\xi_i^+| < \pi/2 - c(\varepsilon)$, $|A'_{i(p)}\xi_i^-| > \pi/2 + c(\varepsilon)$, where $A'_{i(p)} = \bigcup_{\gamma \in \Gamma_i(p)} A'_{i\gamma}$. Therefore we can apply 2.1.1 to the norm $\|v\| = \sum_{i=1}^{k} \varepsilon^{3i}|v_i|$ on $\mathbb{R}^k$.

Let $k=n$ and assume that $f(x) = f(y)$, $x \ne y$, for $x, y$ so close to $p$ that 3.1.3, 4 hold for $x$ or $y$ instead of $p$ with the same $W$. Assume $|W_x| \le |W_y|$. If $x, y$ are sufficiently close comparing to $|pW|$, $|pA_{i\gamma}|$ $(\gamma \in \Gamma_i(p))$ then we have $\measuredangle Wxy > \pi/2 - \delta$, $\measuredangle A_{i\gamma}xy > \pi/2 - \delta$ for $\gamma \in \Gamma_i(x)$. We get a contradiction to 2.2 for $\Sigma_x$. ∎

3.3. <u>Proposition</u>. A level set of noncritical map has locally an intrinsic metric which is equivalent to the induced one. More precisely, let $f: \mathcal{U} \subset M^n \to \mathbb{R}^k$ be $(\varepsilon,\delta)$-noncritical at $p \in \mathcal{U}$. Let $\Pi = f^{-1}(f(p))$, $\rho_0 = \min\{\varepsilon\rho(p), \delta \cdot |W(p)p|, \delta \cdot |A_{i\gamma}p|$ $(1 \le i \le k, \gamma \in \Gamma_i(p))\}$, $q, r \in \Pi \cap B_p(\rho_0)$. Then there is a curve on $\Pi$ of length $\le C(\varepsilon)|qr|$ with endpoints $q, r$.

<u>Proof</u>. Assume that $|W(p)q| \le |W(p)r|$. Then the comparison inequality implies that $|W'(p)r'| > \pi/2 - \varkappa(\delta)$ in $\Sigma_q$. Moreover, we have $|A'_{i\gamma}W'(p)| > \pi/2 + c(\varepsilon)$, $|A'_{i\gamma}r'| > \pi/2 - \varkappa(\delta)$ $(\gamma \in \Gamma_i(q))$, and $|A'_{i\alpha}A'_{j\beta}| > \pi/2 - \varkappa(\delta)$ $(i \ne j, \alpha \in \Gamma_i(q), \beta \in \Gamma_j(q))$. We apply 2.3.b to $\Sigma_q$ and find a direction $\xi \in \Sigma_q$ such that $|A'_{i(q)}\xi| = \pi/2$ $\left(A'_{i(q)} = \bigcup_{\gamma \in \Gamma_i(q)} A'_{i\gamma}\right)$ and $|r'\xi| < \pi/2 - c(\varepsilon)$.

Hence by 2.1.3 there is a point $q_1 \in \Pi$ near $q$ such that $|rq_1| < |rq| - c(\varepsilon)|qq_1|$. Now the construction of the required curve on $\Pi$ is standard. ∎

3.4. Let $f: \mathcal{U} \subset M^n \to \mathbb{R}^k$ be $(\varepsilon,\delta)$-noncritical at

$p\in\mathcal{U}$. Assume that $V_{n-1}(\Sigma_p)\geqslant\varepsilon$. It follows from the volume comparison theorem 2.5.1 that $V_{n-1}(B_{w'}(\varepsilon/2))\geqslant c(\varepsilon)$ $(w'\in W'(p))$. Thus for a very small number $\omega$, $0<\omega<\delta$, we can construct a set of points $W_\alpha\in\mathcal{U}$ such that $\#\{W_\alpha\}\geqslant L\omega^{1-n}$, $L=c(\varepsilon)$, $\angle W_\alpha p W_\beta>\omega$ $(\alpha\neq\beta)$, $\angle W_\alpha p A_{i\gamma}>\pi/2+\varepsilon/2$ $(\gamma\in\Gamma_i(p), 1\leq i\leq k)$.

Let a neighborhood $V$ of $p$ be so small that $\angle W_\alpha x W_\beta>\omega$ $(\alpha\neq\beta)$, $\angle W_\alpha x A_{i\gamma}>\pi/2+\varepsilon/2$ $(\gamma\in\Gamma_i(p), 1\leq i\leq k)$, $\angle A_{i\alpha} x A_{j\beta}>\pi/2-\delta$ $(i\neq j, \alpha\in\Gamma_i(p), \beta\in\Gamma_j(p))$, $|xp|<\delta\cdot\min_{\alpha,i;\gamma\in\Gamma_i(p)}\{|pW_\alpha|,|pA_{i\gamma}|\}$ and $f_{i\gamma}(x)>f_i(x)+\rho$ $(\gamma\notin\Gamma_i(p))$ for all $x\in V$. Let $\sigma(x)$ denote the mean value of $|W_\alpha x|$.

<u>Assertion 1</u>. Let $x,y\in V$ be such that $|f(x)f(y)|<\delta\cdot|xy|$. Then either the map $(f,|x,\cdot|):V\to\mathbb{R}^{k+1}$ is $(c(\varepsilon),\varkappa(\delta))$-noncritical at $y$ or $\sigma(y)-\sigma(x)>c(\varepsilon)|xy|$.

<u>Assertion 2</u>. Let $x,y\in V$ be such that $|f(x)f(y)|<\delta|xy|$ and $x$ be a point of a local maximum of the function $\sigma|_{f^{-1}(f(x))}$. Then $\angle W_\alpha y x>\pi/2+c(\varepsilon)$ for some $\alpha$.

<u>Proof of 1</u>. The conditions 3.1.1, 2, 3 for $(f,|x,\cdot|)$ are clearly satisfied. Take a point on a shortest line $W_\alpha y$ close to $y$ as a candidate for $W(y)$. To satisfy 3.1.4 it suffices to choose $\alpha$ such that $|x'W'_\alpha|>\pi/2+c(\varepsilon)$ in $\Sigma_y$. On the other hand, we have $\sigma(y)-\sigma(x)>c(\varepsilon)|xy|$ provided mean value of $\cos|x'W'_\alpha|$ is greater than $c(\varepsilon)$. Since $\#\{W_\alpha\}\geqslant L\omega^{1-n}$ and (by the volume estimate 2.5.2) $\#\{W_\alpha : ||W'_\alpha x'|-\pi/2|<a\}<C_n a\omega^{1-n}$ $(a>\omega)$, one of the conditions above on $|W'_\alpha x'|$ must be satisfied. ∎

<u>Proof of 2</u>. It suffices to check that for some $\alpha$ we have $|W'_\alpha y'|<\pi/2-c(\varepsilon)$ in $\Sigma_x$. Take $a=c(\varepsilon)$, $b=c(\varepsilon)$ such that $C_n a<(L-C_n a-C_n b)\cdot\sin b$, where $C_n$ is from 2.5.2. Assume $|W'_\alpha y'|\geqslant\pi/2-a$ for all $\alpha$ and let $\mathfrak{A}_1=\{\alpha:|W'_\alpha y'|\leq\pi/2+a\}$, $\mathfrak{A}_2=\{\alpha:|W'_\alpha y'|>\pi/2+a\}$, $W'=\bigcup_{\alpha\in\mathfrak{A}_2}W'_\alpha$. Apply 2.3.a to the sets $W', y', A'_{1(x)},\dots,A'_{k(x)}$ in $\Sigma_x$ $(A'_{i(x)}=\bigcup_{\gamma\in\Gamma_i(x)}A'_{i\gamma})$, and find a direction $\xi\in\Sigma_x$ such that $|\xi A'_{i(x)}|=\pi/2$, $|\xi W'|=\pi/2+c(a)$. Let $\mathfrak{A}_3=\{\alpha\in\mathfrak{A}_2 : |W'_\alpha\xi|>\pi/2+b\}$. Then

$$\#(\mathfrak{A}_1\cup\mathfrak{A}_2)\,\sigma'_{(x)}(\xi)<-\#\mathfrak{A}_3\cdot\sin b+\#\mathfrak{A}_1<\big(C_n a-(L-C_n a-C_n b)\sin b\big)\omega^{1-n}<0,$$

and 2.1.3 gives a contradiction to the local maximality assumption. ■

3.5. A map $f: \mathcal{U} \subset M^n \to \mathbb{R}^k$ is called $(\varepsilon,\delta)$-complementable at $p$, iff there is a function $g$ such that the map $(f,g)$ is $(\varepsilon,\delta)$-noncritical at $p$.

<u>Proposition</u>. Let $f: \mathcal{U} \subset M^n \to \mathbb{R}^k$ be $(\varepsilon,\delta)$-noncritical at $p \in \mathcal{U}$, $\mathrm{Vr}_{n-1}(\Sigma_p) \geq \varepsilon$. Then either $f$ is $(c(\varepsilon), \varkappa(\delta))$-complementable at $p$ or for sufficiently small $R>0$ there exists a continuous function $h$ in $\mathcal{U}_1 = f^{-1}(I^k_{f(p)}(\delta^5 R)) \cap \cap \bar{B}_p(R)$ such that

1. $h(\mathcal{U}_1) = [0,R]$, $h(x) = |px|$ if $|px| > R/2$.
2. $f$ is injective on $S = h^{-1}(0)$.
3. $f$ is $(c(\varepsilon), \varkappa(\delta))$-complementable at any point of $\mathcal{U}_1 \setminus S$.
4. $(f,h)$ is $(c(\varepsilon), \varkappa(\delta))$-noncritical at any point $x \in \mathcal{U}_1$ such that $|f(x)f(S)| < \frac{1}{3}\delta^5 h(x)$.

<u>Proof</u>. Let $R>0$ be so small that general assumptions of 3.4 hold in $\mathcal{U}_1$. Using 3.4.1 choose $M = c(\varepsilon)$ in such a way that $f$ is $(c(\varepsilon), \varkappa(\delta))$-complementable at any point of $\mathcal{U}_1 \setminus S$ where $S = \{x \in \mathcal{U}_1 : \sigma(x) - \sigma(y) \geq M\,|xy|$ for all $y \in \mathcal{U}_1$ satisfying $|f(x)f(y)| < \delta\,|xy|\}$. Clearly $S$ is compact and nonempty provided $f$ is not $(c(\varepsilon), \varkappa(\delta))$-complementable at $p$. Obviously, $f$ is injective on $S$ and moreover, it follows from $c(\varepsilon)$-openness of $f$ that $|f(x)f(y)| > M_1\,|xy|$ for all $x,y \in S$, where $M_1 = c(\varepsilon)$. In particular, $S \subset B_p(c(\varepsilon)\delta^5 R)$.

Define a sequence of finite subsets $S_j \subset S$ in a following way: $S_0 = \{p\}$, $S_j \supset S_{j-1}$, $f(S_j)$ is a maximal $\delta^{j+5}R$-net in $f(S)$ **). Define $h(x) = \inf\limits_{p_\gamma \in \bigcup_j S_j} h_\gamma(x)$

where

$$h_\gamma(x) = \varphi_{\delta^{j+1}R}(|p_\gamma x|) + \sum_{\ell=1}^{k} 10\,M_1^{-1}\,|f_\ell(x) - f_\ell(p_\gamma)| \quad \text{for } p_\gamma \in S_j \setminus S_{j-1},\ j \geq 1,$$

$$h_\gamma(x) = \min\left\{\varphi_{\delta R}(|px|) + \sum_{\ell=1}^{k} 10\,M_1^{-1}\,|f_\ell(x) - f_\ell(p)|,\ \tfrac{1}{2}\varphi_{R/2}(|px|) + R/4\right\} \quad \text{for } p_\gamma = p.$$

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**) that is $|v_1 v_2| \geq \delta^{j+5}R$ for $v_1, v_2 \in f(S_j)$, and $\forall v \in f(S)\ \exists\ v_1 \in f(S_j) : |v v_1| \leq \delta^{j+5}R$.

$$\varphi_r(a) = \begin{cases} a, & a \le r \\ 2a - r, & a \geq r \end{cases}$$

It is clear that $h^{-1}(0) = S$, $h(U_1) = [0,R]$ and $h(x) = |px|$ if $|px| > R/2$ . To check the condition 4 it suffices to prove the following assertion 3 and to refer to 3.4.1.

<u>Assertion 3</u>. For $x \in U_1 \setminus S$ let $\Gamma(x) = \{\gamma : h_\gamma(x) = h(x)\}$. If $|f(x)f(S)| < \frac{1}{3}\delta^5 h(x)$ then there exists $s \in S$ such that $|f(x)f(s)| < \delta^2 |xs|$ and $|p_\gamma s| < \delta \cdot |sx|$ for all $\gamma \in \Gamma(x)$ . Moreover $\# \Gamma(x) \le C$ . $|f(p_\gamma) f(s)| < \delta^2 |xs|$

<u>Proof</u>. There exists $j$ and $p_\gamma \in S_j$ such that

(1) $\quad |f(x)f(p_\gamma)| \le 10\delta^{j+5}R, \quad |xp_\gamma| \geq \delta^{j+1}R$

Indeed, the case $f(x) \in f(S)$ is clear. Otherwise choose $y \in S$ such that $|f(x)f(y)| < \frac{1}{3}\delta^5 h(x)$ , $j$ such that $\delta^{j+6}R < |f(y)f(x)| \le \delta^{j+5}R$ and $p_\gamma \in S_j$ such that $|f(y)f(p_\gamma)| \le \delta^{j+5}R$ . Then $|f(x)f(p_\gamma)| \le 10\,\delta^{j+5}R$ and $3\delta^{j+6}R < \delta^5 h(x) < 2\delta^5|p_\gamma x| + \delta^5 \cdot k \cdot 10\,M_1^{-1}\,|f(x)f(p_\gamma)| < 2\delta^5|p_\gamma x| + 100\,k\,M_1^{-1}\delta^{j+10}R$ , hence $|p_\gamma x| > \delta^{j+1}R$.

Let $j_0$ be the minimal value of $j$ that agrees with (1), and $s = p_{\gamma_0}$ be the corresponding point of $S$ . Then $\gamma \in \Gamma(x)$ implies $p_\gamma \in S_{j_0}$ . Indeed, let $p_\gamma = S \setminus S_{j_0}$ . Then

$$h_\gamma(x) - h_{\gamma_0}(x) \geq \varphi_{\delta^{j_0+2}R}(|p_\gamma x|) - \varphi_{\delta^{j_0+1}R}(|p_{\gamma_0}x|) + \sum_{l=1}^{k} 10\,M_1^{-1}\big(|f_l(p_\gamma) - f_l(x)| -$$

$$- |f_l(p_{\gamma_0}) - f_l(x)|\big) \geq \varphi_{\delta^{j_0+2}R}(|p_{\gamma_0}x| - |p_{\gamma_0}p_\gamma|) - \varphi_{\delta^{j_0+1}R}(|p_{\gamma_0}x|) + 10\,M_1^{-1}|f(p_\gamma)f(p_{\gamma_0})| -$$

$$- 20\,k\,M_1^{-1}|f(p_{\gamma_0})f(x)| \geq \delta^{j_0+1}R - \delta^{j_0+2}R - 2|p_{\gamma_0}p_\gamma| + 10|p_{\gamma_0}p_\gamma| - 200\,k\,M_1^{-1}\delta^{j_0+5}R > 0$$

the same $\delta^{j_0+1}R$ Assume now $\gamma \in \Gamma(x)$ and $p_\gamma \in S_{j_0} \setminus S_{j_0-1}$ . Then

$0 \geq h_\gamma(x) - h_{\gamma_0}(x) \geq -2|p_{\gamma_0}p_\gamma| + 10\,M_1^{-1}|f(p_\gamma)f(p_{\gamma_0})| - 20\,k\,M_1^{-1}(f(p_{\gamma_0})f(x)|$,

hence $|f(p_\gamma)f(p_{\gamma_0})| \le 25\,k\,\delta^{j_0+5}R$ and all points $p_\gamma \in S_{j_0} \setminus S_{j_0-1}$ satisfy our assertion.

If $j_0 > 0$ then there is a point $p_{\gamma_1} \in S_{j_0-1}$ such that $|f(p_{\gamma_0})f(p_{\gamma_1})| \le \delta^{j_0+4}R$ . Observe that the choice of $j_0$

implies $|xp_{\gamma_1}| < \delta^{j_0}R$ . Assume that $p_\gamma \in S_{j_0-1}$ and $\gamma \in \Gamma(x)$ . Then $0 \geqslant h_\gamma(x) - h_{\gamma_1}(x) \geqslant |p_\gamma x| - |p_{\gamma_1}x| + 10 M_1^{-1} |f(p_\gamma) f(p_{\gamma_1})| - 20 k M_1^{-1} | f(p_{\gamma_1}) f(x)| \geqslant 9 M_1^{-1} | f(p_\gamma) f(p_{\gamma_1})| - 27 k M_1^{-1} \delta^{j_0+4} R$,
hence $|f(p_\gamma) f(p_{\gamma_1})| \leqslant 3k\, \delta^{j_0+4} R$ and all points $p_\gamma \in S_{j_0-1}$ satisfy our assertion. ∎

3.6. <u>Proposition</u>. Let $f : \mathcal{U} \subset M^n \to \mathbb{R}^k$ be $(\varepsilon,\delta)$ - noncritical at $p \in \mathcal{U}$ , $Vr_{n-1}(\Sigma_p) \geqslant \varepsilon$ . Let $R > 0$ be so small that general assumptions of 3.4 hold in $\mathcal{U}_1 = \bar{B}_p(R) \cap \cap f^{-1}(I^k_{f(p)}(\delta^5 R))$ . Suppose that

(2) For all $x \in \mathcal{U}_1$ such that $\delta R \leqslant |px| \leqslant R$ , holds $\sigma(p) - \sigma(x) \geqslant M|px|$ , $M = c(\varepsilon)$ . Then for any $v \in I^k_{f(p)}(\delta^5 R)$ there exists a continuous function $h_v : \mathcal{U}_1 \to [0,R]$ and a point $O_v \in \mathcal{U}_1 \cap f^{-1}(v)$ such that

1. For $x \in f^{-1}(v) \cap \mathcal{U}_1$ hold $h_v(x) = R \Leftrightarrow |px| = R$ , $h_v(x) = 0 \Leftrightarrow x = O_v$ .

2. $(f, h_v)$ has no $(c(\varepsilon), \varkappa(\delta))$ -critical points on $\mathcal{U}_1 \cap f^{-1}(v) \setminus \{O_v\}$.

<u>Proof</u>. Let $O_v$ be the point where $\sigma|_{f^{-1}(v) \cap \mathcal{U}_1}$ attains its maximum. Since $f$ is $c(\varepsilon)$-open, it follows from (2) that $|pO_v| < \delta R$ . Define R o.k.

$$h_v(x) = \min\{\varphi_{\delta R}(|O_v x|), \tfrac{1}{2}\varphi_{\frac{R}{2}}(|px|) + R/4\} \qquad \text{where } \varphi_r(a) = \begin{cases} a, & a \leqslant r \\ 2a - r, & a \geqslant r. \end{cases}$$

The first assertion is now obvious. The second assertion follows easily from 3.4.2.1. ∎ both not needed

3.7. The Propositions 3.4, 3.5, 3.6 justify the following

<u>Definition</u>. Let $\mathcal{U} \subset M^n$ be a domain in Alexandrov's space, and let $\varepsilon_0 = \inf_{p \in \mathcal{U}} Vr_{n-1}(\Sigma_p) > 0$ (This is always true if $\bar{\mathcal{U}}$ is compact, see [I,9.7]). A map $f : \mathcal{U} \to \mathbb{R}^k$ $(0 \leqslant k \leqslant n+1)$ is called noncritical at $p$ if it is $(\varepsilon,\delta)$-noncritical at $p$ in the sense of 3.1, with $\varepsilon < \varepsilon_0$ , $\delta < \Delta_{n,k}(\varepsilon)$ , where $\Delta_{n,k}(\varepsilon)$ is a positive function, defined inductively (using reverse induction on $k$ , starting from $k = n+1$ ) in such a way that $(\varepsilon,\delta)$ -noncritical maps $f : \mathcal{U} \to \mathbb{R}^k$ with

$\varepsilon<\varepsilon_0$ , $\delta<\Delta_{n,k}(\varepsilon)$ satisfy 3.2-3.6 and the pairs $(c(\varepsilon), \varkappa(\delta))$ appearing in the formulations of 3.4., 3.5., 3.6. satisfy $\varkappa(\delta)<\Delta_{n,k+1}(c(\varepsilon))$ .

It is clear that noncritical maps satisfy all the conditions 1.3.

## 4. The stability theorem and its corollaries.

### 4.1. Canonical neighborhoods and framed sets.

Fix $\varepsilon_0>0$ . Let $\mathcal{U}\subset M^n$ be a domain in Alexandrov's space, such that $\mathrm{Vol}_{n-1}(\Sigma_p)\geqslant\varepsilon_0$ for any $p\in\mathcal{U}$ . A subset $\mathcal{U}_1\subset\mathcal{U}$ is called an $(\varepsilon,\delta)$ -canonical neighborhood of $p\in\mathcal{U}$ of rank $k$ $(0\leq k\leq n)$ if $\mathcal{U}_1=\bar{B}_p(R)\cap\cap f^{-1}(I^k_{f(p)}(\delta^5 R))$ , where $f:\mathcal{U}_1\to\mathbb{R}^k$ is $(\varepsilon,\delta)$-noncritical at $p$ , and $R>0$ is so small that general assumptions of 3.4 and the second alternative of 3.5 hold true in $\mathcal{U}_1$ . A canonical neighborhood of rank $k$ is an $(\varepsilon,\delta)$ -canonical neighborhood of rank $k$ with $\varepsilon<\varepsilon_0$, $\delta<\Delta_{n,k}(\varepsilon)$. metric concept
It follows from 3.5 that any point $p\in\mathcal{U}$ has a canonical neighborhood of some rank. (possibly 0)

A compact subset $P\subset\mathcal{U}$ is called $k$ -framed if it is covered by a finite set of open domains $\mathring{\mathcal{U}}_\alpha\Subset\mathcal{U}$ , such that each $\mathcal{U}_\alpha$ is a canonical neighborhood of some $p_\alpha\in P$ of rank $\geqslant k$ , and $P\cap\mathcal{U}_\alpha=\mathcal{U}_\alpha\cap f_\alpha^{-1}(H_\alpha\cap(\bigcup_{j\in J_\alpha}\bar{O}_j))$ , where $H_\alpha$ is an affine coordinate plane in $\mathbb{R}^k$ , containing $f_\alpha(p_\alpha)$ and each $O_j$ is an ortant in $\mathbb{R}^k$ with apex $f_\alpha(p_\alpha)$ . Clearly $I^k_{f_\alpha(p_\alpha)}\cap H_\alpha\cap(\bigcup_{j\in J_\alpha}\bar{O}_j)$ is an MCS-space, hence by 1.4.2.b,a and 1.4.3, $P$ is an MCS-space.

We say that the framing $\{\mathcal{U}_\alpha\}$ respects a map $f:\mathcal{U}\to\mathbb{R}^\ell$ on a compact subset $K\subset P$ if the first $\ell$ coordinate functions of $f_\alpha$ coincide with $f$ on $\mathcal{U}_\alpha$ provided $\mathcal{U}_\alpha\cap K\neq\varnothing$.

### 4.2. Correspondence.

Let $\mathcal{M}^n$, $\tilde{\mathcal{M}}^n$ be (complete) Alexandrov's spaces with the same lower bound of curvatures, $\theta : M^n \to \widetilde{M}^n$ satisfy $\big|\,|xy| - |\theta(x)\theta(y)|\,\big| < \nu$ for $x, y \in \mathcal{U}$, where $\mathcal{U} \subset M^n$ is a fixed domain with compact closure. We call $\theta$ a $\nu$-approximation on $\mathcal{U}$. Let $\widetilde{\mathcal{U}} = \{\tilde{x} \in \widetilde{M}^n : \exists x \in \mathcal{U} : |\tilde{x}\theta(x)| < \nu\}$. If $\nu > 0$ is sufficiently small then there is a positive lower bound for $Vz_{n-1}(\Sigma_{\tilde{p}})$ $(\tilde{p} \in \widetilde{\mathcal{U}})$ and $Vz_{n-1}(\Sigma_p)$ $(p \in \mathcal{U})$, which is independent of $\widetilde{\mathcal{M}}, \theta, \nu$ (Indeed, by [I,9.7] it suffices to have a positive lower bound for $Vz_n(\widetilde{\mathcal{U}})$. But the existence of a $n$-strained point in $\mathcal{U}$ implies (when $\nu > 0$ is small enough) the existence of a domain in $\widetilde{\mathcal{U}}$, which is bilipschitz equivalent to euclidean ball of radius bounded away from zero, hence $Vz_n(\widetilde{\mathcal{U}})$ is also bounded away from zero). Let $\varepsilon_0$ denote this lower bound.

Let $f : \mathcal{U} \to \mathbb{R}^k$ be $(\varepsilon,\delta)$-noncritical at $p \in \mathcal{U}$, $\varepsilon < \varepsilon_0$, $\delta < \Delta_{n,k}(\varepsilon)$. Define a corresponding map $\tilde{f} : \widetilde{\mathcal{U}} \to \mathbb{R}^k$ using the same formulas with $\widetilde{A}_{i\gamma}$ instead of $A_{i\gamma}$, where $\widetilde{A}_{\gamma} \subset \widetilde{\mathcal{U}}$ is a compact set such that the Hausdorff distance between $\widetilde{A}_{i_\gamma}$ and $\theta(A_{i\gamma})$ is less than $\nu$. (We assume that $A_{i\gamma} \subset \mathcal{U}$). If $\nu > 0$ is small enough (depending on $M^n$, $f, \mathcal{U}, p$) then there exists a point $\tilde{p} \in \widetilde{\mathcal{U}}$ such that $\tilde{f}(\tilde{p}) = f(p)$ and $|\tilde{p}\,\theta(p)| < C(\varepsilon)\nu$, and $\tilde{f}$ is $(\varepsilon,\delta)$-noncritical at any such point. This follows from $C(\varepsilon)$-openness of noncritical maps. If $\mathcal{U}(p) = \overline{B}(p) \cap f^{-1}(\mathrm{I}^k_{f(p)}(\delta^5 R))$ is an $(\varepsilon,\delta)$-canonical neighbourhood of $p$, then we let $\widetilde{\mathcal{U}}(\tilde{p}) = \overline{B}_{\tilde{p}}(R) \cap \tilde{f}^{-1}(\mathrm{I}^k_{\tilde{f}(\tilde{p})}(\delta^5 R))$ for a point $\tilde{p}$ satisfying $\tilde{f}(\tilde{p}) = f(p)$, $|\tilde{p}\,\theta(p)| < C(\varepsilon)\cdot\nu$. Clearly, $\widetilde{\mathcal{U}}(\tilde{p})$ satisfies general assumptions of 3.4 (use $\widetilde{W}_\alpha = \theta(W_\alpha)$ instead of $W_\alpha$) but may satisfy the first alternative of 3.5 instead of the second one. However it satisfies the assumptions of 3.6.

Let $P \subset \mathcal{U}$ be $k$-framed by the covering $\{\mathring{\mathcal{U}}_\alpha(p_\alpha)\}$. Then a compact subset $\widetilde{P} \subset \widetilde{\mathcal{U}}$ is corresponding to $P$ if it is covered by $\{\mathring{\widetilde{\mathcal{U}}}_\alpha(\tilde{p}_\alpha)\}$ and $\widetilde{P} \cap \widetilde{\mathcal{U}}_\alpha = \widetilde{\mathcal{U}}_\alpha \cap \tilde{f}^{-1}(H_\alpha \cap (\bigcup_{j \in J_\alpha} \overline{O}_j))$. Clearly a compact Alexandrov's space $M^n$ admits a $0$-framing and $\widetilde{M}^n$ is corresponding to $M^n$ if $\theta : M^n \to \widetilde{M}^n$ is a $\nu$-approximation and $\nu$ is small enough. Now we are in a position to prove the following generalization of the stability Theorem 0.3.

4.3. Theorem. Corresponding subsets are homeomorphic. More precisely, let $\mathcal{M}^n, \widetilde{\mathcal{M}}^n$ be complete Alexandrov spaces with the same lower bound of curvatures, $P \subset \mathcal{U} \subset M^n$ be a $k$-framed compact subset, $\widetilde{P} \subset \widetilde{\mathcal{U}} \subset \widetilde{M}^n$ be corresponding to $P$ w.r.t. $\nu$-approximation $\theta$. Then there exists a homeomorphism $\theta': P \to \widetilde{P}$ which is $\mathfrak{X}(\nu)$-close to $\theta$, $\mathfrak{X}$ depending on $M^n, P$. Moreover, if the framing of $P$ respects a map $f: \mathcal{U} \to \mathbb{R}^\ell$ on $P$ and a map $(f,h): \mathcal{U} \to \mathbb{R}^{\ell+1}$ on a compact subset $K \subset P$, then $\theta'$ can be chosen to satisfy $f \equiv \widetilde{f} \circ \theta'$ on $P$, $(f,h) \equiv (\widetilde{f}, \widetilde{h}) \circ \theta'$ on $K$, where $\mathfrak{X}$ depends now on $M^n, P, f, h, K$.

Proof. We are going to use the complement to the Theorem B. First observe that any two close points of $\widetilde{P}$ can be connected in $\widetilde{P}$ by a curve of small diameter. Indeed, since $\widetilde{f}_\alpha$ are $c(\varepsilon)$-lipschitz and $c(\varepsilon)$-open in $\widetilde{\mathcal{U}}_\alpha$, this assertion follows easily from 3.3. Thus to apply the complement to the Theorem B it suffices to construct homeomorphisms $\theta_\alpha: (\mathcal{U}_\alpha, \mathring{\mathcal{U}}_\alpha) \to$ $\to (\widetilde{\mathcal{U}}_\alpha, \mathring{\widetilde{\mathcal{U}}}_\alpha)$, $\mathfrak{X}(\nu)$-close to $\theta$, such that $\widetilde{f}_\alpha \circ \theta_\alpha \equiv f_\alpha$. If $k=n$ then we can take $\theta_\alpha = \widetilde{f}_\alpha^{-1} \circ f_\alpha$. Otherwise we use reverse induction on $k$.

Let $\mathcal{U}_\alpha = \mathcal{U}_\alpha(p) = \overline{B}_p(R) \cap f_\alpha^{-1}(I^k_{f(p)}(\delta^5 R))$ be an element of the $k$-framing of $P$, $h_\alpha: \mathcal{U}_\alpha \to [0,R]$ be the function constructed in 3.5. Fix a number $\nu_1 > 0$ and consider a preliminary finite cell decomposition of $\mathcal{U}_\alpha$, constructed in 1.5, such that each cell of type IV has diameter $< \nu_1$. Let $P_1$ denote the union of closed cells of types I, II, III, $K_1$ denote the union of the cells of type I. Then there exists a $(k+1)$-framing of $P_1$ that respects $f_\alpha$ on $P_1$ and respects $(f_\alpha, h_\alpha)$ on $K_1$. Consider the corresponding cell decomposition of $\widetilde{\mathcal{U}}_\alpha = \widetilde{\mathcal{U}}_\alpha(\widetilde{p})$ and let $\widetilde{P}_1, \widetilde{K}_1$ be cell-corresponding to $P_1, K_1$. By inductional assumption we can construct a homeomorphism $\theta_\alpha': P_1 \to \widetilde{P}_1$ which is $\mathfrak{X}(\nu)$-close to $\theta$ and satisfies $\widetilde{f}_\alpha \circ \theta_\alpha' \equiv f_\alpha$ on $P_1$, $(\widetilde{f}_\alpha, \widetilde{h}_\alpha) \circ \theta_\alpha' \equiv (f_\alpha, h_\alpha)$ on $K_1$. Now $\theta_\alpha'$ can be extended to the cells of type $\overline{IV}$ to get the required homeomorphism $\theta_\alpha$ $(\varkappa(\nu)+\nu_1)$ -close to $\theta$ provided these cells and the corresponding cells in $\mathcal{U}_\alpha$ satisfy $(\zeta_\gamma, \varsigma_\gamma) \approx (\overline{K}(\Sigma)\times I^\ell, \Sigma \times I^\ell)$ respecting $f_\alpha\ (\tilde{f}_\alpha)$, where $\Sigma = \partial B_p(R) \cap f_\alpha^{-1}(f_\alpha(p))$. The last condition follows from 3.6 and the inductional assumption, that guarantees that $\partial B_{\tilde{p}}(R) \cap \tilde{f}_\alpha^{-1}(\tilde{f}_\alpha(\tilde{p})) \approx \Sigma$. (Use the arguments of 1.5 - the proof of Assertion 3 and the description of the topology of the cells of type $\overline{III}$ ). ■

4.4. Proof of Theorem 0.1 on spherical neighborhood.

Let $p \in M^n$ be a point in Alexandrov's space. Theorem 1.4.1 implies that $(\overline{B}_p(R), \partial B_p(R)) \approx (\overline{K}(\partial B_p(R)), \partial B_p(R))$ for small $R>0$. Indeed, the function $|p,\cdot|$ is noncritical at points close to $p$, excluding $p$ itself. It remains to show that $\partial B_p(R) \approx \Sigma_p$ for small $R>0$. This is a corollary of 4.3 applied to 1-framed compact subset $\Sigma_p \subset K(\Sigma_p)$ as $P$ and the corresponding subset $\partial B_p(1)$ in $(R^{-1}\cdot M^n, p)$, which converges to $(K(\Sigma_p), p)$ in Gromov-Hausdorff sence as $R \to 0$. ■

4.5. Theorem. A complete Alexandrov's space $M^n$ with curvatures $\geqslant 1$ and with $diam(M^n) > \pi/2$ is homeomorphic to a suspension on a compact $(n-1)$-dimensional Alexandrov's space with curvatures $\geqslant 1$.

(This is a direct generalization of the Diameter sphere Theorem of Grove and Shiohama [GSh] )

Proof. Let $pq$ be a diameter of $M^n$. Then clearly

$\varepsilon \to 0$ $\tilde{\angle} pxq > \pi/2 + \varepsilon$ for some $\varepsilon > 0$, depending on $|pq|$, and

as for all $x \neq p,q$. Hence the function $|p,\cdot|$ is noncritical

$x \to p, q$ in $M^n \setminus \{p,q\}$ and by 1.4.1 $M^n \approx S(\partial B_p(R))$ for any $0 < R < |pq|$. But $\partial B_p(R) \approx \Sigma_p$ for small $R>0$, hence $M^n \approx S(\Sigma_p)$. ■

4.6. Theorem. The boundary points of an Alexandrov's space are distinguished from the interior ones by the topology of their conical neighborhoods. The boundary of Alexandrov's space is closed.

Proof. It suffices to establish the following characterization of the boundary points: A point belongs to the boundary (to the interior) of Alexandrov's space iff its conical neighborhood is homeomorphic to $\mathbb{R}^\ell \times K(\Sigma)$, for some $\ell$, where $\Sigma$ is a compact Alexandrov's space with curvatures $\geqslant 1$ with nonempty (empty) boundary. Thus our theorem is reduced to the following.

Assertion. If $\Sigma, \Sigma_1$ are compact Alexandrov's spaces, $\mathbb{R}^\ell \times K(\Sigma) \approx \mathbb{R}^{\ell_1} \times K(\Sigma_1)$ and $\Sigma$ has nonempty boundary, then $\Sigma_1$ also has nonempty boundary.

Proof of the Assertion. We use the induction on the dimension of $\Sigma$, and the second induction on $\dim \Sigma_1$ to establish the base of the first induction. The base of the second induction is clear: $\mathbb{R}^\ell \times K(I)$ is not homeomorphic to $\mathbb{R}^\ell \times K(S^1)$. Assume that $\mathbb{R}^\ell \times K(I) \approx \approx \mathbb{R}^{\ell_1} \times K(\Sigma_1)$, where $\ell_1 < \ell$ and $\Sigma_1$ has empty boundary. Then there is a point in $\mathbb{R}^\ell \times K(\partial I)$, such that the corresponding point in $\mathbb{R}^{\ell_1} \times K(\Sigma_1)$ does not lie in $\mathbb{R}^{\ell_1} \times \{apex\}$. Considering the conical neighborhoods of this point we get $\mathbb{R}^\ell \times K(I) \approx \mathbb{R}^{\ell_1+1} \times K(\widetilde{\Sigma}_1)$, where $\widetilde{\Sigma}_1$ is a compact Alexandrov's space with empty boundary, $\dim \widetilde{\Sigma}_1 = \dim \Sigma_1 - 1$.

At last, assume that $\mathbb{R}^\ell \times K(\Sigma) \approx \mathbb{R}^{\ell_1} \times K(\Sigma_1)$, and $\Sigma_1$ has empty boundary. Take again a point in $\mathbb{R}^\ell \times K(\partial\Sigma)$ and the corresponding point in $\mathbb{R}^{\ell_1} \times K(\Sigma_1)$ and consider their conical neighborhoods. We get either $\mathbb{R}^{\ell+1} \times K(\widetilde{\Sigma}) \approx \mathbb{R}^{\ell_1} \times K(\Sigma_1)$, or $\mathbb{R}^{\ell+1} \times K(\widetilde{\Sigma}) \approx \mathbb{R}^{\ell_1+1} \times K(\widetilde{\Sigma}_1)$, where $\widetilde{\Sigma}_1, \widetilde{\Sigma}$ are compact Alexandrov's spaces, $\widetilde{\Sigma}$ has nonempty boundary, $\widetilde{\Sigma}_1$ has empty boundary, and $\dim \widetilde{\Sigma} = \dim \Sigma - 1$.

4.7. Corollary. Let $M^n$ be Alexandrov's space, $p \in \partial M^n$. Then $(R^{-1} \cdot M^n, \partial(R^{-1} \cdot M^n), p)$ converge to $(K(\Sigma_p), \partial K(\Sigma_p) = K(\partial\Sigma_p), p)$ in Gromov-Hausdorff sense as $R \to 0$. A small spherical neighborhood of $p$ in $\partial M^n$ is homeomorphic to $K(\partial\Sigma_p)$. ∎

## 5. The Doubling theorem.

5.1. Let $\mathcal{M}^n$ be a complete Alexandrov's space with boundary $N \neq \emptyset$. Let $\varphi : M^n \to M_1^n$ be an isometry. It follows from 4.6 that $\partial M_1^n = \varphi(N)$. The doubling $\bar{M}^n$ of $M^n$ is defined to be the quotient $\bar{M}^n = M^n \cup M_1^n / \sim$, where $x \sim y$ iff $x \in N, y = \varphi(x)$ or $y \in N, x = \varphi(y)$. To simplify the notation we view points of $N$ as lying in $M^n \cap M_1^n$. We define the canonical metric on $\bar{M}^n$ by $\rho(x,y) = \begin{cases} |xy|, & x,y \in M^n \text{ or } x,y \in M_1^n \\ \min_{z \in N} |xz| + |yz|, & x \in M^n, y \in M_1^n \end{cases}$. This is obviously an intrinsic metric.

5.2. The Doubling theorem. The doubling $\bar{M}^n$ of $M^n$ is a complete Alexandrov's space (with the same lower bound of curvatures) with empty boundary.

Proof. We proceed by induction on $n$, the case $n=1$ being trivial. Observe that a shortest line in $M^n$ can touch the boundary $N$ by its endpoints only (unless it lies on $N$). This is a corollary of 4.6 since the tangent cone varies continuously (in Gromov-Hausdorff topology) when its base point moves within a shortest line (see [I,7.15]). Therefore, a simple reflection argument shows that a shortest line in $\bar{M}^n$ can go through the common boundary of $M^n$ and $M_1^n$ only once.

Let $pA, pA_1$ be two shortest lines in $\bar{M}^n$, $p \in N$. For local consideration near $p$ we may assume that each of them lies in $M^n$ or $M_1^n$ and has a direction $A'$ $(A_1')$ in $\Sigma_p$ or $\Sigma_{1p}$. We are going to prove that

$$(1) \qquad \measuredangle A_p A_1 := \lim_{\substack{x, x_1 \to p \\ x \in pA,\, x_1 \in pA_1}} \inf \tilde{\measuredangle}\, x p x_1 = |A'A_1'|$$

where the distance is taken in $\bar{\Sigma}_p$ - the doubling of $\Sigma_p$. Clearly, it suffices to check this identify for $A' \in \Sigma_p \setminus \partial\Sigma_p$, $A_1' \in \Sigma_{1p} \setminus \partial\Sigma_{1p}$. Let $x \in pA \subset M^n$, $x_1 \in pA_1 \subset M_1^n$, $y = xx_1 \cap N$. Then $\tilde{\measuredangle}\, x p x_1 \geqslant \tilde{\measuredangle}\, x p y + \tilde{\measuredangle}\, y p x_1 \geqslant \measuredangle\, x p y + \measuredangle\, y p x_1 - 2\nu \geqslant |A'\xi| + |A_1'\xi| - 4\nu$ for some $\xi \in \partial\Sigma_p$, where $\nu > 0$ can be made as small as we like, taking $x, x_1$ sufficiently close to $p$ - this is a consequence of 4.7. Hence $\measuredangle A p A_1 \geq |A'A_1'|$. On the other hand, let $\xi \in \partial\Sigma_p$ and $y \in N$ satisfy $|A'A_1'| + \nu \geq$ $\geq |A'\xi| + |A_1'\xi|$, $|y'\xi| \leq \nu$, $|A'A_1'| < \pi - 4\nu$. Then we can choose $x \in pA$, $x_1 \in pA_1$ in such a way that $\tilde{\measuredangle} x p x_1 \leq$ $\leq \tilde{\measuredangle} x p y + \tilde{\measuredangle} y p x_1 \leq |A'\xi| + |A_1'\xi| + 2\nu \leq |A'A_1'| + 3\nu$, hence $\measuredangle A p A_1 \leq |A'A_1'|$.

It follows from (1) that if $A p A_1$ is a shortest line then $|A'A_1'| = \pi$ and since by inductional assumption $\overline{\Sigma}_p$ is a compete space with curvatures $\geq 1$, we have

(2) $|A'\xi| + |\xi A_1'| \leq \pi$ for any $\xi \in \overline{\Sigma}_p$.

In particular, if $B \in N$, and $B' \in \Sigma_p$, $B_1' \in \Sigma_{1p}$ are directions of symmetric shortest lines $pB$ in $M^n$ and $M_1^n$ respectively, then

(3) $|A'B'| + |A_1'B_1'| \leq \pi$,

since clearly $|A_1'B'| \geq |A_1'B_1'|$.

Now we are going to prove the angle comparison inequality for a triangle $BAA_1$ with $B \in N$, $A \in M^n \setminus N$, $A_1 \in M_1^n \setminus N$ Let $p = AA_1 \cap N$, $A' \in \Sigma_p$, $A_1' \in \Sigma_{1p}$ be the directions of the shortest lines $pA, pA_1$, and $B' \in \Sigma_p$, $B_1' \in \Sigma_{1p}$ be the directions of symmetric shortest lines $pB$. Then $\tilde{\measuredangle} B p A + \tilde{\measuredangle} B p A_1 \leq |A'B'| + |A_1'B_1'| \leq \pi$ (by (3)), hence by Alexandrov's lemma (see [I, the bottom of p.6])

$\tilde{\measuredangle} BAA_1 \leq \tilde{\measuredangle} BAp \leq \measuredangle BAp = \measuredangle BAA_1$, and

$\tilde{\measuredangle} BA_1A \leq \tilde{\measuredangle} BA_1p \leq \measuredangle BA_1p = \measuredangle BA_1A$.

Now it is easy to see that $\liminf\limits_{t \to +0} \dfrac{\tilde{\measuredangle} A_1BA(t) - \tilde{\measuredangle} A_1BA}{t} \geq 0$,

where $A(t) \in AB$, $|AA(t)| = t$. Since this is true for all such triangles, it follows that $\measuredangle ABA_1$ exists and satisfies the angle comparison inequality.

Now we may conclude by (1) that for any $p \in N$ the space of directions of $\overline{M}^n$ at $p$ exists and coincides with $\overline{\Sigma}_p$.

At last, the angle comparison inequality for general triangle $ABC$ (say $A, B \in M^n \setminus N$, $C \in M_1^n \setminus N$) follows from Alexandrov's lemma. Indeed, if $p = AC \cap N$, $A', B', C' \in \overline{\Sigma}_p$ denote the directions of shortest lines $pA, pB, pC$, then $\measuredangle A p B + \measuredangle B p C \leq |A'B'| + |B'C'| \leq \pi$ by (2).

$\bar{M}^n$ has empty boundary since the described above spaces of directions at points of $\bar{M}^n$ have empty boundaries by inductional assumption.

## 6. Convex sets and complete noncompact spaces of nonnegative curvature.

6.1. <u>Theorem</u>. Let $M^n$ be a complete Alexandrov's space with curvatures $\geq 0$ $(\geq k > 0)$ with boundary $N \neq \emptyset$. Then the distance function $f(\cdot) = |N, \cdot|$ is (strictly) convex (that is $f$ becomes (strictly) convex being restricted to any shortest line).

<u>Proof</u>. We consider the case of curvatures $\geq 0$; the case of curvatures $\geq k > 0$ is similar. Let $xy$ be a shortest line, $q$ lie within $xy$. Clearly $f'_{(q)}(x') + f'_{(q)}(y') \leq 0$ where $x', y' \in \Sigma_q$ denote the directions of shortest lines $qx, qy$. Thus it suffices to prove that

$$\lim_{t \to +0} \sup \frac{f(q(t)) - f(q) - t\, f'_{(q)}(x')}{t^2} \leq 0$$

where $q(t) \in xq$, $|q(t)q| = t$. Assume that for a sequence $t_i \to +0$ we have $f(q(t_i)) \geq f(q) + t_i f'_{(q)}(x') + \varepsilon t_i^2$, $\varepsilon > 0$. Clearly $q \notin N$ (see the beginning of the proof of 5.2). Let $p \in N$ be the closest to $q$ point of $N$, $q' \subset \Sigma_p$ be the set of directions of shortest lines $pq$. Then it follows from 4.7 that $|q' \partial \Sigma_p| \geq \pi/2$. If $q'_1$ is the image of $q'$ under reflection w.r.t. $\partial \Sigma_p$ in $\overline{\Sigma}_p$, then we have $|q' q'_1| \geq$ $\geq \pi$. Hence $q'$ is a point, $\overline{\Sigma}_p$ is the spherical suspension on $\partial \Sigma_p$, $|q' \xi| = \pi/2$ for any $\xi \in \partial \Sigma_p$. Let $q'_i \in \Sigma_p$ denote the direction of a shortest line

$pq(t_i)$, $\xi_i \in \partial\Sigma_p$ be the projection of $q_i'$ onto $\partial\Sigma_p$ (that is $q_i' \in q'\xi_i$ ), $\xi \in \partial\Sigma_p$ be a limit point of $\xi_i$, $\alpha_i = \measuredangle q(t_i)pq = |q_i' q'|$, $\beta_i = \measuredangle q(t_i)pq$; $p_i \in N$ satisfy $|pp_i| = |q(t_i)p| \sin\beta_i$, $p_i' \xrightarrow[i\to\infty]{} \xi$ (see 4.7). We estimate $\cos|p_i'q_i'| \geq \cos|\xi_i q_i'| \cos|p_i'\xi_i|$, hence $|p_i'q_i'| \leq \pi/2 - \alpha_i + o(\alpha_i)$ and $\measuredangle p_i p q(t_i) \leq \measuredangle p_i p q(t_i) = |p_i'q_i'| \leq \pi/2 - \alpha_i + o(\alpha_i) \leq \pi/2 - \beta_i + o(t_i)$. Now considering the quadrangle made up from triangles on the plane $\tilde{p}_i \tilde{p} \tilde{q}(t_i)$ and $\tilde{q}(t_i)\tilde{p}\tilde{q}$ we conclude that

$$f(q(t_i)) \leq |p_i q(t_i)| \leq |pq| - t_i \cos \measuredangle q(t_i) q p + o(t_i^2) \leq f(q) + t_i f'_{(q)}(x') + o(t_i^2)$$

- a contradiction. ∎

6.2. Let $M^n$ be a compact Alexandrov's space with curvatures $\geq 0$ with boundary $N \neq \emptyset$. Then the distance function $f(\cdot) = |N, \cdot|$ has a maximal value $a > 0$. It follows from 6.1 that $S_1 = f^{-1}(a)$ is a convex subset of $M^n$, and clearly $\dim S_1 < n$. $S_1$ itself can be considered as compact nonnegatively curved Alexandrov's space. If $S_1$ has nonempty boundary then we can repeat the operation and obtain a convex subset $S_2 \subset S_1$ with $\dim S_2 < \dim S_1$. After finite number of steps we get a convex subset $S$ without boundary, that can be called a soul of $M^n$. Clearly $f$ is noncritical in $f^{-1}(\varepsilon, a-\varepsilon)$ for any $\varepsilon > 0$, hence by the Stability theorem 0.3, 4.7, 1.4.1 $(M^n, N) \approx (f^{-1}[\varepsilon, a],$ $f^{-1}(\varepsilon)) \approx (f^{-1}[a-\varepsilon, a], f^{-1}(a-\varepsilon))$. We prove in 6.3 that $S_1$ is a deformation retract of $M^n$ and therefore $S$ is a deformation retract of $M^n$.

The same construction can be applied to a complete noncompact nonnegatively curved $M^n$, using the minimum of a suitable combination of Busemann functions instead of $f$ on the first step. In this case $M^n \approx f^{-1}(a-\varepsilon, a]$ and $S$ is a deformation retract of $M^n$.

Let $M^n$ be compact Alexandrov's space with curvatures $\geq k > 0$. Then 6.1 implies that $S_1 = S$ is a point. In this case $(M^n, N) \approx (K(\Sigma_S), \Sigma_S)$. To prove this assertion by reference to 1.4.1 take a function $f_1 = \min\{f, \varphi(|\partial B_S(R), \cdot|) + c_1\}$

where $R, \varphi, c_1$ are chosen in such a way that for some $0 < R_1 < R_2 < R$ $f_1(x) \equiv f(x)$ if $|Sx| \geq R_2$; $f_1(x) = \varphi(R-R_1)+c_1$ on $\Pi_{R_1} = \{x \in B_S(R): |x, \partial B_S(R)| = R-R_1\}$; $\Pi_{R_1} \approx \Sigma_S$ and $f_1$ is noncritical in $f_1^{-1}[0, \varphi(R-R_1)+c_1]$. To make such a choice find $\nu > 0$ such that 4.3 is applicable to the 1-framed subset $\partial B_S(1) \subset K_S(\Sigma_S)$, considered as a level set $\{x \in B_S(2): |x\, \partial B_S(2)| = 1\}$ Now take $R > 0$ so small that for any $x \in B_S(R)$ there exists $y \in \partial B_S(R)$ such that $\measuredangle\, xS_y < \nu$. It follows that for $R_1 > 0$ sufficiently small the level set $\Pi_{2R_1} = \{x \in B_S(R): |x\, \partial B_S(R)| = R - 2R_1\}$ is $\nu R_1/2$-close to $\partial B_S(2R_1)$ and there exists a $\nu/2$-approximation $\theta: B_S(3) \cap K_S(\Sigma_S) \to B_S(3) \cap R_1^{-1} \cdot M^n$. Hence the level set $\Pi_{R_1}' = \{x \in B_S(R): |x\, \partial B_S(R)| = R - R_1\} = \{x \in B_S(2R_1): |x \Pi_{2R_1}| = R_1\}$ is homeomorphic to $\Sigma_S$. To check the noncriticality of $f_1$ at $x \in f^{-1}(0, \varphi(R-R_1)+c_1)$ take $W(x)$ near $x$ on the shortest line $xS$. Other conditions are easy to satisfy provided $R_1$ is small enough.

6.3. The Sharafutdinov's retraction.

Let $M^n$ be a compact Alexandrov's space with curvatures $\geq 0$, with boundary $N \neq \emptyset$, $f(\cdot) = |N, \cdot|$, $f(M^n) = [0, a]$, $S_1 = f^{-1}(a)$. Let $x \in M^n \setminus S_1$, $M_x = f^{-1}[f(x), a]$. By 6.1 $M_x$ is a compact nonnegatively curved Alexandrov's space with boundary $N_x = f^{-1}(f(x))$. The space of directions $\Sigma_x$ of $M_x$ at $x$ is a compact Alexandrov's space with curvatures $\geq 1$, with nonempty boundary, hence it contains the soul $\xi_x$.

<u>Assertion 1</u>. $|\xi_x \xi| \leq \pi/2$ for any $\xi \in \Sigma_x$.

<u>Proof</u>. It follows from 5.2 that $|\xi_x \partial \Sigma_p| \leq \pi/2$. Let $\eta \in \partial \Sigma_p$ be (one of) the closest to $\xi_x$ point of $\partial \Sigma_p$. Then $\Sigma_\eta$ is a half of the spherical suspension on $\partial \Sigma_\eta$ with apex $\xi_x' \in \Sigma_\eta$ (see a similar argument in 6.1). Hence for any $\xi \in \Sigma_x$ we have $\measuredangle\, \xi \eta \xi_x \leq \pi/2$. On the other hand $\measuredangle\, \xi \xi_x \eta \leq \pi/2$ for at least one such $\eta$ since $\xi_x$ is the soul. Now the assertion follows from the angle comparison inequality. ∎

**Assertion 2.** $f'_{(x)}(\xi_x) = \sin|\xi_x \partial\Sigma_x| \geq c\,(Vr_{n-1}(\Sigma_x))^2$.

**Proof.** Let $|\xi_x \partial\Sigma_x| = \varepsilon$. Let $\eta' \subset \Sigma_{\xi_x}$ denote the set of directions of shortest lines $\xi_x\eta$, such that $\eta \in \partial\Sigma_x$, $|\xi_x\eta| = \varepsilon$; $\Sigma_1 = \{\zeta \in \Sigma_{\xi_x} : |\zeta\eta'| < \pi/2 - \sqrt{\varepsilon}\}$, $\Sigma_2 = \{\zeta \in \Sigma_{\xi_x} : \frac{\pi}{2} - \sqrt{\varepsilon} \leq |\zeta\eta'| \leq \frac{\pi}{2}\}$. Clearly $\Sigma_{\xi_x} = \Sigma_1 \cup \Sigma_2$. The angle comparison inequality implies $|\zeta\xi_x| \leq c\sqrt{\varepsilon}$ provided the direction of the shortest line $\xi_x\zeta$ lies in $\Sigma_1$. We apply volume estimates 25.2, [I, 9.2, 9.3] and get $Vr_{n-2}(\Sigma_2) < c\sqrt{\varepsilon}$ and $Vr_{n-1}(\Sigma_x) < c\sqrt{\varepsilon}$. ■

Fix $\nu > 0$ and consider paths $x_0 x_1 \dots x_m$ made up from shortest lines $x_i x_{i+1}$ of two types. Segments of the first type must satisfy $|\xi_{x_i} x'_{i+1}| \leq \nu$ in $\Sigma_{x_i}$, $|x_i x_{i+1}| \leq \nu^2 a$, $f(x_{i+1}) \geq f(x_i) + \frac{1}{2} f'_{(x_i)}(\xi_{x_i}) |x_i x_{i+1}|$, segments of the second type must satisfy $f(x_{i+1}) > f(x_i)$ and the sum of their lengths must be $\leq \nu\,(f(x_m) - f(x_0))$.

. It is easy to see that starting from arbitrary point $x_0 \in M^n$ one can construct such a path with $f(x_m) > a - \nu$. (Otherwise assume $\sup f(x_m) = b \leq a - \nu$ and come to a contradiction).

**Assertion 3.** Let $x_0, x_1 \dots x_m$ and $y_0, y_1 \dots y_\ell$ be paths as above, $z \in M^n$, $f(z) \geq f(x_{m_1})$. Then

$$|zx_i| < |zx_j| + 10\,\nu\,\varepsilon^{-1} a \qquad \text{for} \quad 0 \leq j \leq i \leq m_1 + 1$$

$$|x_i y_j| < |x_0 y_0| + 2\varepsilon^{-1} |f(x_i) - f(y_j)| + 10\, a\nu\varepsilon^{-1} \quad \text{for} \quad 0 \leq i \leq m,\ 0 \leq j \leq \ell,$$

where $\varepsilon = \inf f'_{(x)}(\xi_x)$ for $x \in M^n$: $f(x) \leq f(x_i), f(y_j)$.
(Assertion 2 implies that $\varepsilon > 0$ provided $f(x_i), f(y_j) < a$)

**Proof.** For the segments $x_\alpha x_{\alpha+1}$ of the first type we have $|z x_{\alpha+1}| \leq |z x_\alpha| + 2\nu\,|x_\alpha x_{\alpha+1}| \leq |z x_\alpha| + 4\nu\varepsilon^{-1}(f(x_{\alpha+1}) - f(x_\alpha))$ provided $|z x_\alpha| \geq \nu a$ since $\angle z x_\alpha x_{\alpha+1} \leq \pi/2 + \nu$. For segments of the second type we have $|z x_{\alpha+1}| \leq |z x_\alpha| + |x_\alpha x_{\alpha+1}|$. Summing up we get $|z x_i| < |z x_j| + 10\, a\nu\varepsilon^{-1}$. The proof of the second inequality is similar.

Assertion 3 implies that paths with fixed starting point x converge (as $\nu \to 0$) to a continuous curve $\gamma_x(t)$, $f(x) \le t \le a$, such that $f(\gamma_x(t)) = t$. Moreover, $|xy| \geq |\gamma_x(t)\gamma_y(t)|$ provided $\max\{f(x), f(y)\} \le t \le a$, and $|\gamma_x(t)y| \le |xy|$ provided $f(x) \le t \le f(y) \le a$. Therefore we may define a deformation $\bar{\gamma}(x,t) = \begin{cases} x, & 0 \le t \le f(x) \\ \gamma_x(t), & f(x) \le t \le a \end{cases}$, that satisfies

$$|\bar{\gamma}(x,t_1)\bar{\gamma}(y,t_1)| \geq |\bar{\gamma}(x,t_2)\bar{\gamma}(y,t_2)| \quad \text{for} \quad 0 \le t_1 \le t_2 \le a; \quad \bar{\gamma}(x,a) \in S_1.$$

6.4. In contrast with the case of Riemannian manifolds, there exists a nonnegatively curved complete noncompact Alexandrov's space which is not homeomorphic to a (locally trivial) bundle over its soul. For example, consider the natural orthogonal projection $\pi: K_p(\mathbb{C}P^2) \to K_p(\mathbb{C}P^1)$, $\pi(\tau; z_1, z_2, z_3) = (\tau'; z_1, z_2)$, where $\tau^2(|z_1|^2 + |z_2|^2) = \tau'^2(|z_1|^2 + |z_2|^2 + |z_3|^2)$, and take $M^5 = \pi^{-1}(\bar{B}_p(1))$ (we assume that $\mathbb{C}P^2$ has canonical metric with sectional curvatures between $1$ and $4$). It is easy to see that $M^5$ is a convex subset of $K_p(\mathbb{C}P^2)$, hence it is a complete noncompact nonnegatively curved Alexandrov's space. The doubling $\bar{M}^5$ of $M^5$ has the doubling $S$ of $\bar{B}_p(1) \subset K_p(\mathbb{C}P^1)$ as its soul. But $\bar{M}^5$ can not be homeomorphic to a fiber bundle over $S$ since $S$ is homeomorphic to the $3$-sphere, and $\bar{M}^5$ has two singular points.

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