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Extremal questions for surfaces of a given topological type, II.

In work(1) (p 146, th 1) was obtained a bound

$$r > R\left(\frac{\sqrt{6}}{2} - 1\right) \qquad (1)$$

for the radius of the greatest ~~ball~~ enclosed in an arbitrary 2-dim'l surface in E^3 homeomorphic to a sphere, such that the princ. rad. of curv. $\geq R$. In the ~~foregoing~~ present paper we construct examples proving the sharpness of bound (1).

Let Φ be a smooth 2-sided surface without self-#, $\bar{n}_\Phi(M)$ the continuous unit normal field. Denote by ρ_Φ the internal Riem. distance. $\Phi \in \Phi_R$ if for every convergent sequence $M_n \to M_0$ on Φ

$$\overline{\lim} \frac{|\bar{n}_\Phi(M_n) - \bar{n}_\Phi(M_0)|}{\rho_\Phi(M_n, M_0)} \leq \frac{1}{R}. \qquad (2)$$

The equivalence of the latter relation with formula (1:1,15) of work(2) leads to the inclusion $F_R^2 \subset \Phi_R$, where F_R^2 is the class of closed 2-dim'l C^2 surfaces, having all principal rad. of curv. $\geq R$. It is easy to prove that many properties of surfaces of class Φ_R coincide with corresponding properties of surfaces $\in F_R^2$. In particular, an analysis of the proofs of lemmas 1:2, 2:1; 2:2 of (2) shows that any surface $\Phi \in \Phi_R$ enjoys the following properties.

a) The connected component $\Phi(M_0, R)$ of the intersection of Φ with the solid cylinder of radius R and center line along $\bar{n}_\Phi(M_0)$ projects bijectively on the ~~plane~~ disk ($T_{M_0}\Phi \cap$ cylinder)

b) $\Phi(M_0, R)$ does not contain interior points of the balls of radius R tangent at M_0.

10.27 c) Normal lines of length $< R$ constructed at any two distinct points of the component $\Phi(M_0, \frac{1}{2}R)$ don't intersect.

It is not difficult to establish also the following:

d) the surface parallel to $\Phi \in \Phi_R$ and at distance $\mu < R$ from Φ is a surface of class $\Phi_{\frac{R^2}{R+\mu}}$.

Surfaces of class Φ_R can be approximated by surfaces of the family $\bigcup_{0 \le \mu \le \mu_0} F^2_{R-\mu}$, $\mu_0 < R$; namely there holds

Lemma 1 Let $\Phi(\nu)$ be the tube about Φ of distance $\le \nu > 0$. Then there is a function $\mu(\nu) > 0$ such that $\mu(\nu) \to 0$ as $\nu \to 0$ and for any $\nu > 0$ there is $F \subset \Phi(\nu)$, $F \in F^2_{R-\mu(\nu)}$, homeo to Φ.

Proof. We take $\nu > 0$. We introduce in E^3 a cartesian coordinate system. Then Φ will correspond to equation $\varphi(x^1, x^2, x^3) = 0$. Evidently, there is $\nu_\Phi > 0$ such that the intersection of Φ with the ~~sphere~~ ball of radius ν_Φ and center at any point of Φ is homeomorphic to a disk.

We set
$$\nu_1 = \min\left(\nu, \tfrac{1}{3}\nu_\Phi, \tfrac{1}{2}R\right). \tag{3}$$

We are given on Φ the vector function $\bar{n}_\Phi(M)$, $M \in \Phi$.

Due to the choice of ν_1 and property c), the distance from any point $N \in \Phi(\nu_1)$ from the surface Φ is realized by a segment NM, $M \in \Phi$, normal to Φ. We redefine the function φ in the set $\Phi(\nu_2)$, setting

$$\varphi(x^1, x^2, x^3) = \pm \text{ length } N(x^1, x^2, x^3)M, \qquad (4)$$

where the sign is plus (minus) in the case where the direction of the vector $\overrightarrow{MN}$ coincides (opposes) the direction of $\bar{n}_\Phi(M)$. The function (4), as is not hard to show, is C^1 (analogous to the considerations put forth in the work (2) on page 313). Due to the definition of the function (4), we have

$$\operatorname{grad} \varphi(x^1, x^2, x^3) = \bar{n}_\Phi(M). \qquad (5)$$

Designating by $U(N)$ the cube with center $N(x^1, x^2, x^3)$ and edges of length $\frac{1}{4}\nu_1$ parallel to the coordinate axes, we introduce the function

$$f(x^1, x^2, x^3) \equiv f(N) = \frac{1}{\operatorname{mes} U(N)} \int_{U(N)} \varphi(\xi^1, \xi^2, \xi^3)\, d\xi^1 d\xi^2 d\xi^3 \qquad (6)$$

It is easy to see that as N moves along segment $\lambda \bar{n}_\Phi(M_0)$, $M_0 \in \Phi$, $-\frac{1}{2}\nu_1 \leq \lambda \leq \frac{1}{2}\nu_1$ in the direction of the vector
1028 $\bar{n}_\Phi(M_0)$ the function $f(N)$ is monotonically increasing, using on the ends of the segment the meaning of the different signs. Whence it is found that the equation

$$f(x^1, x^2, x^3) = 0 \qquad (7)$$

defines a surface $F' \subset \Phi(\nu_1)$, homeomorphic, due to property c), to the surface Φ. Since the function (4) is C^1, from the relation (6) it is seen that the surface F' is C^2. The surface F' is closed and does not have singularities, therefore the curvature of normal sections assumes its maximum on F'

$$\frac{1}{R-\mu(\nu)} \in (0,\infty).$$

Taking into account (5), (6), we obtain:

$$\begin{aligned} \big||\mathrm{grad}\, f(N)| - 1\big| &= \big|\,|\mathrm{grad}\, f(N)| - |\mathrm{grad}\,\varphi(N)|\,\big| \\ &\le |\mathrm{grad}\, f(N) - \mathrm{grad}\,\varphi(N)| \\ &= \frac{1}{\mathrm{mes}\, U(N)} \sqrt{\sum_{i=1}^{3}\left\{\int_{U(N)} \left[\frac{\partial\varphi}{\partial\xi^i}(\xi^1,\xi^2,\xi^3) - \frac{\partial\varphi(N)}{\partial x^i}\right] d\xi^1 d\xi^2 d\xi^3\right\}^2} = \varepsilon(\nu_1). \end{aligned} \tag{8}$$

Due to the uniform continuity of the $\frac{\partial\varphi}{\partial x^i}$ on $\Phi(\nu_1)$ and the obvious relation $\lim_{\nu_1\to 0} \mathrm{mes}\, U(N) = 0$, for $\varepsilon(\nu_1)$ we have:

$$\lim_{\nu_1\to 0} \varepsilon(\nu_1) = 0. \tag{9}$$

From relations (8) and (9) comes

$$1 - \varepsilon(\nu_1) \le |\mathrm{grad}\, f(N)| \le 1 + \varepsilon(\nu_1). \tag{10}$$

For nearby points $N_1(x_1^1, x_1^2, x_1^3)$, $N_2(x_1^1+\alpha^1, x_1^2+\alpha^2, x_1^3+\alpha^3)$ of the surface F^1 it is not hard to establish the following relation:

$$|\mathrm{grad}\, f(N_1) - \mathrm{grad}\, f(N_2)|$$

$$= \frac{1}{\mathrm{mes}\, U(N_1)} \sqrt{\sum_{i=1}^{3}\left\{\int_{U(N_1)} \left[\frac{\partial\varphi}{\partial x^i}(\xi^1+\alpha^1, \xi^2+\alpha^2, \xi^3+\alpha^3) - \frac{\partial\varphi}{\partial x^i}(\xi^1,\xi^2,\xi^3)\right] d\xi^1 d\xi^2 d\xi^3\right\}^2}$$

$$\leq \frac{\overline{N_1 N_2}}{R}\left(1 - \frac{\nu_1}{R}\right). \qquad (11)$$

From (10) and (11) there easily follows:

$$\left| \frac{\operatorname{grad} f(N_1)}{|\operatorname{grad} f(N_1)|} - \frac{\operatorname{grad} f(N_2)}{|\operatorname{grad} f(N_2)|} \right| \leq \frac{1}{1-\varepsilon(\nu_1)} \cdot \frac{|N_1 N_2|}{R}\left(1 - \frac{\nu_1}{R}\right) + \frac{1+\varepsilon(\nu_1)}{(1-\varepsilon(\nu_1))^2} \cdot 2\varepsilon(\nu_1).$$

With the aid of (3), (9) and the last inequality we are convinced of the fact that $\mu(\nu) \to 0$ as $\nu \to 0$.

1029 The presence of lemma 1 permits proceeding to the proof of the sharpness of bound (1) to ~~for~~ surfaces constructed

$$\tilde{\Phi} \in \Phi_{R-\mu}, \qquad (12)$$

homeomorphic to the sphere, in which it is impossible to enclose spheres of radii exceeding a number

$$R\left(\frac{\sqrt{6}}{2} - 1\right) + \nu_1, \qquad (13)$$

where μ, ν_1 are arbitrarily small positive numbers.

<u>Lemma 2.</u> There exists a C^3 plane convex curve L^* represented by an equation:

$$f^*(x) = \begin{cases} -kx, & -a \leq x < b, \\ \varphi(x), & -b \leq x \leq b, \\ kx & b < x \leq c, \end{cases}$$

where k, a, b, c are arbitrary positive numbers and $\varphi(x)$ is an even function.

The proof of the lemma is easily carried out by using steklov (smoothing?) means on functions, the graphs of which are constructed from pieces of cubic parabolas connected in vertices with rectilinear segments.

Lemma 3. For a piece G of a surface let there hold the following conditions:

1) G is bijectively projected on ~~the~~ a disk $K(r)$ of radius r in a plane P;

2) G consists of C^2 pieces G_i, $i=1,\dots,n$; $n=1,2,\dots$, glued along smooth arcs l_i, $i=1,\dots,n$, issuing from a point M_0, which projects to the center O of $K(r)$;

3) the projections l_i' of l_i on P are included in equal sectors ω_i of $K(r)$, intersecting only at O;

4) the normals at interior points to the piece G_i form angles with P not less than a number $\beta_0 \in (0, \pi/2)$.

Then there exists a positive number ρ such that:

first, $2\rho < \tau$,

second, the sectors $\tilde{\omega}_i \supset \omega_i$ obtained from ω_i by rotating the radii bounding ω_i by angle $\frac{1}{2}\rho$, do not intersect;

third, there exists a piece $\tilde{G}$ of surface, satisfying the following requirements:

α) $\tilde{G}$ is bijectively projected on $K(r)$;

β) on the ring $K(r, r-\rho)$ of $K(r)$ consisting of points of $K(r)$ lying at distance from O, not less than $r-\rho$, the surface G coincides with $\tilde{G}$;

γ) the part of $\tilde{G}$, projecting on the subset of $K(r)$ obtained by cutting out all points of the sectors $\tilde{\omega}_i$ lying at distance from O
1030 not less than $r-2\rho$, is C^2;

δ) the points of the surfaces G, $\tilde{G}$ lying on the same normal to P, are at distance from one another not exceeding a number $\lambda(\rho)$ for which $\lim_{\rho \to 0} \lambda(\rho) = 0$;

ε) the surface $\tilde{G}$ is smooth everywhere except at the points which project to $K(r, r-\rho) \cap \{\bigcup_{i=1}^{n} l_i'\}$;

ζ) The normals at all points of $\tilde{G}$, where they exist, form an angle with P not less than β_0.

<u>Proof.</u> The validity of the first two assertions of the lemma are obvious. We prove the validity of the third assertion. We introduce a cartesian coordinate system $XYZO$, the origin of which we put at the center O of the disk $K(r)$, and the axis OZ directed perpendicular to P. Then the surface G will have some equation:

$$z = g(x, y), \qquad x^2 + y^2 \le r^2. \tag{14}$$

Using, for example, an arc of the type L^* (cf. lemma 2), it is not hard to construct a function

$$h_\delta(x,y) = \begin{cases} \delta, & x^2+y^2 < (r-2\rho)^2, \\ \delta \cdot h^*(x,y), & (r-2\rho)^2 \le x^2+y^2 \le (r-\rho)^2, \\ 0, & x^2+y^2 > (r-\rho)^2, \end{cases}$$

where $0 < \delta < \frac{1}{4}\rho$, and $h^*(x,y) \equiv h^*(M(x,y))$, is, first, C^2; second, its gradient at all points of the ~~any~~ ring $(r-2\rho)^2 < x^2+y^2 < (r-\rho)^2$ is directed to the center O; third, $h^*(x,y) = 1$ for $x^2+y^2 = (r-2\rho)^2$, $h^*(x,y) = 0$ for $x^2+y^2 = (r-\rho)^2$; fourth, all possible derivatives of the first and second orders of $h^*(x,y)$

at points lying on the circles $x^2+y^2=(r-2\rho)^2$, $x^2+y^2=(r-\rho)^2$ equal zero.

We put for any continuous function $f(\xi,\eta)$

$$I(x,y;f)=\frac{1}{4h_\delta^2(x,y)}\int_{x-h_\delta(x,y)}^{x+h_\delta(x,y)}\int_{y-h_\delta(x,y)}^{y+h_\delta(x,y)} f(\xi,\eta)\,d\xi\,d\eta .$$

Then the function

$$\tilde{g}(x,y)=\begin{cases} I(x,y;\, I(\xi,\eta;g)), & x^2+y^2<(r-\rho)^2, \\ g(x,y), & x^2+y^2\geqq(r-\rho)^2 \end{cases}$$

as it is not hard to show, corresponds to a piece of surface $\tilde{G}$. Actually, the validity of points α), β), γ), ε) comes from the definition of $\tilde{g}(x,y)$ (it is especially easy to be convinced of this, making
1031 the following ~~change~~ of variables in the expression for $\tilde{g}(x,y)$:

$$\xi = x + h_\delta(x,y)\,\xi^*,$$
$$\eta = y + h_\delta(x,y)\,\eta^*).$$

We turn to points δ), ζ). Proceeding with the function $g(x,y)\equiv g(N(x,y))$, due to conditions 4), it satisfies the relation

$$\frac{|g(N_1)-g(N_2)|}{\text{length } N_1N_2}\leq C_g,\ 0<C_g<\infty, \tag{15}$$

where $N_i\in K(r)$ and C_g is ~~a~~ constant for fixed function $g(x,y)$. It is easy to note that averaging of the type as is used twice in obtaining $\tilde{g}(x,y)$ from $g(x,y)$, does not disturb condition (15).

The validity of points 8), 9) is found from the relation of the form (15) for the function $\tilde{g}(x,y)$. We note that properties a), b), c), d) and also lemmas 1, 3 are generalized literally to the n-dimensional case.

We proceed to the construction of the surface (12).

We take vertices A, B, C, D of an equilateral tetrahedron and its center K (cf fig. 1). The planes, containing triples of points A, K, D; B, K, C we denote, resp., by P, Q.

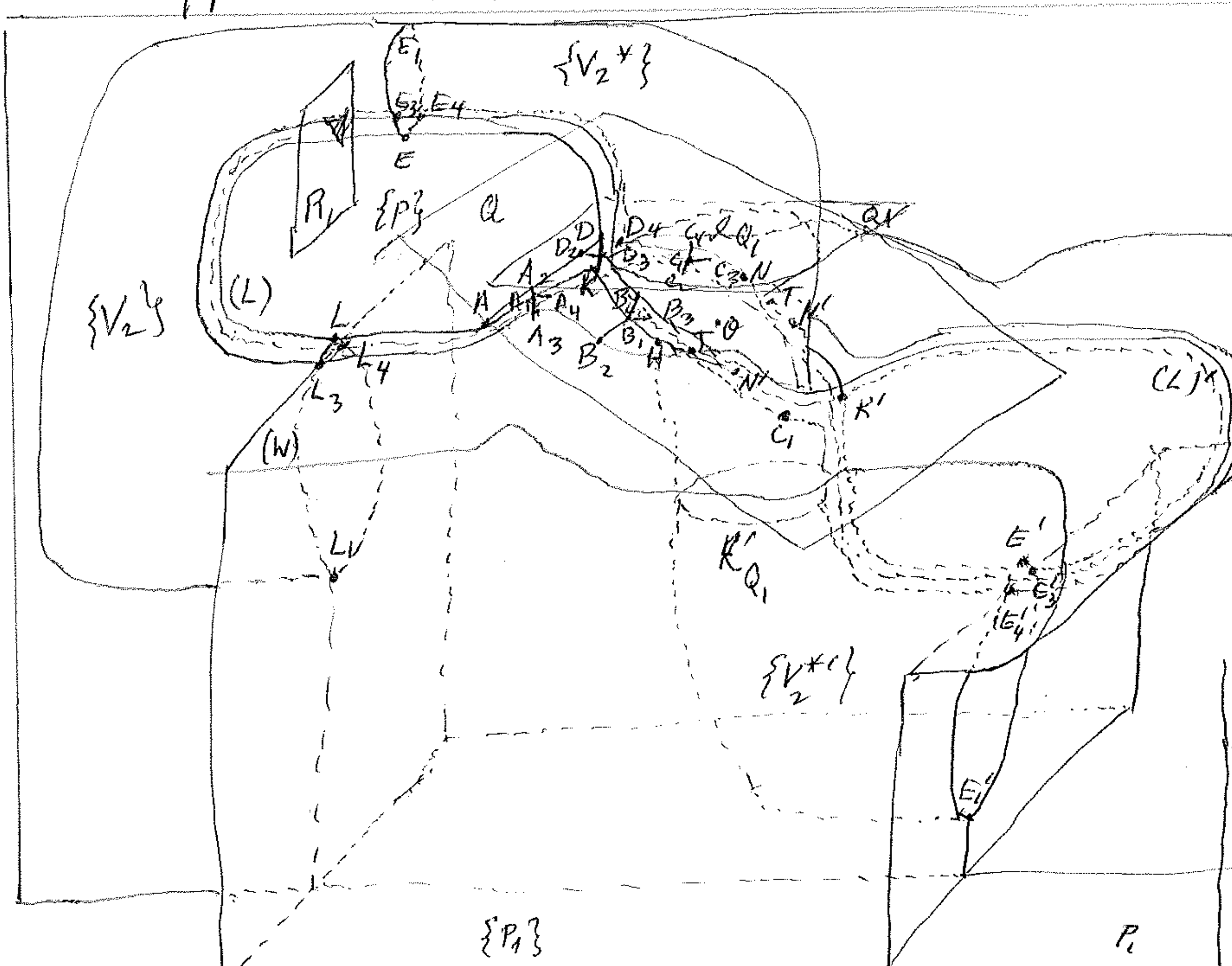


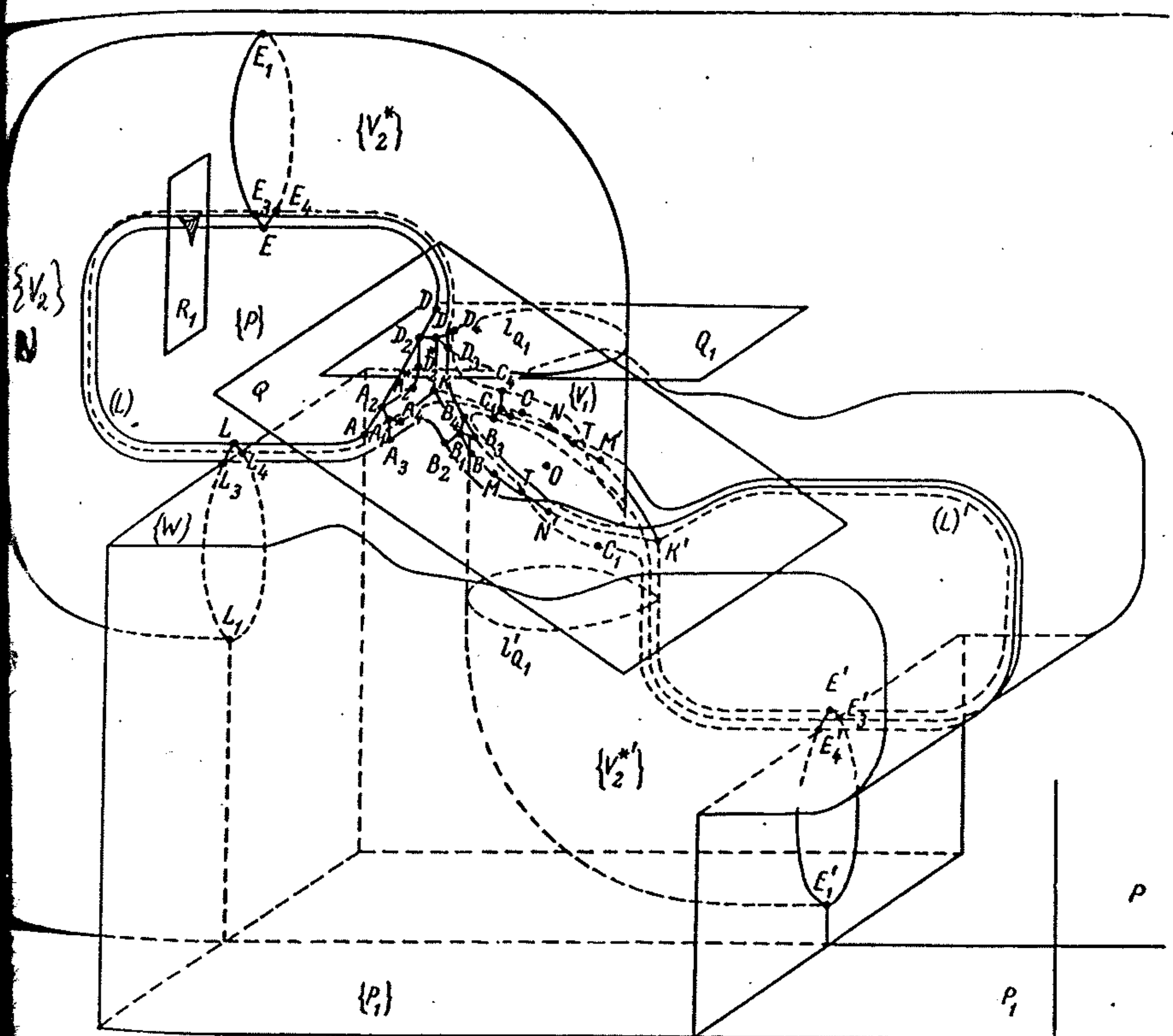
Fig. 1   See enlarged version.

...тки $X_i \in K(r)$, а C_g — константа для фиксированной функции $g(x, y)$. ...заметить, что усреднение такого типа, какое дважды применено при ...нии функции $\tilde{g}(x, y)$ из $g(x, y)$, не нарушает условия (15).

Справедливость пунктов δ), ζ) вытекает из соотношения вида (15) функции $\tilde{g}(x, y)$. Заметим, что свойства a), b), c), d), а также лемма 3 дословно обобщаются на n-мерный случай.

...ступим к построению поверхности (12).

...ьмем вершины A, B, C, D, правильного тетраэдра и его центр K (черт. 1). Плоскости, содержащие тройки точек A, K, D; B, K, C обо-



Черт. 1

1032 Using arcs of type L^*, considered in lemma 2, we complete (supplement) the broken line AKD by a closed C^3, except at K, arc $(L) \subset P$. We supplement the segment KB by an arc $KBMT \subset Q$ of type L^* such that the rectilinear piece MT of arc KBMT should be parallel to the bisector of the angle CKB. We construct arc $KCNT'$ symmetric to arc KBMT relative to the plane P. The length of segment MT we will assume sufficiently great. We denote the midpoint of segment TT' by O. We construct a complex $\{K\}$ from all possible triangles, the common vertex of which is K, and the other vertices are taken from the four points A, B, C, D. We lay out on segments KA, KB, KC, KD, resp., segments KA_1, KB_1, KC_1, KD_1 of the same lengths, less than the length of KA, but arbitrarily near to it. We cut off from the complex $\{K\}$ pieces by the planes perpendicular to the segments KA, KB, KC, KD and passing through A_1, B_1, C_1, D_1, those pieces containing A, B, C, D, and denote the complex obtained by $\{K_1\}$. The intersection of $\{K\}$ with the above-mentioned planes are made up of triples of segments, which we denote by A_1A_2, A_1A_3, A_1A_4; B_1B_2, B_1B_3, B_1B_4 and so forth. In the triangle AKD through A_2, D_2 we draw an arc $A_2A_2^*D_2^*D_2$ of type L^*, the rectilinear parts $A_2A_2^*$, $D_2^*D_2$ of which have the same length and are parallel respectively to segments AK, KD. We discard from the boundary $A_1A_2D_2D_1K$ of the complex $\{K_1\}$ the figure bounded by segment A_2D_2 and arc $A_2A_2^*D_2^*D_2$. We carry out the analogous operation on the other boundaries of $\{K_1\}$. We denote the resulting complex $\{K_2\}$. We will assume that the triple of segments D_1D_2, D_1D_3, D_1D_4 are rigidly fastened to D_1, i.e., all segments of the triple lie in the same plane and the angles between them are identical. We shall move the aforementioned triple with adherence to the following: first, the point D_1 should trace out an arc $D_1 D A_1 \subset (L)$; second, the plane containing the triple of segments at all times remains perpendicular to the arc (L); third, the segment D_1D_2 at all times lies in the plane P. During the tracing out, the motion of the segments D_1D_2, D_1D_3, D_1D_4 sweeps out, respectively, strips

$$\Phi(D_1D_2), \quad \Phi(D_1D_3), \Phi(D_1D_4), \tag{16}$$

being, as is easy to establish, C^2 surfaces, and the ends D_2, D_3, D_4 sweep out C^2 arcs

$$l(D_2), \; l(D_3), \; l(D_4)$$

the tangents of which are always parallel to the plane P. It is easily seen, that, for example, the strip $\Phi(D_1D_3)$ together with the boundary of the complex $\{K_2\}$ containing the corresponding points B_2, B_4, forms a piece of surface everywhere C^2 except at the points lying on the segment KB_1. We stretch on the contour (L)

a piece $\{P\}$ of the plane P. The complex consisting of the strips (16), the complex $\{K_2\}$, the figure $\{P\}$ and arcs $KBMT$, $KCNT'$, we denote by $\{K_3\}$.
1033 We construct complex $\{K_3\}'$, the symmetric complex to $\{K_3\}$ relative to the point O. We agree to denote pairs of elements of our construction symmetric relative to O by the same letter or symbol, attaching a prime to one of these letters or symbols. We consider a triple of segments B_1B_2, B_1B_3, B_1B_4, analogously to the previously considered triple D_1D_2, D_1D_3, D_1D_4. We will move the point B_1 of this triple along the arc $B_1MTN'C_1'$ in conformity with the following conditions: first, the plane in which the triple lies should at all times be perpendicular to the arc $B_1MTN'C_1'$; second, as B_1 moves from its initial position up to point M (i.e., on the curvilinear part of the arc $\cup B_1MTN'C_1'$) the segment B_1B_2 should at all times remain in the plane Q; as B_1 moves on the rectilinear segment MN' the triple of segments monotonely is turned by angle $\frac{2}{3}\pi$ so that when B_1 coincides with N' the segment B_1B_4 should end up on the plane Q; as B_1 is further moved up to C_1' the segment B_1B_4 should at all times go along the plane Q;

besides this, we will assume that the rate of turning of the triple of segments as B_1 moves on the segment MN' is such that, first, the strips

$$\Phi(B_1B_2),\ \Phi(B_1B_3),\ \Phi(B_1B_4), \qquad (17)$$

swept out, respectively, by segments B_1B_2, B_1B_3, B_1B_4 as B_1 moves on the arc B_1MTNC_1', will be a C^2 surface; second, the rate of turning, as a function of arc length ~~measured from~~ traveled by B_1, will be an even function if the origin of that parameter is taken at T. It is not hard to write down the angle of rotation of the triple of segments as a function of arc length ~~of~~ traveled by B_1, in explicit form in order that for this there should hold all the requirements recounted earlier on the rate of rotation of the triple of segments. We denote by $\{K_4\}$ the complex consisting of the strips (17), the strips symmetric to (17) relative to the plane P, and the complexes $\{K_3\}$, $\{K_3\}'$. From the construction of the ~~plane~~ complex $\{K_4\}$ it is seen that it consists of C^2 strips glued ~~by~~ three to three ~~to a~~ one-dimensional cycles:

$$(L),\ (L)',\ KTK'T'K, \qquad (18)$$

and by four to points K_1, K'. It is also not hard to note that the boundary* points of the complex $\{K_4\}$ form a C^2 arc $\{L_4\}$ homeomorphic to a circle.

* By the boundary points of a complex we mean such points of the complex for which a sufficiently small neighborhood in the complex is homeomorphic to a half-disk (i.e., by this essentially the idea of the boundary of a surface is transfered to a complex).

We draw through D_1 the plane Q_1 perpendicular to the segment KD_1. From each point of the arc $D_3B_4C_4D_4 \subset (L_4)$ we drop the perpendicular segment to Q_1. The totality of all these segments form, as it is easy to show, a piece $\{V_1\}$ of C^2 cylindrical surface, the intersection T_{Q_1} of which with
1034 the plane Q_1 being a C^2 arc. Together with the broken line $D_3D_1D_4$ the curve T_{Q_1} forms a closed C^2, in all points except D_1, arc l_{Q_1}. Assuming that the arc l_{Q_1} is rigidly connected to the triple of segments D_1D_2, D_1D_3, D_1D_4, we will translate the specified triple together with l_{Q_1} as was done for the construction of strips (16), with a single difference: let the point D_1 trace only the arc $D_1EL \subset (L)$,

where the point E is the middle of the upper rectilinear part of the arc (L), and the point L is the middle of the lower rectilinear part of the arc. ~~By~~ In the tracing, the motion of arc l_{Q_1} sweeps out a surface $\{V_2\}$, C^2 in all points, except the points of arc D_1EL. (An analogous construction was carried out in the work (3) on p. 875.) We designate the surface $\{V_2^*\} \subset \{V_2\}$ to be that described by the arc l_{Q_1} as D_1 moves along the curve $D_1E \subset D_1EL$. The arcs, obtained from l_{Q_1} when D_1 coincides with points E, L, we denote, respectively, by $EE_3E_1E_4$, $LL_3L_1L_4$, where E_3, E_4, L_3, L_4 belong to (L_4), and E_1 (L_1) is the highest (lowest) point of the arc $EE_3E_1E_4$ $(LL_3L_1L_4)$. We set

$$\{K_5\} = \{K_4\} \cup \{V_1\} \cup \{V_1\}' \cup \{V_2\} \cup \{V_2^*\}'.$$

From E_1' we extend a segment $E_1'H$ in the direction of the vector $\overline{DK}$. From L_1 we extend a ray in the same direction, which intersects with the horizontal ray issuing from H_1 in some point H_2. We construct the plane P_1 parallel to P, not intersecting the complex $\{K_5\}$ and being obtained from P by displacing the latter in the direction of the vector $\overline{OT}$. We consider the closed arc

$$H_1 E_1' E_4' A_3 L_3 L_1 H_2 H_1, \tag{19}$$

a partial arc of which, $E_4' A_3 L_3$, belongs to (L_4). From each point of the arc (19) we drop a segment perpendicular to the plane P_1. The set of all such segments form a piece of cylindrical surface $\{W\}$, C^2 everywhere except at the points of the segments extending through L_3, H_2, H_1, E_4'. The piece of the plane P_1 bounded by the projection of the arc (19) on P_1 we denote by $\{P_1\}$.

Let $\{\tilde{W}\}$, $\{\tilde{P}_1\}$ be the surfaces symmetrically related to the plane P to the surfaces $\{W\}$ and $\{P_1\}$ respectively. We introduce the notation

$$\{K_6\} = \{K_5\} \cup \{W\} \cup \{\tilde{W}\} \cup \{P_1\} \cup \{\tilde{P}_1\},$$

$$(L_6) = (L) \cup (L') \cup \smile KTK'T'K$$

(cf. (18)), and let $\{K_6(\rho < r)\}$ $(\{K_6(\rho \geq r)\})$ be the subset of points of the complex $\{K_6\}$ at distance from (L_6) (in the sense of the metric of E^3) less than (not less than) a positive number r, satisfying the inequality

$$r < \frac{1}{2} \text{ length } B_1B_2 . \tag{20}$$

1035 The subcomplex $\{K_6(\rho \geq r)\}$ is a surface without self-intersections, C^2 everywhere except at the points of a finite number of smooth arcs. It is easily seen that applying to $\{K_6(\rho \geq r)\}$ a finite number of times the smoothing described in lemma 3, it can be transformed to a ~~sub~~complex $\{K_6^*(\rho \geq r)\}$ everywhere C^2, coinciding with $\{K_6(\rho \geq r)\}$ on the set $\{K_6(\rho \geq r)\} \cap \{K_6(\rho < \frac{3}{2}r)\}$, and such that the subcomplex $\{K_6^*(\rho \geq \frac{3}{2}r)\} \subset \{K_6^*(\rho \geq r)\}$ is displaced from the complex $\{K_6(\rho < r)\}$ by a positive distance. We change from the complex $\{K_6\}$ to the complex

$$\{K_7\} = \{K_6(\rho < r)\} \cup \{K_6^*(\rho \geq r)\},$$

homeomorphic to the complex $\{K_6\}$ by construction and coinciding with it on the subcomplex $\{K_6(\rho < r)\}$. On the subcomplex $\{K_7(\rho > \frac{1}{2}r)\}$ the maximum curvature of a normal section is uniquely defined at each point and is represented by a continuous function; the maximum of this function on the subcomplex $\{K_7(\rho \geq r)\}$ we denote by k_1. It is easy to see that on the complex $\{K_7(\rho \leq r)\}$, thanks to its special construction, the maximal curvature of normal sections defined at points of (L_6) as the upper limit as these points are approached in any way on the complex $\{K_7(\rho \leq r)\}$, also assumes its largest value k_2. Let

$$R = \min\left(\frac{1}{2\max(k_1,k_2)}, \frac{1}{2}r\right\}, \qquad (21)$$

where the number r is taken from (20). We denote by $W(R)$, where R is defined by the relation (21), the set-theoretic sum of all ~~spheres~~ balls (considered as sets of points), each of which is tangent to the complex $\{K_7(\rho<r)\}$ in not less than two points (an analogous construction is contained in the work (4) on p. 55). It is easy to note that the ~~supplementary~~ complement in space for the set $W(R)$ consists of two components, for which the bounded component $E(R)$ contains the cycle (L_6) (in figure 1 there is depicted the section of the set $E(R)$ by a rectangle R_1 perpendicular to (L_6)).

From the definition of the set $E(R)$ it is found that there are two greatest balls of radius $R(\sqrt{6}/2-1)$ contained in the closure of $\bar{E}(R)$ of the set $E(R)$, for which the centers of these balls are points K, K'. From the definition of the sets $\{\bar{K}_7\}$, $E(R)$, relation (21) and property d) it is found that the set $\tilde{\Phi}$, consisting of all those and only those points of the space E^3 the distance of which from the set $\{K_7\} \cup E(R)$ equals a sufficiently small

positive number ν_1, is a surface of class $\Phi_{R - \frac{R\nu_1}{R+\nu_1}}$, homeomorphic to a sphere, in which it is impossible to enclose a ball of radius exceeding the number (13). The fact that the surface $\tilde{\Phi}$ is homeomorphic to a sphere, can be discovered, equating the surface $\tilde{\Phi}$ (for which the complex $\{K_\eta\}$ is the central set) with the surface depicted in figure 2 and suggesting
1036 in some sense an intermediate position between the surface $\tilde{\Phi}$ and an ordinary sphere. Since the number ν_1 is arbitrarily small, the sharpness of the bound (1) is proved.

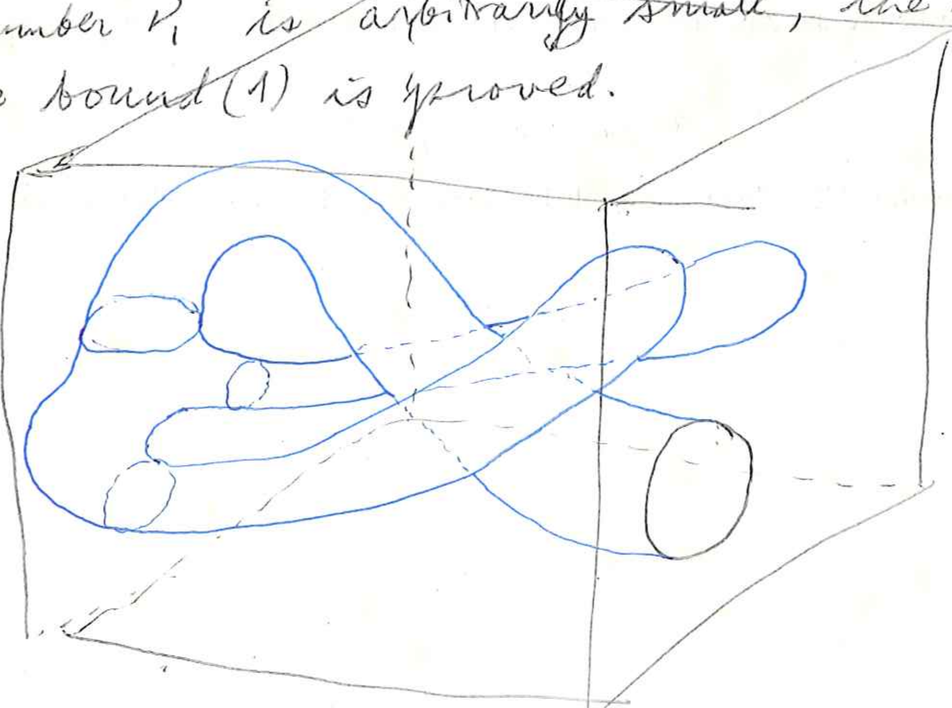


Fig. 2.

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Literature

1. Lagunov V.N. & Fet A. I., Extremal questions for surfaces of a given topological type. I, Siberian Mat. Jour., IV, № 1, (1963), 145-176.

2. Lagunov, V.N., On the greatest ball enclosed in a closed surface, I, Siberian Mat. Jour., I, № 2 (1960), 205-232.

3 ______, ______ II, ibid., II, № 6 (1961), 874-883.

4 Pogorelov A.V., Surfaces of bounded exterior curvature, Khar'kov, 1956.