V. N. Lagunov   Siberian M. J., Vol 1 (1960), 205-232

On the largest ball included in a closed surface.

One of the most natural local limitations, of separate interest for the geometry of a whole class of surfaces, is the requirement that the principal curvatures in each point of the surface do not exceed a given number. The class of surfaces defined in this manner have been studied by V. Blaschke (2), G. Durand (4) and, in recent times, Yu. G. Reshetnyak (8). However, extremal questions of the type considered in this work, were only posed earlier for convex surfaces. Blaschke ((2), pp 115-118 new edition) proved the following theorem

If a circle is tangent internally at an arbitrary point of a closed convex curve L and the radius of the circle does not exceed the minimal radius of curvature of L, then this circle lies in the region bounded by L.

The largest value of the radius of a ball, for which the ball is tangent to a convex surface at ~~an~~ any of its points and lying in the interior of the domain bounded by the surface, equals the least value of the principal radius of curvature of the surface.

For nonconvex closed curves G. I. Pestov in 1954 proved (7) the following theorem, formulated in the form of a proposition by A. I. Fyet.

In the interior of every twice-cont. differentiable non selfintersecting closed curve in the plane, the radius of curvature of which at each point ~~does not exceed~~ is no less than R, there can be put a circle of radius R.

A. I. Fyet also posed the question of possible generalizations of the theorem of Pestov to n-dimensional surfaces of (n+1)-dim'l Euclidean space. The question can be posed thus: we consider the class F_R^n of twice-continuously differentiable closed nonselfintersecting n-dim'l surfaces F^n in (n+1)-dim'l Euclidean space E^{n+1}, in each point of which the principal radii of curvature are no ~~more~~ less than R; the bounded connected component of the set $E^{n+1} \setminus F^n$ we designate by $T(F^n)$ and we will say that an open (n+1)-dim'l ball K_r^{n+1} of radius r is imbedded in the surface F^n if $K_r^{n+1} \subset T(F^n)$; it is required to obtain the largest radius of a ball which can be imbedded in any surface of the class F_R^n.

The essential difference between our question and the question solved by Blaschke is the extension of the class of surfaces in which the extremum is sought: within that same hypothesis relative to radius of curvature we consider all, not only convex surfaces. In connection with this our methods of proof are completely different from the methods of Blaschke and depend on topological considerations.

It is shown that a direct transformation of the theorem of Pestov on surfaces of class F_R^n for $n \geq 2$ is impossible, that is, there exist surfaces of class F_R^n in which there is not included any ball K_R^{n+1} of radius R. In the continuation such surfaces will be called <u>flattened</u>. Examples of flattened surfaces were constructed by V. I. Diskant and the author in 1958. Then the author proved(5) that a generalization of the theorem of Pestov ~~could~~ be formulated in the form of the following theorem:

<u>In</u> every surface of class F_R^n for $n \geq 2$ there can be imbedded a ball of $K_{r^*}^{n+1}$ of radius $r^* = R\left(\frac{2}{\sqrt{3}} - 1\right)$; for an arbitrary $\varepsilon > 0$, in class F_R^n one can get a surface in which it is impossible to imbed a ball of radius $r^* + \varepsilon$.

In the present work is proved the first part of the theorem formulated — establishing the possibility of imbedding a ball of radius r^*.

In the course of this work, besides the basic theorem, the proof of the following propositions is incidental.

<u>Let</u> $\underset{\sim}{n_0}$ be a vector with origin at point $M_0 \subset F^n \in F_R^n$, normal to F^n, let n_0 be the line belonging to $\underset{\sim}{n_0}$, let E^n be the plane perpendicular to n_0, let C_R^{n+1} be the part of the space E^{n+1} bounded by the circular <u>cylinder</u> of radius R with axis n_0; then the connected component $F_{M_0}^n$ of the set $F^n \cap C_R^{n+1}$, containing the point M_0, is a surface projecting in the direction $\pm \underset{\sim}{n_0}$ bijectively on the ball $E^n \cap C_R^{n+1}$ and this surface $F_{M_0}^n$

is included between the two n-dimensional spheres $S_R^{n'}$, $S_R^{n''}$ of radius R tangent on either side of F^n at the point M_0.

The normals of length $< R$ constructed at any two points of ~~the~~ a geodesic disk of radius $R/2$ of the surface $F^n \in F_R^n$ do not intersect.

The diameter of any surface of class F_R^n is not less than $2R$.

If the diameter of surface $F^n \in F_R^n$ equals $2R$, then F^n is a sphere

In §1 is considered some properties of plane sections of surfaces of class F_R^n needed for the proof of the basic theorem. In §2 is studied properties of cylindrical sections of surfaces of class F_R^n, by result of which is proved (the?) theorem on the extremal property of spheres. In §3 is considered the introduction of our central set of a surface F^n and there is established the connection between the local metric properties of the surface F^n and the topological properties of the central set of the surface F^n; as a result there is obtained a proof of the first assertion of the basic theorem.

In the case $n=2$ it is possible to topologically classify the closed surfaces of class F_R^2. It is found that limited to the class of surfaces F_R^2 of genus $k \geq 1$ it is not possible to improve the bound presented above: it is sharp for each topological class of such surfaces. The author obtained examples showing sharpness of the bound for genus $k \geq 2$. For genus 1 a corresponding example was communicated to the author by V. I. Diskant.

For the class of surfaces F_R^2 of genus zero, and also for surfaces F_R^n of any dimension, homeomorphic to a sphere, the bound on the radius of inscribed ball can be improved. Moreover, the bound can be improved by imposing a requirement of "topological simplicity" on the bodies $T(F_R^n)$ bounded by the surfaces F_R^n. For a series of such cases the sharp bound is also obtained, which will be published in a joint work of A. I. Fyet and the author.

§1. Plane sections of surfaces of class F_R^n.

1.1 We introduce initially some known formulas of $(n+1)$-dimensional geometry in a form needed by us later. Let a surface $F^n \subset E_R^n$ be given in a neighborhood of a point M by equations

$$\underset{\sim}{r} = \underset{\sim}{r}(u^1, \ldots, u^n),$$

where the vector-function $\underset{\sim}{r}(u^1, \ldots, u^n)$ is twice-continuously-differentiable and

$$\frac{\partial r}{\partial u^i} = r_i \ , \ i = 1, \dots, n \qquad (1:1,1)$$

are linearly independent in each point of the surface; let

$$n = n(u^1, \dots, u^n)$$

be the unit vector normal to the surface F^n. Infinitely small displacement from point M on the surface at distance ds corresponds to differentials

$$dr = r_i du^i \ , \ dn = n_i du^i , \qquad (1:1,2)$$

lying in the tangent plane E^n, attached to F^n at the point M (here $r_i = \frac{\partial r}{\partial u^i}$, $n_i = \frac{\partial n}{\partial u^i}$).

We define the linear vector function

$$dn = A(dr)$$

by the condition

$$A(r_i) = n_i \ , \ i = 1, \dots, n \qquad (1:1,3)$$

A is a symmetric vector-function. Concerning this, differentiating the relations

$$n r_i = 0 \ , \ n r_j = 0 ,$$

we get

$$n_j r_i + n r_{ij} = 0 , \ n_i r_j + n r_{ji} = 0$$

where $r_{ij} = \frac{\partial^2 r}{\partial u^i \partial u^j}$, hence

$$n_j r_i = n_i r_j . \qquad (1:1,4)$$

Combining (1:1,3), (1:1,4), we get equations

$$r_i A(r_j) = r_i n_j = n_i r_j = A(r_i) r_j ,$$

from which, in view of the linear independence of the systems (1:1,1) and (1:1,2), is obtained the symmetry of A. But then the linear vector function A has n orthogonal characteristic directions, for which we construct in E^n the unit vectors

$$\underset{\sim}{t}_i, \ i=1,\dots,n; \quad \underset{\sim}{t}_i \underset{\sim}{t}_j = 0 \text{ for } i \neq j.$$

For the vectors $\underset{\sim}{t}_i$ there will hold relations

$$A(\underset{\sim}{t}_i) = \lambda_i \underset{\sim}{t}_i, \quad i=1,\dots,n.$$

The following expression is known for the curvature of a curve:

$$k = \left|\frac{d^2\underset{\sim}{r}}{ds^2}\right| = |\ddot{\underset{\sim}{r}}|, \quad \ddot{\underset{\sim}{r}} = k\underset{\sim}{\nu}, \tag{1:1,5}$$

where $\underset{\sim}{\nu}$ is the principal normal to the curve. From (1:1,5) we have for any twice-continuously-differentiable curve on the surface F^n extending from M:

$$\ddot{\underset{\sim}{r}}\,\underset{\sim}{n} = k\cos(\sphericalangle\,\underset{\sim}{\nu},\underset{\sim}{n}). \tag{1:1,6}$$

If $\underset{\sim}{\nu} = \underset{\sim}{n}$, then from (1:1,6) we obtain the curvature $\tilde{k}_n$ of a normal section:

$$\tilde{k}_n = \ddot{\underset{\sim}{r}}\,\underset{\sim}{n}. \tag{1:1,7}$$

On the other hand, we have:

$$\dot{\underset{\sim}{r}}\,\underset{\sim}{n} = 0, \quad \ddot{\underset{\sim}{r}}\,\underset{\sim}{n} + \dot{\underset{\sim}{r}}\,\dot{\underset{\sim}{n}} = 0, \quad \ddot{\underset{\sim}{r}}\,\underset{\sim}{n} = -\dot{\underset{\sim}{r}}\,\dot{\underset{\sim}{n}},$$

therefore (1:1,6), (1:1,7) can be rewritten as:

$$k = -\frac{\dot{\underset{\sim}{r}}\,\dot{\underset{\sim}{n}}}{\cos(\sphericalangle\,\underset{\sim}{\nu},\underset{\sim}{n})}$$

$$\tilde{k}_n = -\dot{\underset{\sim}{r}}\,\dot{\underset{\sim}{n}}. \tag{1:1,8}$$

We choose a system of coordinates $(u^1, \dots, u^n)$ in order that in the given point

$$\underset{\sim}{r}_i = \underset{\sim}{t}_i, \quad i = 1, \dots, n. \qquad (1:1,9)$$

If $d\underset{\sim}{r} = ds\,\underset{\sim}{t}_i$, then $\dot{\underset{\sim}{r}} = \underset{\sim}{t}_i$ and, accounting for (1:1,8), (1:1,9) we get the curvature k_i of the principal normal section in the direction $\underset{\sim}{t}_i$.

$$k_i = -\dot{\underset{\sim}{t}}_i \underset{\sim}{n}_i, \qquad (1:1,10)$$

$$\dot{\underset{\sim}{n}} = \frac{d\underset{\sim}{n}}{ds} = \underset{\sim}{n}_j \frac{du^j}{ds} = \underset{\sim}{n}_j \cos\alpha^j, \qquad (1:1,11)$$

where $\cos\alpha^j$ is the projection on ~~the vector~~ t_j of the vector $\frac{dr}{ds}$, along which the displacement proceeds. From (1:1,10), (1:1,11) we get:

$$k_i = -\underset{\sim}{t}_i \underset{\sim}{n}_j \cos\alpha^j. \qquad (1:1,12)$$

If $\frac{dr}{ds} = \underset{\sim}{t}_i$, then $\cos\alpha^i = 1$, $\cos\alpha^j = 0$ for $j \neq i$, and in this case (1:1,12) is rewritten in the form:

$$k_i = -\underset{\sim}{t}_i \underset{\sim}{n}_i = -\underset{\sim}{t}_i A(\underset{\sim}{t}_i).$$

Since

$$\underset{\sim}{n}_i = A(\underset{\sim}{r}_i) = A(\underset{\sim}{t}_i) = \lambda_i \underset{\sim}{t}_i \qquad (1:1,13)$$

$$k = -\lambda_i \underset{\sim}{t}_i \underset{\sim}{t}_i = -\lambda_i .$$

Now from (1:1,8) we get Euler's formula for the curvature of any normal section:

$$\tilde{k}_n = -\dot{\underset{\sim}{r}}\dot{\underset{\sim}{n}} = -\underset{\sim}{r}_i \cos\alpha^i \, \underset{\sim}{n}_j \cos\alpha^j = -\underset{\sim}{r}_i \underset{\sim}{n}_j \cos\alpha^i \cos\alpha^j$$

$$= -\lambda_i \underset{\sim}{t_i} \underset{\sim}{t_j} \cos\alpha^i \cos\alpha^j = -k_i \cos^2\alpha^i.$$

Accounting for (1.1,13), we are led to:

$$\dot{\underset{\sim}{n}} = \underset{\sim}{n_i} \cos\alpha^i = -k_i \underset{\sim}{t_i} \cos\alpha^i, \qquad (1.1,14)$$

$$|\dot{\underset{\sim}{n}}| = \sqrt{k_i^2 \cos^2\alpha^i}.$$

On a surface $F^n_R \in F^n_R$ the inequality $k_i \le \frac{1}{R}$ holds, therefore from (1.1,14) we get an inequality important for ~~the~~ our work:

$$\left|\frac{d\underset{\sim}{n}}{ds}\right| \le \frac{1}{R}. \qquad (1.1,15)$$

In case $d\underset{\sim}{r} = ds\, \underset{\sim}{t_i}$ we come to inequality:

$$d\underset{\sim}{n} = A(d\underset{\sim}{r}) = A(ds\, \underset{\sim}{t_i}) = \lambda_i\, ds\, \underset{\sim}{t_i} = \lambda_i\, d\underset{\sim}{r}.$$

Changing in the latter relation, according to (1.1,13), λ_i to $-k_i$, we get Rodrigues's formula:

$$d\underset{\sim}{n} = -k_i\, d\underset{\sim}{r}, \quad i = 1, \ldots, n. \qquad (1.1,16)$$

<u>Lemma 1.2</u> Let $F^n \in F^n_R$, $\underset{\sim}{n_0}$ be a unit vector, normal to F^n at point $M_0 \in F^n$, $\underset{\sim}{t_0}$ an arbitrary unit vector with the origin at M_0, tangent to F^n, E^2 the plane spanned by the vectors $\underset{\sim}{n_0}, \underset{\sim}{t_0}$, (x, y) cartesian coordinates of a point $M \in E^2$ defined by relation

$$\overline{M_0 M} = x\underset{\sim}{t_0} + y\underset{\sim}{n_0}.$$

$P(x_0 \le x \le x_1)$, $P(x_0 \le x < x_1)$, $P(x_0 < x \le x_1)$, $P(x_0 < x < x_1)$ are the sets of points in the plane E^2 defined by the corresponding inequalities.

Then:

(a) in the plane E^2 there exists a simple C^2 arc l of length $\pi \frac{R}{2}$, proceeding from point M_0 in direction $\underset{\sim}{t}_0$, bijectively projected on the half interval $0 \le x < c$ of the x-axis and coinciding with the connected component $D_{M_0}(0 \le x < c)$ of the set $F^n \cap P(0 \le x < v)$, containing M_0;

(b) if in the plane E^2 we construct open disks $K_R^{2'}$, $K_R^{2''}$ of radius R, bounded by circles $S_R^{1'}$, $S_R^{1''}$, tangent on different sides of the x-axis at M_0, then l does not contain points of the disks $K_R^{2'}$, $K_R^{2'}$), i.e. $l \cap (K_R^{2'} \cup K_R^{2''}) = \emptyset$;

c) in the chosen system of coordinates on E^2, l has equations of the form:

$$y = f(x), \quad 0 \le x < c,$$

where $c \ge R$, and $f(x)$ is C^2, defined on the half interval $[0, c)$;

d) if Q is the connected component of the set $P(0 \le x < R) \setminus (K_R^{2'} \cup K_R^{2''})$, containing the point M_0, then the set

$$l(P) = l \cap P(0 \le x < R)$$

coincides with $D_{M_0}(0 \le x < R)$ and belongs to Q.

Proof. (a) With the use of an orthonormal basis $t_0, e_2, \dots, e_n, n_0$ we introduce in a nbhd of M_0 cartesian coordinates $x, x^2, \dots, x^n, y$. In the chosen system of coordinates the surface F^n in a nbhd of point M_0 has equations of the form:

$$y = \Phi(x, x^2, \dots, x^n),$$

where $\Phi(x, x^2, \dots, x^n) \in C^2$ defined in the ball

$$(x)^2 + \sum_{i=2}^{n} (x^i)^2 \leq \varepsilon^2, \quad y = 0,$$

where ε is some positive number. The section of F^n in the plane E^2

$$x^2 = \dots = x^n = 0$$

in a nbhd of M_0 will be a simple C^2 arc $\tilde{l}$:

$$y = \Phi(x, 0, \dots, 0) \equiv f(x), \quad -\varepsilon < x < \varepsilon,$$

where $f(x)$ is defined on the interval $(-\varepsilon, \varepsilon)$; $\tilde{l}$ is bijectively projected on the interval $(-\varepsilon, \varepsilon)$ of the x-axis. Let $K_\delta^{n+1}(M_0)$ be the open ball of radius δ with center M_0. For sufficiently small $\delta > 0$ the set $F^n \cap E^2 \cap K_\delta^{n+1}(M_0)$ is an arc $\tilde{l}_\delta \subset \tilde{l}$, but the set $F^n \cap P(0 \leq x < R) \cap K_{\delta/2}^{n+1}(M_0)$ is an arc $l_{s_1} \subset \tilde{l}_\delta$, where $s_1 > 0$ is the length of l_{s_1}. Arc l_{s_1}, by construction, is simple C^2, going out from point M_0 in the direction t_0; l_{s_1} bijectively projects on some halfinterval $[0, r_1)$ of the x-axis; by construction $r_1 < \varepsilon$. The strip $P(0 \leq x < r_1)$, as is easily seen, is decomposed in a union of nonintersecting terms:

$$P(0 \le x < \gamma_1) = \{P(0 \le x < \gamma_1) \cap K^{n+1}_{\delta/2}(M_0)\}$$
$$\cup \{P(0 \le x < \gamma_1) \cap [K^{n+1}_{\delta}(M_0) \setminus K^{n+1}_{\delta/2}(M_0)]\}$$
$$\cup \{P(0 \le x < \gamma_1) \setminus K^{n+1}_{\delta}(M_0)\}. \qquad (1.2,1)$$

We show that the middle term does not contain points of F^n. Regarding this, the set $F^n \cap P(0 \le x < R) \cap K^{n+1}_{\delta}(M_0)$ is the arc $\tilde{l}'_\delta = \tilde{l}_\delta \cap P(0 \le x < R)$; arc $\tilde{l}'_\delta$ is decomposed in a union of two nonintersecting terms:

$$\tilde{l}'_\delta = [\tilde{l}'_\delta \cap K^{n+1}_{\delta/2}(M_0)] \cup [\tilde{l}'_\delta \cap K^{n+1}_{\delta}(M_0) \setminus K^{n+1}_{\delta/2}(M_0)],$$

for which the first term is the arc $l_{s_1} \subset K^{n+1}_{\delta/2}(M_0)$, but the second term, in view of the bijectivity of the projection of l_δ on the x-axis, is contained in the strip $P(\gamma_1 \le x < R)$.

We show that the set $D_{M_0}(0 \le x < \gamma_1) \setminus l_{s_1}$ is empty. If not, let point $M' \in D_{M_0}(0 \le x < \gamma_1) \setminus l_{s_1}$. By what has been shown, M' cannot belong to either the first or the second term of the union (1.2,1), hence it follows that M' belongs to the third term. The closures of the first and third terms do not intersect, by construction, but then the points M' and M_0 cannot belong to the same connected component of $D_{M_0}(0 \le x < \gamma_1)$. Thus, $D_{M_0}(0 \le x < \gamma_1) \subset l_{s_1}$. The ~~same~~ opposite inclusion is evident, so that

$$l_{s_1} = D_{M_0}(0 \le x < \gamma_1).$$

We showed that for sufficiently small $s_1 > 0$ in the plane E^2 there exists a simple C^2 arc l_{s_1} of length s_1 issuing from M_0 in the direction $\underline{t}_0$, bijectively projecting on some halfinterval $[0, \gamma_1)$ of the x-axis and coinciding with the connected component $D_{M_0}(0 \le x < \gamma_1)$ of the set $F^n \cap P(0 \le x < \gamma_1)$, containing M_0. Let s_0 be the lub of numbers s_1 enjoying the specified properties. We show that

$$s_0 \ge \frac{\pi}{2} R. \qquad (1.2,2)$$

Let, on the contrary,

$$S_0 < \frac{\pi}{2} R \qquad (1.2,3)$$

We take a sequence S_n converging, increasing to s_0 from the left:

$$S_n \to s_0, \quad S_n < S_0. \qquad (1.2,4)$$

The arc l_{s_n} is bijectively projected on the halfinterval $[0, \gamma_n)$ of the x-axis. We show that

$$\gamma_n < \gamma_{n+1} \qquad (1.2,5)$$

Concerning this, if $\gamma_n = \gamma_{n+1}$, then $P(0 \le x < \gamma_n) = P(0 \le x < \gamma_{n+1})$ and then

$$l_{s_n} = D_{M_0}(0 \le x < \gamma_n) = D_{M_0}(0 \le x < \gamma_{n+1}) = l_{s_{n+1}},$$

which is impossible for arcs l_{s_n}, $l_{s_{n+1}}$ of different lengths.

If $\gamma_n > \gamma_{n+1}$, then

$$l_{s_n} = [D_{M_0}(0 \le x < \gamma_n) \cap P(0 \le x < \gamma_{n+1})] \cup [D_{M_0}(0 \le x < \gamma_n) \cap P(\gamma_{n+1} \le x < \gamma_n)],$$

for which the first term is the arc $l_{s_{n+1}}$, and the second term is nonempty, and $l_{s_{n+1}}$ is found to be a part of arc l_{s_n}, which is impossible as $S_n < S_{n+1}$. Thus, inequality (1.2,5) is shown. But then from the relation

$$l_{s_{n+1}} = \left[D_{M_0}(0 \le x < \gamma_{n+1}) \cap P(0 \le x < \gamma_n)\right] \cup \left[D_{M_0}(0 \le x < \gamma_{n+1}) \cap P(\gamma_n \le x < \gamma_{n+1})\right] = l_{s_n} \cup \left[D_{M_0}(0 \le x < \gamma_{n+1}) \cap P(\gamma_n \le x < \gamma_{n+1})\right]$$

comes the fact that l_{s_n} is part of the arc $l_{s_{n+1}}$:

$$l_{s_n} \subset l_{s_{n+1}}. \qquad (1.2,6)$$

Using the criterion of Cauchy for the convergence of sequences it is easily shown that the end L_{s_n} of arc l_{s_n} converges to some point $L_{s_0} \in F^n$. By what has been shown,

$$l_{s_0} = \bigcup_{i=1}^{\infty} l_{s_i} \qquad (1:2,7)$$

is a simple C^2 arc of length s_0, not containing the limit point L_{s_0}, bijectively projecting on the half-interval $[0, \gamma_0)$ of the x-axis, where $\gamma_0 = \lim_{n \to \infty} \gamma_n$.

We take $s < s_0$, $s > 0$ and lay off on l_{s_0} from the point M_0 the arc l_s of length s. On the semicircle $S_R^1 \cap P(0 \le x \le R)$ we lay off from M_0 an arc m_s of the same length s. The ends of the arcs l_s, m_s we designate, respectively, L_s, M_s. Let $n_F(s)$ be the unit vector normal to F^n at point L_s, $n_K(s)$ the unit vector principal normal to S_K^1 at point M_s.* Vectors n_0, $n_F(s)$ we choose in order that $n_0 = n_F(0) = n_K(0)$; for $s > 0$ the direction of $n_F(s)$ is defined by continuity. According to formula (1:1,15), on l_{s_0}

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* The vector $n_K(s)$ is directed to the center of circle S_K^1.

the relation
$$\left|\frac{d\,\underset{\sim}{n}_F}{ds}\right| \le \frac{1}{R} = \left|\frac{d\,\underset{\sim}{n}_K}{ds}\right| \qquad (1{:}2,8)$$
is valid.

We show now that for every $s' < s_0$ there holds the inequality
$$\measuredangle(\underset{\sim}{n}_0, \underset{\sim}{n}_F(s')) \le \measuredangle(\underset{\sim}{n}_0, \underset{\sim}{n}_K(s')) = \frac{s'}{R} < \frac{\pi}{2}. \qquad (1{:}2,9)$$
Concerning this, we translate the origin of vectors $\underset{\sim}{n}_F(s)$, $\underset{\sim}{n}_K(s)$ to the center of the unit n-dimensional sphere S_1^n. Let s monotonically increase from 0 to s'; then the ends of vectors $\underset{\sim}{n}_K(s)$ trace an arc of a great circle of length $\frac{s'}{R}$. The ends of vectors $\underset{\sim}{n}_F(s)$ ~~as this is~~ traces, as is not hard to show, a rectifiable arc. According to (1:2,8) the length of this arc is
$$\int_0^{s'} |d\underset{\sim}{n}_F| \le \frac{s'}{R}. \qquad (1{:}2,10)$$
But $\measuredangle(\underset{\sim}{n}_0, \underset{\sim}{n}_K(s)) = \frac{s'}{R}$, it is evident also that $\measuredangle(\underset{\sim}{n}_0, \underset{\sim}{n}_F(s')) \le \int_0^{s'} |d\underset{\sim}{n}_F|$, where the integral is taken along $l_{s'}$; from this relation and from (1:2,10) follows (1:2,9). For $s=0$ the vector $\underset{\sim}{n}_F(0) = \underset{\sim}{n}_0$ lies in E^2; for $s = s'$ $\measuredangle(\underset{\sim}{n}_0, \underset{\sim}{n}_F(s')) \le \frac{s'}{R} < \frac{s_0}{R}$, therefore at point $L_{s'}$ the angle between $\underset{\sim}{n}_F(s')$ and E^2 is no greater than $\frac{s_0}{R} < \frac{\pi}{2}$. Since s' is an arbitrary number, satisfying the inequality $0 < s' < s_0$, and $\underset{\sim}{n}_F(s)$ depends

continuously on s, the angle between normal $\underset{\sim}{n}_F(s_0)$ ~~and~~ to the surface F^n at point L_{s_0} and E^2 is not greater than $\frac{s_0}{R} < \frac{\pi}{2}$ also.

Therefore ~~Since~~ there exists $\delta > 0$ such that the set $F^n \cap E^2 \cap K_\delta^{n+1}(L_{s_0})$ is a simple C^2 arc l_0. We note that the tangent to l_0 at point L_{s_0} cannot be parallel to $\underset{\sim}{n}_0$, since in the opposite case it would happen that $\measuredangle(\underset{\sim}{n}_0, \underset{\sim}{n}_F(s_0)) = \frac{\pi}{2}$, contrary to the inequality shown above $\measuredangle(\underset{\sim}{n}_0, \underset{\sim}{n}_F(s_0)) \leq \frac{s_0}{R} < \frac{\pi}{2}$. Therefore for sufficiently small δ the tangent to l_0 in no point of the arc l_0 is parallel to $\underset{\sim}{n}_0$. Hence it is obtained that l_0 is bijectively projected on some interval ~~of the~~ $(\gamma_0 - \gamma', \gamma_0 + \gamma'')$ of the x-axis. For sufficiently large n' $L_{s_{n'-1}} \in K_\delta^{n+1}(L_{s_0})$, $L_{s_{n'}} \in K_\delta^{n+1}(L_{s_0})$, inasmuch as $F^n \cap E^2 \cap K_\delta^{n+1}(L_{s_0}) = l_0$, $L_{s_{n'-1}} \in l_0$, $L_{s_{n'}} \in l_0$. The arc l_0 is projected on the x-axis bijectively and, consequently, the point $L_{s_{n'}}$ divides l_0 into two arcs, one of which l_0' is projected on the interval $(\gamma_0 - \gamma', \gamma_{n'})$, and the other l_0'' on the half-interval $(\gamma_{n'}, \gamma_0 + \gamma'')$. [s,2]

Since arc $l_{s_{n'-1}}$ is part of arc $l_{s_{n'}}$, the abscissae $x_{n'-1}$, respectively, $x_{n'}$ of points $L_{s_{n'-1}}$, respectively, $L_{s_{n'}}$ are connected by inequality $x_{n'-1} < x_{n'}$.

Therefore $L_{s_{n'-1}} \in l_0$, $l_0' \cap l_{s_{n-1}} \neq 0$ and, consequently, arc $l_0' \subset D_{M_0}(0 \leq x < \gamma_{n'}) = l_{s_{n'}}$. Hence it follows that l_0' is a part of arc $l_{s_{n'}}$. The intersection $l_{s_{n'}} \cap l_0 = l_0'$ is C^2, from which comes that $l_{s_{n'}} \cap l_0$, evidently, is bijectively projected on the half interval $[0, \gamma_0 + \gamma_0'')$ of the x-axis. The part of the arc of the curve $l_{s_{n'}} \cup l_0$ projecting on segment $[0, \gamma_0 + \frac{\gamma''}{2}]$ we designate l_{s_m}''. Evidently the length of l_{s_m}'' is strictly greater than s_0. We show that

$$D_{M_0}(0 \leq x < \gamma_0 + \frac{\gamma''}{2}) = l_{s_m}''.$$

For each point $M \in l_{s_m}''$ there exists a ~~segment~~ neighborhood $K_\eta^{n+1}(M)$ such that $F^n \cap E^2 \cap K_\eta^{n+1}(M)$ itself represents part of the arc of the curve $l \subset l_{s_{n'}} \cup l_0$. Due to the compactness of l_{s_m}'', there exists a finite union W of neighborhoods $K_\eta^{n+1}(M)$ covering l_{s_m}''; the intersection $W \cap P(0 \leq x < \gamma_0 + \frac{\gamma''}{2})$ is a neighborhood $U(l_{s_m}'')$ relative to $P(0 \leq x \leq \gamma_0 + \frac{\gamma''}{2})$ such that $U(l_{s_m}'') \cap F^n = l_s''$. Therefore $D_{M_0}(0 \leq x < \gamma_0 + \frac{\gamma''}{2})$ cannot contain points not belonging to l_{s_m}'' and, consequently, In this passage a notation of boldface l's used, which has been transcribed $\underset{\sim}{l}$. I suspect there are misprints.

coincides with $\underset{\sim}{l}_s''$. Further, evidently,

$$D_{M_0}\left(0 \le x < \gamma_0 + \frac{\gamma''}{2}\right) \subset D_{M_0}\left(0 \le x \le \gamma_0 + \frac{\gamma''}{2}\right) = \underset{\sim}{l}_s'';$$

? on the other hand, the arc $\underset{\sim}{l}_s''$, obtained from $\underset{\sim}{l}_s''$ by cutting off the right end, belongs to $D_{M_0}\left(0 \le x < \gamma_0 + \frac{\gamma''}{2}\right)$. This means

$$D_{M_0}\left(0 \le x < \gamma_0 + \frac{\gamma''}{2}\right) = l_s''.$$

We have shown that l_s'' is a simple C^2 arc of length $s'' > s_0$, issuing from M_0 in the direction $\underset{\sim}{t}_0$, bijectively being projected on the half interval $[0, \gamma_0 + \frac{\gamma''}{2})$ of the x-axis, coinciding with the connected component $D_{M_0}(0 \le x < \gamma_0 + \frac{\gamma''}{2})$ of the set $F'' \cap P(0 \le x < \gamma_0 + \frac{\gamma''}{2})$, containing M_0; but this contradicts the definition of the number s_0. In this way, inequality (1:2,3) leads to a contradiction, and we have shown inequality (1:2,2). If $s_0 > \frac{\pi}{2}R$, we put $l = l_{\frac{\pi}{2}R}$; if $s_0 = \frac{\pi}{2}R$, then we put in (1:2,4) $s_0 = \frac{\pi}{2}R$. Then formula (1:2,7) defines arc $l_{\frac{\pi}{2}R} = l$.

Evidently, l is a simple C^2 arc, bijectively projecting on some halfinterval $0 \le x < c$. If we show that $l = D_{M_0}(0 \le x < c)$, then the arc l will satisfy all the requirements of point (a)

For each point $L_s \in l$ a ball $K^{n+1}_{\eta(s)}(L_s)$ is obtained such that $K^{n+1}_{\eta(s)}(L_s) \cap F^n \cap E^2$ represents a partial arc of the arc $\tilde{l} \cup l$. It can be assumed besides that $\lim_{s \to \frac{\pi}{2}R} \eta(s) = 0$, and that $K^{n+1}_{\eta(s)}(L_s) \cap P(c \leq x < \infty) = 0$.

There can be chosen a sequence of balls $K_j = K^{n+1}_{\eta_j}(L_{s_j})$ ($\eta_j = \frac{\eta(s_j)}{2}$, $j = 1, 2, \cdots$) such that $l \subset \bigcup_{j=1}^{\infty} K_j$ and each strip $P(0 \leq x < c')$, $c' < c$, is intersected only by a finite number of balls K_i. Then the set

$$V(l) = \left[\bigcup_{j=1}^{\infty} K_j \right] \cap P(0 \leq x < c)$$

is open in the relative topology of $P(0 \leq x < c)$ and $V(l) \cap F^n = l$. Consequently, l is an open subset of the set $F^n \cap P(0 \leq x < c)$ in the relative topology of the latter. Evidently,

$$\overline{V(l)} = \left[\bigcup_{j=1}^{\infty} \overline{K_j} \right] \cap P(0 \leq x < c),$$

(the closure is taken in the relative topology of $P(0 \leq x < c)$. Since, by construction, $K_j \cap F^n \cap E^2$ is a subset of the arc $\tilde{l} \cup l$, $\overline{V(l)} \cap F^n = l$. Consequently, l is a closed subset of the set $F^n \cap P(0 \leq x < c)$ in the relative topology of the latter. Thus, l is the connected component of $F^n \cap P(0 \leq x < c)$, containing M_0, so $l = D_{M_0}(0 \leq x < c)$.

Proposition (a) is completely shown.

(b) We pass an n-dimensional plane E_1^n, normal to l at the point L_s $(s < \frac{\pi}{2}R)$; we designate by q the line of intersection of E_1^n with E^2. It is clear that q and $\underset{\sim}{n}_F(s)$ are orthogonal to l and lie in a threedimensional plane E^3, spanned by E^2 and $\underset{\sim}{n}_F(s)$. We showed in point (a) that for $s < \frac{\pi}{2}R$ the angle between $\underset{\sim}{n}_F(s)$ and E^2 is less than $\frac{\pi}{2}$; this angle equals one of the angles formed by q with $\underset{\sim}{n}_F(s)$ in E^3. Therefore there can be constructed a unit vector $\underset{\sim}{n}_l(s)$, belonging to q and such that

$$\measuredangle\, \underset{\sim}{n}_l(s), \underset{\sim}{n}_F(s) < \frac{\pi}{2}. \qquad (1.2,11)$$

On the other hand, in point (a) it was shown that

$$\measuredangle\, \underset{\sim}{n}_0, \underset{\sim}{n}_F(s) < \frac{\pi}{2}, \qquad (1:2,12)$$

from which it is easily seen that

$$\measuredangle\, \underset{\sim}{n}_0, \underset{\sim}{n}_l(s) \neq \frac{\pi}{2}.$$

Since the latter angle depends continuously on s, and for $s = 0$ reduces to 0,

$$\measuredangle\, \underset{\sim}{n}_0, \underset{\sim}{n}_l(s) < \frac{\pi}{2}. \qquad (1:2,13)$$

We translate vector $\underset{\sim}{n}_0$ to the point L_s and we consider the three vectors $\underset{\sim}{n}_0$, $\underset{\sim}{n}_F(s)$, $\underset{\sim}{n}_l(s)$ in the three-dimensional plane E^3. These vectors are the edges of a threefold angle with vertex at L_s, lying in E^3. According to (1.2,11), (1:2,12), (1:2,13), for the constructed threefold angle the plane angles at the vertex L_s are acute, and the twofold angle between the planes of vectors $\underset{\sim}{n}_0$, $\underset{\sim}{n}_i(s)$, respectively, $\underset{\sim}{n}_\ell(s)$, $\underset{\sim}{n}_F(s)$ is a right angle. With the aid of an elementary construction it is proved that under this conditions

$$\measuredangle \underset{\sim}{n}_0, \underset{\sim}{n}_\ell(s) \leq \measuredangle \underset{\sim}{n}_0, \underset{\sim}{n}_F(s). \qquad (1:2,14)$$

Combining (1:2,9) and (1:2,14) it is shown that for $0 < s < \frac{\pi}{2}R$

$$\measuredangle \underset{\sim}{n}_0, \underset{\sim}{n}_\ell(s) \leq \measuredangle \underset{\sim}{n}_0, \underset{\sim}{n}_K(s). \qquad (1:2,15)$$

We put in correspondence the ends L_s, M_s of arcs l_s ; m_s $(0 < s < \frac{\pi}{2}R)$, defined in point (a).
210 and also arcs on l, respectively, on $S_R^1 \cap P(0 \leq x \leq R)$, bounded by points L_{s_1}, L_{s_2}, respectively, M_{s_1}, M_{s_2}. And by (1.2,15) it follows that the projection of the element of arc ds_ℓ of the curve $l \subset E^2$ onto the x axis is not less than the projection of the corresponding element of arc ds_m of the circle S_R^1 on the x-axis, from which we get:

$$pr_x\, m_s \leq pr_x\, l_s. \qquad (1:2,16)$$

For the projections of m_s and l_s on the y-axis we will have, evidently, the opposite inequality:

$$pr_y\, m_s \geq pr_y\, l_s, \qquad (1:2,17)$$

where by the projection of l_s on the y-axis is understood the projection of the enclosing vector $\overline{M_0L_s}$.

The coordinates $x_M(s)$, $y_M(s)$ of the point M_s are equal to, respectively, $pr_x m_s$, $pr_y m_s$; the coordinates $x_L(s)$, $y_L(s)$ of the point L_s are equal, respectively, $pr_x l_s$, $pr_y l_s$. Inequalities (1: 2, 16), (1: 2, 17) now can be rewritten as:

$$x_L(s) \geq x_M(s), \qquad (1:2,18)$$
$$y_L(s) \leq y_M(s),$$

from which it is seen that for any $s \in (0, \frac{\pi}{2} R)$ the point L_s will lie in the part $P(s)$ of the plane E^2, defined by inequalities:

$$x \geq x_M(s),$$
$$y \leq y_M(s).$$

But for $0 < s < \frac{\pi}{2} R$, $P(s) \cap K_R^{2'} = 0$, from which it follows, that no point of the curve l belongs to the open disk $K_R^{2'}$. Completely analogously it is proved that no point of l belongs to $K_R^{2''}$, and so $l \cap (K_R^{2'} \cup K_2^{2''}) = 0$, and proposition (b) is shown.

(c) From (1:2,18) and the obvious relation

$$\lim_{s \to \frac{\pi}{2} R} x_M(s) = R$$

is obtained

$$\lim_{s \to \frac{\pi}{2} R} x_L(s) \geq R,$$

that is, the arc l is projected at least on all of the halfinterval $[0, R)$. Therefore $c \geq R$ and proposition (c) comes from (a).

(d) By virtue of proposition (c), $l(P)$ is a partial arc of arc l, projecting on halfinterval $[0, R)$. Therefore

$$l(P) \subset D_{M_0}(0 \leq x < R).$$

On the other hand,

$$D_{M_0}(0 \leq x < R) \subset D_{M_0}(0 \leq x < c) = l,$$

217 from which

$$D_{M_0}(0 \leq x < R)$$

$$= [D_{M_0}(0 \leq x < R) \cap P(0 \leq x < R)] \subset [D_{M_0}(0 \leq x < c) \cap P(0 \leq x < R)]$$

$$= l \cap P(0 \leq x < R) = l(P).$$

Thus,

$$l(P) = D_{M_0}(0 \leq x < R).$$

In point (b) it was proved that

$$l \cap (K_R^{2'} \cup K_R^{2''}) = 0;$$

from which it follows that

$$l(P) \subset [P(0 \leq x < R) \setminus (K_R^{2'} \cup K_R^{2''})].$$

Lemma 1.2 is completely proved.

<u>Lemma 1.3</u> Let

$$r \leq R\left(\operatorname{cosec}\frac{\alpha'}{2} - 1\right), \quad \frac{\pi}{3} < \alpha' < \pi, \quad 0 < \alpha \leq \alpha'. \tag{1.3.1}$$

Then there is no sphere $S_r^n(O)$ of radius r which is tangent to the surface $F^n \in F_R^n$ at two **points** M_1, M_2 for which the angle between the radii OM_1, OM_2 of the sphere $S_r^n(O)$ equals α.

Proof Let there be a sphere $S^n_r(O)$ enjoying the properties recounted. The two-dimensional plane containing points O, M_1, M_2 we designate E^2. We designate $S^1_r(O)$ the intersection of the sphere $S^n_r(O)$ with the plane E^2, and the open disk in E^2 bounded by the circle $S^1_r(O)$ we designate $K^2_r(O)$, and its closure $\overline{K^2_r}(O)$. We take on $S^1_r(O)$ the arc $M_1O'M_2$ of positive length $\alpha r < \pi r$ with midpoint O' (cf. fig. 1 p.26). By condition (1:3,1) $r < R$; this circumstance is to be kept in view throughout the following proof. We construct in E^2 closed disks $\overline{K^2_R}(O_i), \overline{K^2_R}(O_i')$, $i = 1, 2$, of radius R with centers O_i, O_i'; they are bounded, respectively, by circles $S^1_R(O_i), S^1_R(O_i')$, placed so that $S^1_r(O)$ is tangent to $S^1_R(O_i)$ at the point M_i ~~exteriorly~~, and $S^1_R(O_i')$ ~~at~~ the same point interiorly. The line OO' we designate g. By construction the disk $\overline{K^2_r}(O)$ and also the sets $\overline{K^2_R}(O_1) \cup \overline{K^2_R}(O_2)$, $\overline{K^1_R}(O_1') \cup \overline{K^2_R}(O_2')$ are symmetric relative to the line g.

We show that the inequality

$$r \leq R\left(\operatorname{cosec}\frac{\alpha}{2} - 1\right) \qquad (1{:}3,2)$$

is necessary and sufficient in order that

$$\overline{K^2_R}(O_1) \cap \overline{K^2_R}(O_2) \neq 0, \qquad (1{:}3,3)$$

It is easy to see that the quadrangle $O_1' O_2' O_1 O_2$ is an equal-sided trapezoid, symmetric relative to q, the diagonals $O_1 O_1'$, $O_2 O_2'$ of which intersect in the point O. From the similarity of the triangles $O_1 O O_2$, $O_1' O O_2'$ we get

$$\frac{O_1 O_2}{O_1' O_2'} = \frac{O_1 O}{O_1' O} = \frac{R+r}{R-r} > 1,$$

from which for the basic trapezoid $O_1 O_2' O_1' O_2$ we obtain the inequality

$$O_1 O_2 > O_1' O_2'.$$

It is easy to see that the condition

$$O_1 O_2 \leq 2R \qquad (1.3,4)$$

is necessary and sufficient for relation (1:3,3) to hold. The midpoint of $O_1 O_2$ we designate N. Evidently $N \in q$ and ON is the bisector of the isosceles triangle $O_1 O O_2$. From triangle $O_1 N O$ we get

$$\sin\frac{\alpha}{2} = \frac{O_1 N}{O_1 O} = \frac{O_1 N}{R+r} = \frac{O_1 O_2}{2(R-r)},$$

from which is seen that relation (1:3,2) is equivalent to condition (1:3,4). Hence, condition (1:3,2) is necessary and sufficient for (1:3,3) to hold.

We suppose, beginning with this place, that (1:3,2) holds. We designate by A_1 (respectively, A_2) the point of intersection of segment $O_1 O_2$ with the circle $S_R^1(O_2)$ (respectively, $S_R^1(O_1)$).

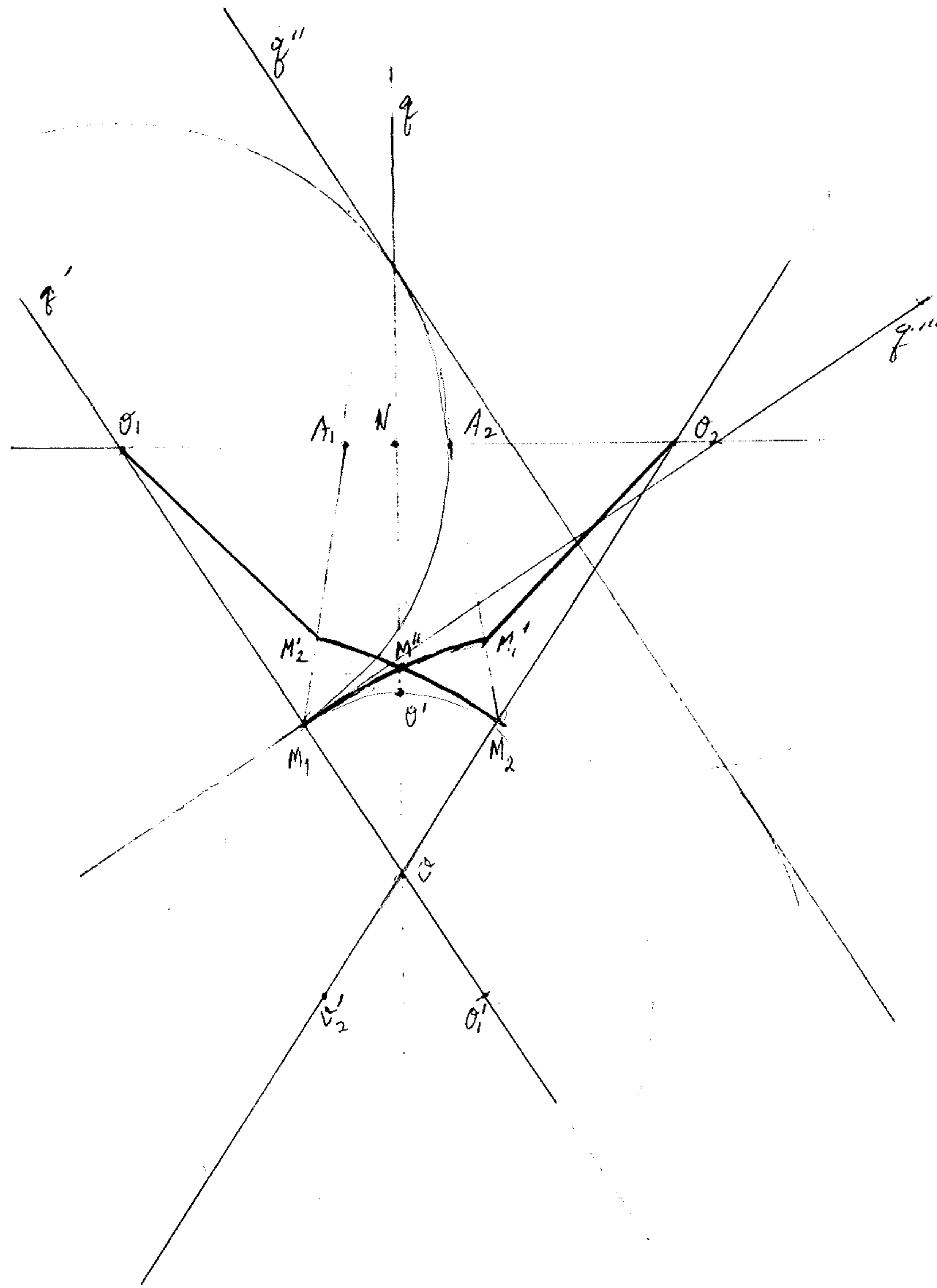

Fig. 1

Since in the isosceles triangle $O_1 O O_2$ the angle $\sphericalangle O_2 O_1 O$ at the base is acute, the arc $\cup M_1 A_2$ of the circle $S^1_R(O_1)$, included in interior angle $\sphericalangle O_2 O_1 O$, is less than $\frac{\pi}{2} R$. Analogously it is proved that arc $\cup M_2 A$ is less than $\frac{\pi}{2} R$. We construct, as in lemma (1·2), sets Q_1 and Q_2, in the boundary of which runs, respectively, arcs of the circles $S^1_R(O_1)$, $S^1_R(O_1')$ and $S^1_R(O_2)$, $S^1_R(O_2')$ and which contain, respectively, arcs $\cup M_1 A_2$ and $\cup M_2 A_1$.

Segments $O_2 A_2$, $O_2 M_2$, as is seen from consideration of the trapezoid $O_1 O_2 O_1' O_2'$ and its diagonals, do not intersect the segment $O_1 O_1'$. On the other hand, by considering the triangle $O_1 O_2 O_1'$ it is seen that segments $O_2 A_2$, $O_2 M_2$ cannot intersect the line q', drawn through O_1 and O_1', outside the segment $O_1 O_1'$. This means that points A_2, M_2 lie in the same halfplane P bounded by the line q'. We construct the tangent q'' to the circles $S^1_R(O_1)$, $S^1_R(O_1')$, entirely belonging to P, and the common tangent q''' to circles $S^1_R(O_1)$, $S^1_R(O_1')$ at the point M_1. It is clear that disks $\overline{K}^2_R(O_1)$, $\overline{K}^2_R(O_1')$ lie on different sides of q'''. Consequently, if we designate by G_1, G_2 the domains into which the strip bounded by q' and q''

is divided by g''', then the points $A_2 \in \overline{K}^2_R(O_1')$ and $M_2 \in \overline{K}^2_R(O_1')$, not coinciding with M_1, lie in different domains G_i. Therefore the segment A_2M_2 intersects g''' in some point $C \in P$. But the distances from A_2 and from M_2 to g' are less than R. Therefore $C \in Q_1$. We take points C' and C'' lying, respectively, on $S^1_R(O_1)$ and $S^1_R(O_1')$, nearest to C. Then the segment $C'C''$ is a section of the Jordan domain $\overline{Q}_1$. We consider in the domain Q_1 the nonselfintersecting Jordan arc l_1, issuing from M_1 and belonging to F'' (cf. lemma 1.2). This arc begins at M_1 and has a limit point on g'', consequently, l_1 contains points of both domains into which section $C'C''$ divides $\overline{Q}_1$. But then l_1 intersects $C'C''$ and, what is more, a segment of A_2M_2 containing $C'C''$. Analogously we consider in the domain Q_2 the arc $l_2 \subset E^2 \cap F''$, issuing from the point M_2 and intersecting symmetrically relative to g the segment A_1M_1. Evidently, the points of the set Q_1, sufficiently close to M_1, belong to the triangle $M_1A_2M_2$. Since $M_1A_2 \in \overline{K}^2_R(O_1)$, $M_1M_2 \in \overline{K}^2_R(O_1')$, $M_2 \in K^2_R(O_1')$, then l_1 can intersect the boundary of the triangle $M_1A_2M_2$ only in the point M_1 and in a point of the side A_2M_2 different from M_2. We designate by l_1' the part of the arc of arc l_1 between the point M_1 and the first point of intersection M_1' of arc l_1 with segment $A_2 M_2$, including M_1.

Analogously we construct a partial arc $l_2' \subset l_2$ with ends M_2, $M_2' \in M_1 A_1$. By construction, l_1' belongs to triangle $M_1 A_2 M_2$, and l_2' to triangle $M_1 A_1 M_2$.

We consider now triangle $O_1 O O_2$. Joining to arcs l_1', l_2', respectively, segments $M_1' O_2$, $M_2' O_1$, we obtain arcs $\tilde{l}_1$, $\tilde{l}_2$, respectively. Arc $\tilde{l}_1$ joins vertex O_2 of triangle $O_1 O O_2$ with interior point M_1 of side OO_1. Arc $\tilde{l}_2$ joins vertex $\tilde{O}_1$ with interior point M_2 of side OO_2. Since l_1 is a section of triangle $O_1 O O_2$, $\tilde{l}_1$ divides triangle $O_1 O O_2$ into two domains H_1, H_2, for which arc $\tilde{l}_2$ contains points of both of these domains (points of $\tilde{l}_2$ near to O_1, respectively, to M_2 belong different domains H_i). This means, $\tilde{l}_2$ intersects $\tilde{l}_1$. By construction, arc l_1' lies in triangle $M_1 A_2 M_2$, segment $O_2 M_1'$ in triangle $O_2 M_2 A_2$; this means $\tilde{l}_1$ belongs to quadrangle $M_1 A_2 O_2 O$. Yet segment $O_1 M_2'$ belongs to triangle $O_1 A_1 M_1$. The intersection of quadrangle $M_1 A_2 O_2 O$ with triangle $O_1 A_1 M_1$ is either the point M_1 (when $A_1 \neq A_2$) or the segment $M_1 A_2$; therefore the halfopen segment $O_1 M_2' \setminus M_2'$ does not intersect $\tilde{l}_2$. Each arc $\tilde{l}_i$ can be represented in the form of a union of nonintersecting terms:

$$\tilde{l}_1 = l_1' \cup [O_2 M_1' \setminus M_1'],$$
$$\tilde{l}_2 = l_2' \cup [O_1 M_2' \setminus M_2'].$$

Since the intersections $\tilde{l}_2 \cap [O_1 M_2' \setminus M_2']$, $\tilde{l}_2 \cap [O_2 M_1' \setminus M_1']$, $\{O_2 M_1' \setminus M_1'\} \cap \{O_1 M_2' \setminus M_2'\}$ are empty, $[\tilde{l}_1 \cap \tilde{l}_2] \subset [l_1' \cap l_2']$ and from the nonemptiness of the intersection $\tilde{l}_1 \cap \tilde{l}_2$ we obtain that $l_1' \cap l_2'$ is not empty. Take the point $M'' \in [l_1' \cap l_2']$, nearest to M_1 on the arc l_1'. Since $M_2' \in K_R^2(O_1)$, $M_2' \in l_1'$, $M_2' \neq M''$, therefore from the point M'' issue three simple arcs $\smile M''M_1 \subset l_1'$,
220 $\smile M''M_2 \subset l_2'$, $\smile M''M_2' \subset l_2'$, pairwise intersecting only in the point M''. But this contradicts lemma (1:2), according to which in ~~each~~ some neighborhood of each interior point of the arc l_2 the set $F^n \cap E^2$ reduces to one simple Jordan arc. The contradiction obtained proves the lemma.

<u>Lemma 1:4</u> For any $0 < r_1 < R$, $0 < \alpha < \frac{\pi}{3}$, there does not exist a sphere of radius r_1 which is tangent to a surface $F^n \in F_R^n$ in two points for which the angle between the radii of the sphere directed to these points of tangency is equal to α.

Proof. We take an angle α', $\frac{\pi}{3} < \alpha' < \pi$ in order that $R\left(\operatorname{cosec}\frac{\alpha'}{2} - 1\right) > r_1$; this is possible since $r_1 < R$. Since $\alpha < \alpha'$, we can use lemma 1:3, from which lemma 1:4 immediately results.

§2. Cylindrical sections of surfaces of class F^n_R.

Lemma 2:1 In lemma 2:1 the notation of lemma 1:2 is preserved. Let E^n be ~~the~~ n-dimensional plane tangent to $F^n \in F^n_R$ at the point M_0. The strip $P(0 \leq x < R)$ introduced in lemma 1:2 belongs to a two-dimensional plane E^2 spanned by the vector $\underset{\sim}{n_0}$ and an arbitrary unit vector $\underset{\sim}{t_0}$ with origin at the point M_0, orthogonal to $\underset{\sim}{n_0}$. We will designate this strip by $P_{\underset{\sim}{t_0}}(0 \leq x < R)$. We put

$$C^{n+1}_R = \bigcup P_{\underset{\sim}{t_0}}(0 \leq x < R),$$

where the union is taken over all unit vectors $\underset{\sim}{t_0}$ orthogonal to $\underset{\sim}{n_0}$. We designate, in an analogous way, the set Q introduced in lemma 1:2 by $Q_{\underset{\sim}{t_0}}$. We put:

$$Q^{n+1} = \bigcup Q_{\underset{\sim}{t_0}}.$$

Analogously we define the set

$$F^n_{M_0} = \bigcup l(P_{\underset{\sim}{t_0}}).$$

We introduce cartesian coordinates $x^1, -, x^{n+1}$ with the help of a frame $\underset{\sim}{e_1}, -, \underset{\sim}{e_{n+1}}$ chosen so that $\underset{\sim}{e_{n+1}} = \underset{\sim}{n_0}$.

Lemma 2:1 The set $F^n_{M_0}$ is a surface which in the chosen system of coordinates is represented by a C^2 function of the form:

$$x^{n+1} = \Phi(x^1, \cdots, x^n),$$

defined on the ball $K^n_R(M_0) = E^n \cap C^{n+1}_R$:

$$\sum_{i=1}^{n} (x^i)^2 < R^2, \quad x^{n+1} = 0.$$

The connected component of the set $F^n \cap C^{n+1}_R$, containing M_0, coincides with $F^n_{M_0}$, for which $F^n_{M_0} \subset Q^{n+1}$.

Proof. We construct the map Φ of the open ball $K^n_R(M_0)$ on $F^n_{M_0}$ in the following way. If point $M' \in K^n_R(M_0)$, $M' \neq M_0$, then we construct
221 unit vector $\underset{\sim}{t_0}(M') = \frac{\overline{M_0 M'}}{|\overline{M_0 M'}|}$ and we designate by $E^2(M')$ the two-dimensional plane spanned by vectors $\underset{\sim}{n_0}$ and $\underset{\sim}{t_0}(M')$. It is easy to see that the plane $E^2(M')$ depends continuously on M' $(M' \neq M_0)$. The strip $P(0 \leq x < R) \subset E^2(M')$, containing M', we designate $P(M')$; in this strip lies the arc $l(P(M'))$, which we will designate, for brevity, $l(M')$. We lay out on the arc $l(M')$ from point M_0 the partial arc projecting on the segment M_0M' (cf. the proof of lemma 1:2); the end of this arc we designate $\Phi(M')$.

We put, finally, $\Phi(M_0) = M_0$. We show that Φ is a continuous map of $K^n_R(M_0)$ on $F^n_{M_0}$. Let $M'_k \to M' \neq M_0$, $M'_k \in K^n_R(M_0)$, $M' \in K^n_R(M_0)$. Then the planes $E^2(M'_k)$ converge to plane $E^2(M')$, by which the corresponding vectors $\underset{\sim}{t}_0(M'_k)$ converge to the vector $\underset{\sim}{t}_0(M')$. We suppose that the points $\Phi(M'_k) = M_k$ do not converge to $\Phi(M')$; chosing, in case it is necessary, a subsequence from the bounded sequence M_k, it may be assumed that M_k converge to some point $M_* \neq \Phi(M')$. Since the distances $\rho(M'_k, x^{n+1})$ of the points M'_k from the x^{n+1}-axis converge to $\rho(M', x^{n+1}) < R$, so too $\rho(M_k, x^{n+1})$ is less than some $R - \varepsilon < R$, so that $\rho(M_*, x^{n+1}) < R$. Therefore $M_* \in C^{n+1}_R$. Moreover, the points $M_k \in E^2(M'_k)$, and the planes $E^2(M'_k)$ converge to the plane $E^2(M')$; consequently, $M_* \in E^2(M')$.

Since the vectors $\overline{M_0 M_k}$ project to vectors $\overline{M_0 M'_k} \to \overline{M_0 M'}$, the vector $\overline{M_0 M_*}$ projects to $\overline{M_0 M'}$; consequently, $M_* \in P_{\underset{\sim}{t}_0(M')}(0 \leq x < R)$, ~~which~~ in that M_* projects to M'. The arcs $l(M'_k)$ can be represented by equations of the form $x^{n+1} = g_k(x)$, $0 \leq x < R$, where x is the distance from the x^{n+1}-axis.

From the proof of lemma 1:2 it is seen (in points (b), (c)) that on each segment $0 \leq x \leq R_1$, $R_1 < R$ the functions $g_k(x)$ are uniformly bounded and have uniformly bounded derivatives. By the theorem of Arzela the sequence $\{g_k(x)\}$ contains a subsequence uniformly converging on each segment $0 \leq x \leq R_1$, $R_1 < R$; we will designate this subsequence as before by $\{g_k(x)\}$ and the limit function by $g_0(x)$. In the plane $E^2(M')$ the equation $x^{n+1} = g_0(x)$ $(0 \leq x < R)$ forms a curve l_* with origin at the point M_0. All points of l_*, as limit points of points of the arcs $l(M'_k)$, belong to F^n; finally, M_0, M_*, as is easily seen, belong to l_*. Arc l_* contains point M_0, consequently, $l_* \subset D_{M_0}(0 \leq x < R)$ in the plane $E^2(M')$ (in the notation of lemma 1:2); according to proposition (d) of lemma 1:2, $l_* \subset l(M')$, from which results $M_* \in l(M')$. But by construction, M_* projects to M'; consequently, $M_* = \Phi(M')$, contrary to the definition of point M_*. We have proved that Φ is continuous at each point $M' \in K^n_R(M_0)$, $M' \neq M_0$. Now let $M'_k \to M_0$, $M_k = \Phi(M'_k)$. By definition of the sequence $\{M'_k\}$, $\overline{M_0 M'_k} \to 0$; since in some segment $[0, R_1]$, $0 < R_1 < R$, the functions $g_k(x)$

have uniformly bounded derivatives, then from $\overline{M_0 M_k'} \to 0$ it follows $M_k' M_k \to 0$, from which $\overline{M_0 M_k} \to 0$, $M_k \to M_0$.

Hence, $\Phi(M')$ is a continuous map of $K^n_R(M_0)$ to $F^n_{M_0}$. By this each arc $l(P) \subset F^n_{M_0}$ is the image of some radius of the ball $K^n_R(M_0)$, so that $\Phi(K^n_R(M_0)) = F^n_{M_0}$. We consider concentric balls $K^n_{R_m}(M_0)$, $0 < R_m < R$, where $R_m < R_{m+1}$ and $\lim_{m\to\infty} R_m = R$.
222 Putting $\Phi(K^n_{R_m}(M_0)) = F^n_{R_m}$, we have: $F^n_{R_m} \subset F^n_{R_{m+1}}$, $\bigcup_{m=1}^{\infty} F^n_{R_m} = F^n_{M_0}$. Φ maps $K^n_{R_m}(M_0)$ bijectively and continuously on $F^n_{R_m}$. Due to the compactness of the preimage, Φ is a homeomorphism on $\overline{K}^n_{R_m}(M_0)$; consequently, Φ^{-1} is continuous on $\Phi(K^n_{R_m}(M_0))$ and what is more on $F^n_{R_m}$. Since m is arbitrary, Φ^{-1} is continuous on all of $F^n_{R_0}$. Hence, Φ is a homeomorphic map of $K^n_R(M_0)$ on $F^n_{M_0}$.

$F^n_{M_0}$ is a ~~submanifold~~ subset of F^n, provided with the topology induced from F^n. Consequently, Φ can be considered as a homeomorphic map of the open ball $K^n_R(M_0)$ to the manifold F^n. By the theorem of Brouwer on invariance of interior points ((1), p 196), F^n_M is an open subset of F^n, and likewise, an open subset of $F^n \cap C_R^{n+1}$.

We show now that $F^n_{M_0}$ is a closed subset of $F^n \cap C_R^{n+1}$; for this it suffices to establish that every point $M \in C_R^{n+1}$, a limit for $F^n_{M_0}$, belongs to $F^n_{M_0}$. Let $M_k \to M$, $M_k \in F^n_{M_0}$. Then all points $M'_k = \Phi^{-1}(M_k)$ lie from the x^{n+1}-axis at distance less than some $R' < R$, and, likewise, belong to some ball $\overline{K}^n_{R_m}(M_0)$. Due to the compactness of a closed ball, there can be chosen a subsequence (designated anew by $\{M_k\}$) such that the M'_k converge to some point $M' \in \overline{K}^n_{R_m}(M_0)$. Therefore $\Phi(M'_k) \to \Phi(M')$, $M_k \to \Phi(M')$, then $M = \Phi(M')$, $M \in F^n_{M_0}$. Hence, $F^n_{M_0}$ is open and closed in $F^n \cap C_R^{n+1}$. Since $F^n_{M_0}$ is connected (all points of $F^n_{M_0}$ are joined by arcs $l(P_{t_0})$ with the point M_0), $F^n_{M_0}$ coincides with the component of $F^n \cap C_R^{n+1}$ containing M_0.

Writing out the map $\Phi(M')$ in coordinates $x^1, \dots, x^{n+1}$, we get formulas:

$$y^i = x^i, \quad i = 1, \dots, n$$

$$y^{n+1} = \Phi(x^1, \dots, x^n),$$

where Φ is defined and continuous on the ball $K^n_R(M_0)$. It remains to prove that Φ is C^2. We fix the point $M'(x^1, \dots, x^n, 0) \in K_R^n(M_0)$ and put $\Phi(M') = M$. According to lemma 1:2, the normal to F^n at point M is not orthogonal to $\underset{\sim}{n}_0$. Therefore there is a neighborhood $V(M)$ relative to C_R^{n+1} such that $V(M) \cap F^n$ is a surface homeomorphic to an n-dimensional ball and having equation $x^{n+1} = \chi(x^1, \dots, x^n)$, where χ is C^2. Since $M \in F_{M_0}^n$ and $V(M) \cap F^n$ is connected, $[V(M) \cap F^n] \subset F_{M_0}^n$. But then for every point $(x^1, \dots, x^{n+1}) \in [V(M) \cap F^n]$ there must hold the equation: $x^{n+1} = \Phi(x^1, \dots, x^n)$.

Consequently, $\Phi = \chi$ in a neighborhood of the point M' and Φ is C^2.

According to lemma 1:2, $l(P_{\underset{\sim}{t}_0}) \subset Q_{\underset{\sim}{t}_0}$, from which follows the inclusion $F_{M_0}^n \subset Q^{n+1}$.

Lemma 2:1, with this, is proved.

2.23 <u>Lemma 2:2</u>. Let $F^n \in F_R^n$, point $M_0 \in F^n$ and $F_{M_0}^n(R/2)$ be the connected component of the set $F^n \cap K_{R/2}^{n+1}(M_0)$, containing M_0; then no two normals of length $< R$, contructed at any two different points of $F_{M_0}^n(R/2)$, intersect.

Proof. Utilizing the notation of lemmas 1:2, 2:1, we construct at an arbitrary point $L_s \in P \cap F^n_{M_0}$ a vector $\pm r \cdot \underset{\sim}{n}_F(s)$, and at point M_s a vector $r \underset{\sim}{n}_K(s)$, where $0 < r < R$. The vector $r \underset{\sim}{n}_F(s)$ does not intersect the line n_0 containing the vector $\underset{\sim}{n}_0$. Due to inequality (1:2, 18),

$$\varkappa_L(s) \geq \varkappa_M(s), \qquad (2:2,1)$$

the distance from the origin L_s of the vector $\pm r \underset{\sim}{n}_F(s)$ to the line n_0 is no less that the distance from the origin M_s of the vector $r \underset{\sim}{n}_K(s)$ to n_0. From inequality (1:2, 9):

$$\measuredangle \underset{\sim}{n}_0, \underset{\sim}{n}_F(s) \leq \measuredangle \underset{\sim}{n}_0, \underset{\sim}{n}_K(s)$$

there follows the inequality

$$|pr_{E^n} \, r \underset{\sim}{n}_F(s)| \leq |pr_{E^n} \, r \underset{\sim}{n}_K(s)|, \qquad (2:2,2)$$

where the projection is carried out onto the plane E^n (cf. lemma 2:1). Since $\underset{\sim}{n}_K(s)$ is the principal normal to the circle $S^{1'}_K$, directed to the center of $S^{1'}_K$, from (2:2,1) and (2:2,2) there results that the end of the vector $\pm r \underset{\sim}{n}_F(s)$ is no closer to the line n_0 than the end of vector $r \underset{\sim}{n}_K(s)$. Still the distance from the end of vector $r \underset{\sim}{n}_K(s)$ to the line n_0 is not equal to zero, from which results that vector

$\pm r \underset{\sim}{n}_F(s)$ does not intersect n_0.

We construct in arbitrary points M', M'' of the set $F^n_{M_0}(R/2)$ normals $\underset{\sim}{n}'$, $\underset{\sim}{n}''$ to F^n of length $< R$. According to lemma 2:1, $F^n_{M_0}(R/2) \subset F^n_{M'}$; from the previous considerations it is seen that $\underset{\sim}{n}''$ does not intersect the line containing $\underset{\sim}{n}'$, and moreover does not intersect $\underset{\sim}{n}'$ itself.

Since the geodesic ball of radius $R/2$ with center the point $M' \in F^n \in F^n_R$ on the surface F^n is contained in $F^n_{M'}(R/2)$, from lemma 2:2 there results

Theorem 2:3. Normals of length $< R$ constructed at any two different points of a geodesic ball of radius $R/2$ on the surface $F^n \in F^n_R$ do not intersect.

Theorem 2:4. The diameter of any surface of class F^n_R is no less than $2R$. If the diameter of a surface $F^n \in F^n_R$ equals $2R$, then F^n is a sphere.

Proof. The validity of the first assertion follows in an obvious way from lemma 2:1, since the diameter of F^n is no less than the diameter of $F^n_{M_0}$, which in turn is no less than the diameter of the projection of $F^n_{M_0}$ onto the plane E^n, that is the ball $K^n_R(M_0)$ of radius R.

Let the diameter of surface $F^n \in F^n_R$ equal $2R$ and let $\tilde{F}^{n+1}$ be the convex hull of F^n. Then there is a chord M_1M_2 of the convex hull $\tilde{F}^{n+1}$, equal to $2R$ and perpendicular to parallel support planes E^n_1, E^n_2 of the convex hull $\tilde{F}^{n+1}$, extending between points M_1, M_2
234 ([3], p 51). It is easy to see that the points M_1, M_2 belong to the intersection $F^n \cap \tilde{F}^{n+1}$. In an arbitrary two-dimensional plane E^2, running through M_1, M_2 we consider strips P', P'' bounded by the line M_1M_2 and the lines parallel to it, n', respectively, n'', separated from M_1M_2 at distance R. In the strips P', P'' we construct arcs $l(P'), l(P'')$, issuing from the point M_1 and analogous to the arc $l(P)$ of lemma 1:2. From the fact that the lengths of arcs $l(P'), l(P'')$ are finite (cf the proof of lemma 1:2), it follows that arc $l(P')$ (respectively, $l(P'')$) has a unique limit point belonging to line n' (respectively, n''). We designate ~~these~~ this point M_1' (respectively, M_1''). We put $\bar{l}(P') = l(P') \cup M_1'$, $\bar{l}(P'') = l(P'') \cup M_1''$, $\cup M_1'M_1M_1'' = \bar{l}(P') \cup \bar{l}(P'')$. Analogously we construct

arc $\cup M_2' M_2 M_2''$ running through M_2 and lying in the strip $\bar{P}' \cup \bar{P}''$. From lemma 2 there results that arcs $\cup M_i' M_i M_i''$, $i=1,2$ lie outside of the disk $K_R^2(O) \subset E^2$, the center O of which lies at the midpoint of the segment $M_1 M_2$. We designate the circle bounding $K_R^2(O)$ by $S_R^1(O)$, and the points of tangenty of $S_R^1(O)$ with the lines n', n'', respectively, by A', A''. We suppose that some point L of the arc $\cup M_1 M_1'$ does not lie on $S_R^1(O)$. Then $LO > R$. We extend segment LO to the intersection with n'' at some point L''. If segment OL'' intersects arc $\cup M_2 M_2''$ in some point M'', then $OM'' \geq R$ and the chord $LM'' > 2R$, which contradicts the condition of the theorem. If segment OL'' does not intersect arc $\cup M_2 M_2''$, then the point M_2'' lies on the ray $A''L''$, going out from A'', outside the segment $A''L''$. In the triangle $LL''M_2'$ the angle $\angle LL''M_2'$ is obtuse, Therefore the chord $LM_2' > LL'' > 2R$, which is impossible. Hence, the arc $\cup M_1 M_1'$ must coincide with the arc $\cup M_1 A'$ of the circle $S_R^1(O)$. Proceeding to analogous considerations for arcs $\cup M_1 M_1''$, $\cup M_2 M_2'$, $\cup M_2 M_2''$, we conclude that $[\cup M_1' M_1 M_1''] \cup [\cup M_2' M_2 M_2''] = S_R'(O)$, from which, due to the arbitrariness of the plane E^2, there results that F^n contains the n-dimensional sphere S_R^n with diameter $M_1 M_2 = 2R$. Since F^n is a connected n-dimensional manifold, $F^n = S_R^n$.

§3. The central set. Proof of the basic theorem.

3:1 Let point M lie inside the surface $F^n \in F_R^n$, so that $M \in T(F^n)$, and $\rho(M, F^n) = r$ is the distance from M to F^n. The set $S_r^n(M) \cap F^n$ is not empty; we will say that a segment, joining a point of this set with M, realizes $\rho(M, F^n)$.

The point $M \in T(F^n)$ is called simple if there is only one segment realizing $\rho(M, F^n)$.

The point $M \in T(F^n)$ is called central if there are no less than two segments realizing $\rho(M, F^n)$.

We call the set $Z(F^n)$ of all central points of $T(F^n)$ the central set of the surface F^n.

Lemma 3:2. The set $E^{n+1} \setminus Z(F^n)$ is connected.

Proof. Connectedness of the set $E^{n+1} \setminus T(E^n)$ follows from the connectedness of F^n, due to the theorem of Jordan-Brouwer (cf. (1), p 519).

It remains for us to prove that for any simple point $M \in T(F^n)$ there can be obtained an arc $\cup MM'$, joining M with F^n ($M' \in F^n$) and not intersecting $Z(F^n)$. Such an arc is the segment MM' realizing $\rho(M, F^n)$. Concerning this, if N is an interior point of the specified segment MM', then the sphere $S^n_{NM'}(N)$ of radius NM' with center N intersects F^n in a unique point of tangency M' and, consequently, N is a simple point.

? 3:3. If a point $M \in F^n \in F^n_R$ and $\rho(M, F^n) < R$, then from lemma 2:2 it follows that for the point M there can only be a finite number of segments realizing $\rho(M, F^n)$; this number we cal the multiplicity of the point M.

By the definition of a flattened surface $F^n \in F^n_R$, for any point $M \in T(F^n)$, $\rho(M, F^n) < R$, so that ~~all~~ points of the set $Z(F^n)$ of a flattened surface F^n have some multiplicity.

Henceforth, if we do no say otherwise, we will consider only the subclass $\hat{F}^n_R$ of flattened surfaces of the class F^n_R.

Lemma 3:4. The multiplicity of a point of the central set of an n-dimensional flattened surface of class $\hat{F}^n_R$ is bounded by some number $c(n)$.

Proof. Let M be any point of $Z(F^n)$ of a surface $F^n \in \hat{F}^n_R$, and MM_i, where $i=1,\dots,k$, all segments, realizing $\rho(M,F^n)$. From lemma 1:4 it follows that for any two segments MM_i, MM_j, realizing $\rho(M,F^n)$,

$$\measuredangle\, \overline{MM_i}, \overline{MM_j} > \frac{\pi}{3}.$$

But then the ends G_i of the unit vectors $\frac{\overline{MM_i}}{\rho(M,F^n)}$, $i=1,\dots,k$, on the unit sphere $S^n_1(M)$ are distributed so that around each point G_i in $S^n_1(M)$ can be placed an n-dimensional ball $K^n_{(i)}$ of spherical radius $\frac{\pi}{6}$, for which $K^n_{(i)} \cap K^n_{(j)} = 0$ $(i \neq j)$. Hence it is clear that the multiplicity of M is no greater than the number

$$c(n) = \frac{\text{mes } S^n_1}{\text{mes } K^n_{(i)}}.$$

The lemma is proved.

3:5. Now we can give the definition of the multiplicity of the central set of a flattened surface. By the multiplicity of $Z(F^n)$ for a surface $F^n \in \hat{F}^n_R$ we will mean the greatest multiplicity of points of $Z(F^n)$.

The subset of all points of the central set $Z(F^n)$ of a surface $F^n \in F^n_R$ having multiplicity no less than m, where m is a whole number, no less than two and no greater than the multiplicity of $Z(F^n)$, we designate $Z_m(F^n)$.

Lemma 3:6. The set $Z_m(F^n)$ is closed.

Proof. Let points $M_k \in Z_m(F^n)$ and $M_k \to M_0$. For each point M_k there exist no less than m segments $M_k M_{k,i}$, $i=1,\dots,m$, realizing $\rho(M_k, F^n)$. From the sequence $\{M_k\}$ can be chosen a subsequence $\{M_{k_\ell}\}$ so that the segments $M_{k_\ell} M_{k_\ell,i}$, $i=1,\dots,m$, will converge, respectively, to some limit segment $M_0 M_{0,i}$ realizing, due to the continuity of $\rho(M, F^n)$ relative to M, the distance $\rho(M_0, F^n)$. From lemma 2:2
226 it follows that the segments $M_0 M_{0,i}$, $i=1,\dots,m$, are distinct, that is $M_0 \in Z_m(F^n)$.

For $m=2$ from lemma 3:1 is obtained

Lemma 3:7. The central set $Z(F^n) = Z_2(F^n)$ of a flattened surface F^n is closed.

3:8. Let $g_1, \dots, g_k$ be an arbitrary system of distinct unit vectors of the space E^{n+1}, having a common origin and

$$\beta = \min_{i \neq j} \measuredangle g_i, g_j .$$

We put

$$\alpha^{n+1}(k) = \sup \beta$$

~~where~~ for all possible systems of k unit vectors. $\alpha^{n+1}(k)$ is a nonincreasing function of the argument k. Concerning this, to each system of $k+1$ unit vectors $g_1, \dots, g_{k+1}$, corresponds a system of k, obtained from the first system by omitting one; it is clear also that for each system of k there is a system of $k+1$ from which by omitting one we get the system of k; therefore

$$\beta_k = \min_{\substack{i \neq j \\ i,j \neq k+1}} (\measuredangle g_i, g_j) \geq \min_{i \neq j} (\measuredangle g_i, g_j) = \beta_{k+1}$$

from which

$$\alpha^{n+1}(k) = \sup \beta_k \geq \sup \beta_{k+1} = \alpha^{n+1}(k+1).$$

3:9. We calculate the value $\alpha^{n+1}(3)$, needed below. In a two-dimensional plane a triple g_1, g_2, g_3 can be laid out so that

$$\measuredangle g_1 g_2 = \measuredangle g_1 g_3 = \measuredangle g_2 g_3 = \tfrac{2}{3}\pi,$$

from which

$$\alpha^{n+1}(3) \geq \tfrac{2}{3}\pi. \qquad (3;9,1)$$

We suppose that there exists a triple g_1', g_2', g_3' for which
$$\beta_3 = \min_{i \neq j}(\measuredangle g_i', g_j') > \frac{2}{3}\pi . \quad (3:9,2)$$

We span a 3-dimensional space E^3 by this triple and consider the unit 2-dimensional sphere $S_1^2(O) \subset E^3$, with center O at the origin of the g_i'. On the surface $S_1^2(O)$ are placed, corresponding to these the ends G_1', G_2', G_3'. We construct on $S_1^2(O)$ the 2-dim'l disk K^2 with center G_1' and spherical radius β_3. The points G_2', G_3' should lie in the disk K_*^2 ~~supplementary~~ to K^2 in $S_1^2(O)$, with center at the point G_* diametrically opposite G_1' and with spherical radius
$\beta_3' = \pi - \beta_3 < \frac{\pi}{3}$.

It is easy to see that for any two points M and N, taken from K_*^2, the shortest arc on $S_1^2(O)$, $\cup MN$ will be shorter than $\frac{2}{3}\pi$, but then
$$\measuredangle g_2' g_3' < \frac{2}{3}\pi . \quad (3:9,3)$$

227 Inequality (3:9,3) contradicts (3:9,2), from which, accounting for (3:9,1), we get
$$\alpha^{n+1}(3) = \frac{2}{3}\pi . \quad (3:9,4)$$

Now we will prove lemma 3:10, which has fundamental value for the continuation.

Lemma 3:10. If the set $Z(F^n)$ of surface $F^n \in \hat{F}^n_R$ has multiplicity not less than three, then in F^n can be imbedded an $(n+1)$-dim'l ball of radius r, satisfying the inequality:

$$r > R\left(\frac{2}{\sqrt{3}} - 1\right). \qquad (3:10,1)$$

Proof. If O is a point of $Z(F^n)$ of multiplicity no less than three and OM_i are segments realizing the distance $\rho(O, F^n)$, then according to (3:8) and (3:9,4), at least two of the radii, OM_i, OM_j of the sphere $S^n_{OM_i}(O)$ form angle

$$\measuredangle\, \overline{OM_i}, \overline{OM_j} = \alpha \le \alpha^{n+1}(3) = \frac{2}{3}\pi.$$

Putting $\alpha' = \frac{2}{3}\pi$ in lemma 1:3 we obtain lemma 3:10.

3:11. We choose in lemma 2:1 a frame such that the unit vectors $\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_n$ should be tangent to principal normal sections of $F^n \in F^n_R$ at point M_0 and we consider the surface $f^n_{R'}(M_0, \varepsilon)$ given by the parametric representation:

$$\underset{\sim}{r}(x^1, \ldots, x^n) = \sum_{i=1}^{n} x^i \underset{\sim}{e}_i + \Phi(x^1, \ldots, x^n)\underset{\sim}{e}_{n+1} + R' \frac{\underset{\sim}{e}_{n+1} - \sum_{i=1}^{n} \Phi_{x^i}(x^1, \ldots, x^n)\underset{\sim}{e}_i}{\sqrt{1 + \sum_{i=1}^{n} \Phi^2_{x^i}(x^1, \ldots, x^n)}} \qquad (3:11,1)$$

where $\sum_{i=1}^{n}(x^i)^2 < \varepsilon^2$, $0 < R' < R$ and $\varepsilon > 0$. For sufficiently small ε and $R' < R$, due to lemma 2:2, $f^n_{R'}(M_0, \varepsilon)$ will be a bijective mapping of the surface:

$$F^n_{M_0}(\varepsilon) : x^{n+1} = \Phi(x^1, \dots, x^n), \quad \sum_{i=1}^{n}(x^i)^2 < \varepsilon^2, \quad (3:11,2)$$

if points with the same values of the parameters $x^1, \dots, x^n$ are put in correspondence. The parametric representation (3:11,1), evidently, is C^1, since F^n is a C^2 surface.

We prove that $f^n_{R'}(M_0, \varepsilon)$ is a C^1 surface without singularities (in the sense of differential geometry).

228 Since the construction of $f^n_{R'}(M_0, \varepsilon)$ (motion by R' in the direction of the normal to F^n) does not depend on the choice of M_0, it suffices to prove that the point $\underset{\sim}{r}(0, \dots, 0)$ is not singular. From (3:11,1) it is seen that

$$\underset{\sim}{r}_{x^i}(0, \dots, 0) = \underset{\sim}{e}_i + \Phi_{x^i}(0, \dots, 0)\underset{\sim}{e}_{n+1} + R' \frac{\partial \underset{\sim}{n}}{\partial x^i}(0, \dots, 0),$$

where $\underset{\sim}{n}(x^1, \dots, x^n)$ is the unit vector normal to F^n at ~~the~~ the point with corresponding values of the parameters $x^1, \dots, x^n$. But due to the choice of parameters $(x^1, \dots, x^n)$, $\Phi_{x^i}(0, \dots, 0) = 0$ $(i = 1, \dots, n)$. Besides that, the vectors $\underset{\sim}{e}_i$, by construction coincide with the principal directions of the surface F^n at the point M_0, so that (cf. 1:1,16)

$$\frac{\partial \underset{\sim}{n}}{\partial x^i}(0, \dots, 0) = -k_i \underset{\sim}{e}_i, \quad i = 1, \dots, n.$$

Consequently,

$$\underset{\sim}{r}_{x^i}(0,\dots,0) = (1-R'k_i)\,\underset{\sim}{e}_i\,.$$

But by the condition that all principal curvatures $k_i \le \frac{1}{R} < \frac{1}{R'}$, so that $1-R'k_i > 0$ $(i=1,\dots,n)$.

Thus the frame

$$\{\underset{\sim}{r}_{x^i}(0,\dots,0)\} \quad (i=1,\dots,n)$$

is nondegenerate, which is what was required to prove.

3:12. We will now assume that in the construction of point 3:11 $\underset{\sim}{e}_{n+1}$ is the <u>interior</u> normal to the surface F^n with respect to the body $T(F^n)$. Adhering to the notation of point 3:11, we consider the region $H^{n+1}(M_0, \varepsilon)$ of the space E^{n+1}, swept out by $f^n_{R'}(M_0, \varepsilon)$ as R' varies in the interval $(0, R)$.

? We designate by $\tilde{\rho}(M_0, F^n_{M_0}(\varepsilon))$ (cf. 3:11, 2) the length of the normal MM' extending from M on $F^n_{M_0}(\varepsilon)$ [such a normal exists and besides that only one].

<u>Lemma 3:13</u>. The function $\tilde{\rho}(M, F^n_{M_0}(\varepsilon))$ of the point M has a derivative in any direction $\underset{\sim}{e}$, for which

$$\frac{\partial \tilde{\rho}(M, F^n_{M_0}(\varepsilon))}{\partial e} = \cos \measuredangle\, \underset{\sim}{e}, \underset{\sim}{n}(M), \qquad (3:13,1)$$

where $\underset{\sim}{n}(M)$ is the normal to the surface $f^n_{R'}(M_0, \varepsilon)$ at the point $M \in f^n_{R'}(M_0, \varepsilon)$, directed to the side of increasing parameter R'.

Proof. We make a displacement from the point $M \in f^n_{R'}(M_0, \varepsilon)$ to the point $M_1 \in f^n_{R'+\Delta R'}(M_0, \varepsilon)$, where $0 < R' + \Delta R' < R$; then $|\Delta R'|$ will be equal to the length of the segment $M_1 P$ of the normal to $F^n_{M_0}(\varepsilon)$ included between $f^n_{R'}(M_0, \epsilon)$ and $f^n_{R'+\Delta R'}(M_0, \varepsilon)$.

We designate $\measuredangle \underset{\sim}{e}, \underset{\sim}{n}(M) = \alpha$, then in the triangle MM_1P $\angle MM_1P = \alpha + \delta$, ($\delta \to 0$ as $M_1 \to M$), if $\alpha \le \frac{\pi}{2}$, and $\angle MM_1P = \pi - \alpha + \delta$, ($\delta \to 0$, as $M_1 \to M$), if $\alpha > \frac{\pi}{2}$. We designate also $\angle MPM_1 = \beta$, $\beta \to \frac{\pi}{2}$ as $M_1 \to M$.

In case $\alpha \le \frac{\pi}{2}$, we have: $\Delta R' = \Delta \tilde{\rho} = PM_1$ and

$$\lim_{M_1 \to M} \frac{PM_1}{MM_1} = \lim_{M_1 \to M} \frac{\sin[\pi - (\alpha + \beta + \delta)]}{\sin \beta}$$

$$= \sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha.$$

229 In case $\alpha > \frac{\pi}{2}$, we have: $\Delta R' = \Delta\tilde{\rho} = -PM_1$, ($PM_1$ is the length of the segment) and

$$\lim_{M_1 \to M} \frac{-PM_1}{MM_1} = -\lim_{M_1 \to M} \frac{\sin(\alpha - \beta - \delta)}{\sin\beta}$$

$$= -\sin\left(\alpha - \frac{\pi}{2}\right) = \cos\alpha.$$

Since $\underset{\sim}{n}(M)$ is a continuous function of the point $M \in H^{n+1}(M_0, \varepsilon)$, from lemma 3:13 there follows

Lemma 3:14. The derivative

$$\frac{\partial \tilde{\rho}(M, F^n_{M_0}(\varepsilon))}{\partial e}$$

for a fixed direction $\underset{\sim}{e}$ is a C' function of the point M in the region $H^{n+1}(M_0, \varepsilon)$.

Lemma 3:15. Let the point $N \in E(F^n)$, $F^n \in \hat{F}^n_R$ and NN_i, $i = 1, \cdots, k$ be all the segments realizing the distance $\rho(N, F^n)$, ε an arbitrarily small positive number. Then there is $\delta > 0$ such that if the point $M \in K^{n+1}_\delta(N)$ (sphere), then for all segments MM_j realizing $\rho(M, F^n)$, there holds the relation:

$$M_j \in \bigcup_{i=1}^{k} F^n_{N_i}(\varepsilon) \quad \text{(nbhd of } N_i \text{ in } M\text{)} \qquad (3:15,1)$$

Proof. We suppose that lemma 3:15 is not true; then there is a sequence of points $M_p \to N$ and a sequence of segments $M_p M'_p$ realizing $\rho(M_p, F^n)$ and such that

[This should be its opposite, $\bar{\varepsilon}$]

$$M'_p \in \bigcup_{i=1}^{k} F^n_{N_i}(\varepsilon). \qquad (3:15,2)$$

From the sequence of segments $M_p M'_p$ can be chosen a subsequence $M_{p_i} M'_{p_i}$ converging to some segment NM', for which from (3:15,2) follows that NM' does not coincide with one of the segments NN_i, $i = 1, \cdots, k$ and simultaneously, due to the continuity of the distance function,

$\rho(N, M') = \rho(N, F^n)$.

The contradiction obtained proves the lemma.

3:16. Keeping the notation of lemma 3:15, we choose $\varepsilon > 0$ so small that the family of surfaces $f^n_{R'}(N_i, \varepsilon)$, $i = 1, \dots, k$, $0 < R' < R$ and the fields of normals to $F^n_{N_i}(\varepsilon)$ should enjoy the properties described in lemma (?) 3:11. Thereon we choose $\delta > 0$ sufficiently small in order that lemma 3:15 should hold and in order that the ball $K^{n+1}_\delta(N)$ should belong to the intersection $\bigcap_{i=1}^{k} H^{n+1}(N_i, \varepsilon)$ (cf. 3:12). Then for any point $M \in K^{n+1}_\delta(N)$ $\rho(M, F^n) \geq \rho(M, F^n_{N_i(\varepsilon)})$, $i = 1, \dots, k$, by which for some i $\rho(M, F^n) = \rho(M, F^n_{N_i}(\varepsilon))$. Consequently, the distance from M to $F^n \in \hat{F}^n_R$ equals the length of normals to some of the surfaces $F^n_{N_i}(\varepsilon)$ $(i = 1, \dots, k)$, extended from M. For such a surface $F^n_{N_i}(\varepsilon)$ the inequality $\rho(M, F^n_{N_i}(\varepsilon)) = \tilde{\rho}(M, F^n_{N_i}(\varepsilon))$ (cf. points 3:13, 3:14) should hold. For a central point $N' \in Z(F^n) \cap K^{n+1}_\delta(N)$,

230 by definition, there exist two groups of indices $i_1, \dots, i_t$ $(t \geq 2)$ and $j_1, \dots, j_s$ $(i_\alpha \neq j_\beta$, $\alpha = 1, \dots, t$, $\beta = 1, \dots, s)$ so that $\rho(N', F^n) = \rho(N', F^n_{M_{i_\alpha}}(\varepsilon)) < \rho(N', F^n_{M_{j_\beta}}(\varepsilon))$

$$(\alpha = 1, \dots, t, \ \beta = 1, \dots, s), \qquad (3{:}16, 1)$$

from which it is seen that the set $Z(F^n) \cap K^{n+1}_\delta(N)$

is contained in the union of sets $W_{ij} \subset K_\delta^{n+1}(N)$, $i,j = 1, \dots, k$, $i \neq j$, defined by equations

$$\tilde{\rho}(M, F^n_{N_i}(\varepsilon)) = \tilde{\rho}(M, F^n_{N_j}(\varepsilon)). \qquad (3:16,2)$$

Equations (3:16,2) can be rewritten in the following form:

$$f_{ij}(M) \equiv \tilde{\rho}(M, F^n_{N_i}(\varepsilon)) - \tilde{\rho}(M, F^n_{N_j}(\varepsilon)) = 0. \qquad (3:16,3)$$

The functions $f_{ij}(M)$ are C^1, as proved in lemmas 3:13, 3:14.

We prove that for a point $M \in K_\delta^{n+1}(N)$

$$0 \leq \measuredangle\, \overline{M_i M}, \overline{M M_j} < \pi. \qquad (3:16,4)$$

If $\measuredangle\, \overline{M_i M}, \overline{M M_j}$ were equal to π, then the points M_i, M_j would lie on the same ray issuing from M. By the conditions, points M_i, M_j are distinct. This means that either $M_i \in MM_j$ or $M_j \in MM_i$. In both cases one of the points M_i, M_j is found to be an interior point of segments MM_j or MM_i. Since all interior points of segments MM_i, MM_j, by definition of points M_i, M_j, are points of the open body $T(F^n)$, but points $M_i, M_j \in F^n$, a contradiction is obtained. That is, inequality (3:16,4) is proved.

We take a unit vector $\underset{\sim}{e}_{n+1}$ with origin at the point $N \in Z(F^n)$, lying in the plane of vectors $\overline{N_i N}, \overline{N N_j}$

and forming with $\overline{NN_i}$, $\overline{NN_j}$ the ~~unique~~ same angle:

$$\measuredangle\, e_{n+1}, \overline{N_iN} = \measuredangle\, e_{n+1}, \overline{NN_j} = \varphi.$$

From the inequality (3:16,4) follows the inequality $\varphi \neq \frac{\pi}{2}$; changing, in case it is necessary, the direction of e_{n+1} to the opposite, it can be assumed that $0 \leq \varphi < \frac{\pi}{2}$.

According to lemma 3:13 and relation (3:16,3), we get

$$\frac{\partial f_{ij}(N)}{\partial e_{n+1}} = \cos\varphi - \cos(\pi - \varphi) = 2\cos\varphi > 0. \quad (3:16,5)$$

We join to vector e_{n+1} arbitrary unit vectors $e_1, \ldots, e_n$ forming together with e_{n+1} an orthonormal frame, and we introduce with the aid of the frame $e_1, \ldots, e_{n+1}$ a rectangular system of coordinates $(x^1, \ldots, x^{n+1})$ with origin at the point N. Then, due to (3:16,5) and the special choice of system of coordinates,

$$\frac{\partial f_{ij}(N(0,\ldots,0))}{\partial e_{n+1}} = \frac{\partial f_{ij}(0,\ldots,0)}{\partial x^{n+1}} > 0, \quad (3:16,6)$$

that is, the equations

$$f_{ij}(M(x^1, \ldots, x^{n+1})) \equiv f_{ij}(x^1, \ldots, x^{n+1}) = 0$$

231 define in some neighborhood of the point N a C^1 surface without singular points, which can be represented

by an equation of the form:

$$x^{n+1} = \varphi(x^1, \dots, x^n). \qquad (3:16,7)$$

If $\delta > 0$ is taken sufficiently small, then W_{ij} = the intersection of surfaces (3:16,7) with $K_\delta^{n+1}(N)$ is a C^1 surface without singular points, homeomorphic to an n-dimensional ball.

For any unit vector e_k, $k \neq n+1$, we have:

$$e_k \perp e_{n+1}, \quad \measuredangle\, e_k, \overline{N_i N} = \measuredangle\, e_k, \overline{NN_j}.$$

We designate the common value of the two sequences of angles by φ_k ($k = 1, \dots, n$). Then

$$\frac{\partial f_{ij}(N)}{\partial e_k} = \frac{\partial f_{ij}(0, \dots, 0)}{\partial x^k} = \cos\varphi_k - \cos\varphi_k = 0; \qquad k = 1, \dots, n. \qquad (3:16,8)$$

The relations (3:16,6), (3:16,8) show that e_{n+1} is the normal to W_{ij} at the point N. This circumstance will be used in the second part of our work.

3:17. Let the point N, considered in the preceeding point, have multiplicity two.

Then the ball $K_\delta^{n+1}(N)$ can be represented in the form of a union of three nonintersecting sets:

$$W_{12}:\ \tilde{\rho}(M, F^n_{N_1}(\varepsilon)) = \tilde{\rho}(M, F^n_{N_2}(\varepsilon)),\quad \rho(M,N)<\delta,$$

$$W^{(1)}_{(2)}:\ \tilde{\rho}(M, F^n_{N_1}(\varepsilon)) > \tilde{\rho}(M, F^n_{N_2}(\varepsilon)),\quad \rho(M,N)<\delta,$$

$$W^{(2)}_{(1)}:\ \tilde{\rho}(M, F^n_{N_1}(\varepsilon)) < \tilde{\rho}(M, F^n_{N_2}(\varepsilon)),\quad \rho(M,N)<\delta.$$

From the results of the preceeding point and, in particular, relation (3:16,1), it follows that all points of the sets $W^{(1)}_{(2)}$, $W^{(2)}_{(1)}$ are simple.

Now let $M \in W_{12}$; then $M \in K^{n+1}_\delta(N)$ and, consequently, $\rho(M, F^n)$ equals one of the distances $\rho(M, F^n_{N_1}(\varepsilon)), \rho(M, F^n_{N_2}(\varepsilon))$, for example, $\rho(M, F^n_{N_1}(\varepsilon))$. But then, as was proved in point 3:16, $\rho(M, F^n_{N_1}(\varepsilon)) = \tilde{\rho}(M, F^n_{N_1}(\varepsilon))$ and, according to the definition of W_{12}, $\rho(M, F^n_{N_1}(\varepsilon)) = \rho(M, F^n_{N_2}(\varepsilon))$.

Thus, $W_{12} \subset Z(F^n)$.

Hence, if N is a double point of the central set $Z(F^n)$, $F^n \in \mathcal{F}^n_R$, then a sufficiently small neighborhood $K^{n+1}_\delta(N) \cap Z(F^n)$ of the point N relative to the set $Z(F^n)$ itself represents an n-dimensional C^1 surface W_{12}; for sufficiently small $\delta > 0$ W_{12}, evidently, is homeomorphic to an n-dimensional ball.

3:18 We suppose that the multiplicity of the set $Z(F^n)$ of a surface $F^n \in \hat{F}^n_R$ equals two. According to lemma 3:7, $Z(F^n)$ is a closed set. In point 3:17 it is proved that each point $N \in Z(F^n)$ has a neighborhood

232 $Z(F^n) \cap K^{n+1}_\delta(N)$, homeomorphic to an n-dimensional ball. Consequently, $Z(F^n)$ is an n-dimensional closed manifold lying in euclidean space E^{n+1}

Due to the theorem of Jordan-Brouwer ([1] p 562, [3.411]) $Z(F^n)$ separates ~~decomposes~~ E^{n+1}, which contradicts lemma 3:2. Hence, it is proved that the multiplicity of the set $Z(F^n)$ of any flattened surface F^n is no less than three. Using lemma 3:10, we conclude that in any surface $F^n \in \hat{F}^n_R$ there can be imbedded an $(n+1)$-dimensional ball of radius $r > R(\frac{2}{\sqrt{3}} - 1)$. Since in any surface of class F^n_R which is not flattened, there can be imbedded an n-dimensional ball of radius $R > R(\frac{2}{\sqrt{3}} - 1)$, the first assertion of the basic theorem is proved.

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