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On the largest ball included in a closed surface

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In the work (4) was proved the following extremal property (of a not necessarily convex) closed plane curve:

Interior to each C^2 closed plane curve, the radius of curvature of which in each point is not less than R, can be placed a circle of radius R.

Initially this theorem was proposed in the form of a proposition by A. I. Fyet, putting also the question of the possibility of generalizing this result to the case of a closed surface. It was shown, however, that a direct translation of Pestov's theorem is impossible: there exist surfaces F, the principal radii of curvature in each point of which are no less than R, but in which is not contained any ball of radius R. The corresponding examples were constructed by V. I. Diskant and the author.

In the forgoing work is given a complete solution of the problem on the largest ball included in a surface F.

Under consideration is the class F_R of C^2 closed surfaces F in 3-dim'l space, in each point of which the principal radii of curvature are not less than R. Each such surface bounds a solid T(F); we will say that a ball K is included in the surface F if $K \subset T(F)$.

Theorem. In each surface of class F_R can be included a ball of radius $R\left(\frac{2}{\sqrt{3}}-1\right)$; for $\varepsilon > 0$ there exists in class F_R a surface in which it is impossible to include a ball of radius $R\left(\frac{2}{\sqrt{3}}-1\right)+\varepsilon$.

Lemma 1. Let $F \in F_R$; $M_0 \in F$; P a plane drawn through the normal n_0 to F at M_0; C that part of the surface bounded by the circular cylinder with axis n_0 and radius R. Then the connected component of the set $F \cap P \wedge C$ containing M_0 is a smooth nonselfintersecting Jordan curve l. If the Y-axis is taken along n_0 and the X-axis is taken perpendicular to n_0 and lying in P, then the curve l has equation $y = f(x)$, where $f(x)$ is defined and C^1 on the open interval $(-R, R)$. If disks K_1 and K_2 are constructed of radius R, lying in P and tangent on either side of F at the point M_0, then l does not contain interior points of K_1 and K_2.

Proof. For sufficiently small s there exists an arc l_s of the form $y = f(x)$ of length s with origin at M_0 belonging to $F \cap P$. We take, instead of K_1 and K_2 disks K_1' and K_2' of radius $R - \delta < R$. It is easily seen that for sufficiently small s $l_s \cap (K_1' \cup K_2') = M_0$. We extend on the circumference of K_1' from M_0 the arc m_s of the same length s; the ends of the arcs l_s, m_s we denote, respectively, L_s, M_s. We designate by $n_F(s)$ the unit vector normal to F at L_s, by $n_k(s)$ the unit vector normal to K_1' at M_s, by $n_l(s)$ the principal

1168 unit normal to l at M_s. From the formulas of differential geometry ((1), p. 255, (405)) it follows that

$$\left| \frac{d n_F}{d s} \right| < \frac{1}{R - \delta} = \left| \frac{d n_k}{d s} \right|, \qquad (1)$$

whence it is not hard to conclude that for $s < \frac{1}{2}\pi(R - \delta)$ the angle between n_F and n_0 is less than $\frac{1}{2}\pi$. Therefore for $s < \frac{1}{2}\pi(R-\delta)$ the curve l_s exists, is smooth and has form $y = f(x)$.

From (1), moreover, it is found that the angle $(n_l(s), n_0)$ is less than the angle $(n_k(s), n_0)$; therefore the projection of any arc ds of the curve l_s on the x-axis is greater than the projection of the corresponding arc of the circle $\partial K_1'$, and $f(x)$ exists for $0 \le x < R-\delta$. If it should happen that for some s, $0 < s < \frac{\pi}{2}(R-\delta)$, L_s belong to the circumference of K_1', then l_s would be longer than arc $M_0 L_s$ of the circumference of K_1' and, consequently, L_s would separate M_0 and M_s on the circumference of K_1'; therefore the arc m_s would have greater projection on the X-axis than l_s, which contradicts what was proved above. Analogously, it is proved that $l_s \cap K_s' = M_0$. Passing to the limit $\delta \to 0$ proves the lemma.

Lemma 2. <u>There does not exist a sphere S of radius r, where</u>

$$r \le R\left(\csc\frac{\alpha}{2} - 1\right), \quad \frac{\pi}{3} < \alpha < \pi,$$

<u>which is tangent to the surface $F \in F_R$ in two points M_1, M_2 such that the angle between the radii of the sphere S subtended by these points is no greater than α.</u>

Proof. let there be a sphere S enjoying the properties enumerated. We pass through M_1 and M_2 a construction as in lemma 1; let the corresponding cylindrical solids be C_1 and C_2. Then in the plane P extending through the center of S, M_1 and M_2, we obtain curves l_1 and l_2 analogous to the curve l of lemma 1. Elementary geometrical arguments show that under the conditions of the lemma, $l_1 \cap l_2 \neq 0$; but then neither of the curves l_1, l_2 can be the entire component of the set $F \cap P \cap C_1$, respectively, $F \cap P \cap C_2$, contradicting lemma 1.

Proof of the theorem. Let $F \in F_R$. We call the central set Z of surface F the set of those points $Q \in T(F)$ for which there are $k(Q) \geq 2$ distinct nearest points ~~to~~ on F; the number $k(Q)$ we call the multiplicity of Q. We designate the nearest points to Q on F by $M_1(Q), M_2(Q), \ldots, M_{k(Q)}(Q)$; we set $\rho(Q) = \rho(Q, M_i(Q))$, where $\rho(A,B)$ = distance between A and B and $1 \leq i \leq k(Q)$. It is easy to prove that no interior point of the segment $QM_i(Q)$ can belong to Z; therefore, from the connectedness of F it follows in addition that the set $E^3 \setminus Z$ is connected in three-dim'l space E^3.

From lemma 2 it is found that for a point Q of multiplicity ≥ 3 $\rho(Q) > R(\frac{2}{\sqrt{3}} - 1)$; concerning this, there are two segments $QM_i(Q)$, $QM_j(Q)$, forming at Q an angle not greater than $\frac{2\pi}{3}$, and lemma 2 can be applied with $M_i(Q)$, $M_j(Q)$ instead of M_1, M_2 and

$\frac{2\pi}{3}$ in place of α. Thus, if there exists at least one point Q of multiplicity not less than 3, the ~~of~~ ball of radius $R\left(\frac{2}{\sqrt{3}}-1\right)$ with center Q is included in F, and in this case the first assertion of the theorem is proved.

It remains to prove that the multiplicity of all points of Z cannot be equal to 2. Let $k(Q)=2$ for all $Q \in Z$. We put in correspondence to each $M \in F$ the point Q nearest to M on the interior
1169 normal at M belonging to Z. It is not hard to prove that the ~~re is obtained in this way a~~ map $Q=\varphi(M)$ of F to Z obtained thus is continuous and onto Z. Therefore Z is closed. Without loss of generality it can be assumed that for every $Q \in Z$ $\rho(Q) < R$ (in the contrary case there could be a ball of radius R included in F). Then from the classical theory ~~of~~ on normal fields of sufficiently smooth surfaces it is ~~found~~ that the segments of normals M_1Q_1, M_2Q_2, where $Q_1, Q_2 \in Z$ and M_1 and M_2 are sufficiently close, do not have points in common. Therefore for each point $M \in F$ there is a neighborhood $U(M)$ in which the map φ is homeomorphic. Each $Q \in Z$

has just two preimages on F under φ. Whence it is easy to show that φ is a twofold covering map of F onto Z ((2), ch. VIII), and Z is shown to be a two-dim'l manifold. From the law of Alexander duality, as is known, it is found that any compact two-dim'l manifold in E^3 separates E^3 ((3), p. 562, [3:411]). But above it was proved that Z does not separate E^3; the contradiction obtained proves the first assertion of the theorem.

The second assertion of the theorem is proved by the construction of examples of surfaces $F \in F_R$ in which it is impossible to include a ball of radius $R(\frac{2}{\sqrt{3}}-1)+\varepsilon$. Since examples for this are very complicated, we find it necessary to omit their description here.

We note that surfaces constructed for the examples can be of any genus ≥ 2. Therefore for the more restrictive class $F_R^n \subset F_R$ of surfaces of genus n, $n \geq 2$, the bounds of our theorem cannot be improved. For surfaces of genus 0 and 1 the bound can be improved, and for the classes F_R^0, F_R^1 it can be proved that the bound obtained is sharp. However, in this case there are required more delicate topological considerations than described in this note. We intend to turn to this question in another work.

The methods of the present work can be applied without essential change to the ~~[illegible]~~ problem of including an n-dim'l ball in an (n-1)-dim'l surface of n-dim'l Euclidean space; in this case also the sharp bound is obtained.

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Cited literature

[1] P. K. Rashevskii, Course on differential geometry, 1950.

[2] G. Seifert, V. Threlfall, Topology, 1938

[3] P. C. Alexandrov, Combinatorial topology, 1947

[4] G. Pestov, V. Ionin, DAN, 127 № 6, (1959).