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On the greatest ball imbedded in a closed surface, II.

In the first part of the work (cf. (1)) ~~it was~~ was proved the following assertions: in any surface $F^n \in F^n_R$ there can be imbedded an $(n+1)$-dimensional ball of radius $r > R\left(\frac{2}{\sqrt{3}} - 1\right)$.

As will be ~~showed~~ proved below, this bound is sharp for all classes F^n_R; that is, for any $\varepsilon > 0$ there will be constructed a surface $F^n \in F^n_R$ in which it is impossible to inscribe a ball of radius $R\left(\frac{2}{\sqrt{3}} - 1\right) + \varepsilon$. *

We construct to begin some special pieces of curves and surfaces.

1. Lemma 1. There exists C^3 plane curves $\mathcal{L}_*$, represented by equation $y = f_*(x)$, $-\infty < x < \infty$, where $f_*(x)$ is an even function such that

$$f_*(x) = |x| \quad \text{for } |x| \geq 1. \tag{1}$$

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* After ~~the printing~~ delivering in a speech of the first part of the work (1) V. Ya. Skorobogat'ko communicated to us that the idea of a central set was used by him under the name "bisectoral surface" in the work: Bisectoral surface and its properties, Ukr. matem. J., 9 (1957), 215-219. The results of this work refer to polyhedra and are not connected to the questions considered by us.

Proof. We set

$$f_*(x) = \begin{cases} -x & \cancel{(x \le 1)} \ (x \le -1) \\ a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 & (-1 < x < 1) \\ x & (x \ge 1). \end{cases}$$

It is easy to see that coefficients a_0, a_2, a_4, a_6 can be selected so that $f_*(x)$ will be C^3 on all of the x-axis. Property (1) and the evenness of the function $f_*(x)$ results from the definition of $f_*(x)$.

$$\left[f_*(x) = \frac{1}{16}(5 + 15x^2 - 5x^4 + x^6) \right].$$

2. We take in E^{n+1} an orthonormal frame $e_1, \ldots, e_{n+1}$ and consider the following vector equation of the curve L:

$$u(x) = x e_1 + f_*(x) e_2 . \qquad (2)$$

The vector

$$u'(x) = e_1 + f_*'(x) e_2$$

is not collinear with e_2 for $x \in (-\infty, \infty)$. The vector

$$v(x) = \frac{f_*'(x) e_1 - e_2}{\sqrt{1 + f_*'^2(x)}}$$

is the unit normal to L_*, C^2 for x on $(-\infty, \infty)$.

875 We construct the vector-function

$$w(x, \alpha^2, \ldots, \alpha^{n+1}) = v(x)\alpha^2 + \sum_{i=3}^{n+1} e_i \alpha^i,$$

where $\sum_{i=2}^{n+1} (\alpha^i)^2 = 1$.

For fixed x and variable α^i the end of the vector

$$\underset{\sim}{u}(x) + \underset{\sim}{W}(x, \alpha^2, \ldots, \alpha^{n+1})$$

runs through an $(n-1)$-dimensional unit sphere with center at the point $\underset{\sim}{u}(x)$, lying in the n-dimensional plane normal to the curve L_* at the point $\underset{\sim}{u}(x)$.

Let

$$0 < \varepsilon < \frac{1}{2 \max\limits_{-\infty<x<\infty} |f_*'(x)|} \left[= \tfrac{1}{2} \text{ since } f_*'' \geq 0.\right]$$

It is easy to prove that the equation

$$\underset{\sim}{r} = \underset{\sim}{u}(x) + H\underset{\sim}{W}(x, \alpha^2, \ldots, \alpha^{n+1}),\ x \in (-\infty, \infty), \quad (3)$$

where $0 < H < \varepsilon$, defines a C^2 surface without singular points, not having (in the whole) points of self-intersection.

The part of the surface (3) (curve (2)) corresponding to values $x \in [-1-a, 1+b]$, $a > 0$, $b > 0$, we designate by $S_H(a,b)$ (by $L_*(a,b)$). The surface, obtained from $S_H(a,b)$ by a similarity transformation with coefficient q, we will designate by $S_{H,q}(a,b)$.

The curve, obtained from $L_*(a,b)$ by the same transformation, we will ~~be~~ designate by $L_{*q}(a,b)$.

Let $H < \varepsilon$, $0 \leq \theta < \min(H, \varepsilon - H)$. Then the surfaces $S_{H+\theta,q}(a,b)$, $S_{H-\theta,q}(a,b)$ also will be C^2 without self intersections.

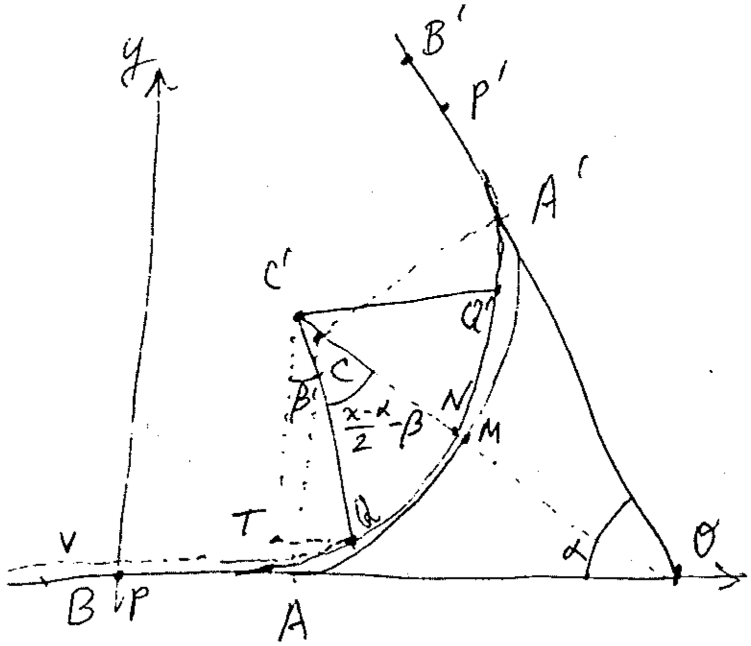
We will call points <u>corresponding</u> on the surfaces $S_{H\pm\theta,q}(a,b)$ when they have the same values of the parameters $x, \alpha^2, \dots, \alpha^{n+1}$. Since the principal curvatures of the surfaces $S_{H,q}(a,b)$ are continuously dependent on H, corresponding principal curvatures of the surfaces $S_{H\pm\theta,q}(a,b)$ (taken at corresponding points) as $\theta \to 0$ and with nonchanging q tend to the corresponding curvatures of the surface $S_{H,q}(a,b)$ at the corresponding points. It is known that for a similarity transformation of a surface with coefficient q the normal sectional curvatures are multiplied by $1/q$. From the two latter assertions follows

<u>Lemma 2. For any R and sufficiently small $\delta > 0$ there exists a pair of surfaces $S_{H\pm\theta,q}(a,b)$ so that the radius of curvature of any normal section in any point of the surfaces will be greater than R, and $\theta < \delta$.</u>

Concerning this, $S_H(a,b)$ has continuous 2nd derivatives and, due to compactness, bounded principal curvatures, by some number A. Subjecting $S_H(a,b)$ to a similarity transformation with underlying coefficient q, we obtain the surface $S_{H,q}(a,b)$

with principal normal curvatures in all points less than $1/2R$. Fixing q and choosing θ sufficiently small, we obtain the surfaces $S_{H\pm\theta,q}(a,b)$ sought for.

87b 3. We consider in the 2-dimensional plane an angle $\angle BOB' = \alpha$, where $0<\alpha<\pi$. We construct the circle of radius $R = AC$, tangent to both sides of the angle at points A and A' (cf. diag. 1). The curve, formed by the ray AB, the arc AA' of the circle and the ray $A'B'$, we designate by $L(\alpha, R)$. At the points A and A' on $L(\alpha, R)$ the curvature does not exist. It is not hard to construct along with $L(\alpha, R)$ a curve $L_\varepsilon(\alpha, R)$ satisfying such conditions



Diag 1.

(a) $L_\varepsilon(\alpha, R)$ has curvature everywhere, not exceeding $1/R$;

b) $L_\varepsilon(\alpha, R)$ is a convex curve, consisting of rays PB, $P'B'$, belonging to the sides of the angle $\angle BOB'$, arc QQ' of a circle of radius $QC' = R$, for which C' lies on the bisector OC of the angle $\angle BOB'$ and two arcs PQ and $P'Q'$, symmetric relative to OC; on which the curvature continuously decreases from the value $1/R$ at the points Q and Q' to the value 0 at the points P and P';

c) for any $\varepsilon > 0$ the distance from O to $L_\varepsilon(\alpha, R)$ exceeds the distance from O to $L(\alpha, R)$ less than $\varepsilon/2$.

4. In the twodimensional plane E^2 we take a curve $L_\varepsilon(\alpha, R)$ and at the points P and P' (cf. diag. 1) we extend normals to $L_\varepsilon(\alpha, R)$ up to the intersection at the point D. We make a similarity transformation with center D and coefficient $1+\omega$, where $\omega > 0$ and such that the distance between the point P and its image P_ω under the transformation equals an arbitrary number $\delta > 0$. The image of the curve $L_\varepsilon(\alpha, R)$ under the transformation we designate by $L_{\varepsilon,\omega}(\alpha, R)$. We lay off from the point P a segment PP_1 of length m in the direction OP. Analogously we lay off segment $P'P_1'$ of length m' in the direction OP'. By intersecting the curves $L_\varepsilon(\alpha, R)$, $L_{\varepsilon,\omega}(\alpha, R)$ with the angle, bounded by the normals to $L_\varepsilon(\alpha, R)$ at the points P_1, P_1' and containing the point O, there are two arcs formed, which we designate correspondingly $L_\varepsilon^{(1)}(\alpha, R)$, $L_\varepsilon^{(2)}(\alpha, R)$.

5. In the space E^{n+1} we take an orthonormal frame $e_1, \dots, e_{n+1}$ and in the plane E^n spanned by e_1, e_2 we lay out the curve $L_\varepsilon^{(1)}\left(\frac{\pi}{2}, R\right)$ ~~...~~ so that the segment PP_1 of the curve $L_\varepsilon^{(1)}(\frac{\pi}{2}, R)$ of length m is in the direction of e_1 and displaced from e_1 by R, and segment $P'P_1'$ of the same curve of length m_1 is in the of vector e_2.

We take e_i along the axes of rectangular coordinates x^i $(i = 1, \dots, n+1)$. We form the arc $L_\varepsilon^{(1)}(\alpha, R)$ by an equation of the form

$$\varphi(x^1, x^2) = 0,$$

where is a C^2 function of x^1, x^2 for which

$$\left(\frac{\partial\varphi}{\partial x^1}\right)^2 + \left(\frac{\partial\varphi}{\partial x^2}\right)^2 > 0$$

877 for all points $L_\varepsilon^{(1)}(\alpha, R)$. The equation

$$\varphi\left(x^1, \sqrt{\sum_{k=2}^{n+1} (x^k)^2}\right) = 0 \qquad (4)$$

defines in E^{n+1}, as is not hard to prove, a C^2 surface without singular points. Surface (4) we designate by ${}^{(1)}S_\varepsilon^n(\alpha, R)$.

Now we examine the principal curvatures of the surface given by equation (4). We choose an arbitrary point M on ${}^{(1)}S^n_\varepsilon(\alpha, R)$. Without limiting generality we can assume that the point M has coordinates $(0, x^2, 0, \cdots, 0)$ (for this it is only needed to completely parallel translate the system of coordinates along the x^1-axis and rotate about this same axis). The unit normal to the surface ${}^{(1)}S^n_\varepsilon(\alpha, R)$ is

$$\underset{\sim}{n} = \lambda\left[\frac{\partial\varphi}{\partial x^1}\underset{\sim}{e_1} + \frac{1}{r}\frac{\partial\varphi}{\partial x^2}\sum_{k=2}^{n+1} x^k \underset{\sim}{e_k}\right], \quad (5)$$

where
$$r = \sqrt{\sum_{k=2}^{n+1}(x^k)^2},$$

and λ is the normalizing multiplier. For displacements in the twodimensional plane $E^2_{1,2}$ containing $\underset{\sim}{e_1}, \underset{\sim}{e_2}$, or in the twodimensional plane E^2_k, containing $\underset{\sim}{n}, \underset{\sim}{e_k}$ $(k = 3, \cdots, n+1)$, all coordinates x^i $(i \neq 1, 2, k)$ equal zero, so that, due to (5), $\underset{\sim}{n}$ remains in the three-dim'l plane E^3_{12k}, containing vectors $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_k}$. The intersection of E^3_{12k} with the surface ${}^{(1)}S^n_\varepsilon(\alpha, R)$ is the surface of revolution in three-dim'l space given by equations

$$\varphi\left(x^1, \sqrt{(x^2)^2 + (x^k)^2}\right) = 0.$$

It is known ((2), p.93) that on such a surface the principal normal sections are the curves

$$\varphi(x^1, x^2) = 0, \quad x^k = 0$$

and the circle
$$\varphi\left(0, \sqrt{(x^2)^2 + (x^k)^2}\right) = 0, \quad x^1 = 0$$

of radius $\sqrt{(x^2)^2+(x^k)^2}$. Consequently, the principal curvatures at point M equal the corresponding curvature of the curve $\varphi(x^1, x^2)=0$ at this same point and the numbers

$$\frac{1}{\sqrt{(x^2)^2+(x^k)^2}} \leq \frac{1}{x^2}, \quad k = 3, \cdots, n+1.$$

[This is not correct. The circle $x^1=0$ is not the normal section unless $\underset{\sim}{n}$ is $\perp \underset{\sim}{e}_1$. To account for the difference we have to multiply $\frac{1}{\sqrt{(x^2)^2+(x^k)^2}}$ (which equals $\frac{1}{x^2}$ since $x^k=0$ at M) by $\underset{\sim}{n}\,\underset{\sim}{e}_1 = \sin \measuredangle\, \underset{\sim}{n}, \underset{\sim}{e}_1$ to get the normal curvature. But the inequality still holds and is all that is needed.]

We will say that the surface ${}^{(1)}S^n_\varepsilon(\alpha, R)$ is obtained by rotating arc $L^1_\varepsilon(\alpha, R)$ around the x^1-axis, and below we will use such methods to construct surfaces.

878 The rotation of arc $L^{(2)}_\varepsilon(\alpha, R)$ around the x^1-axis gives a surface which we designate by ${}^{(2)}S^n_\varepsilon(\alpha, R)$.

The pair of curves $L^{(1)}_\varepsilon(\alpha, R)$, $L^{(2)}_\varepsilon(\alpha, R)$ will be designated below $L_{\varepsilon,\delta}(\alpha, R)$ (the parameter δ is defined in point 4).

The pair of surfaces ${}^{(1)}S^n_\varepsilon(\alpha, R)$, ${}^{(2)}S^n_\varepsilon(\alpha, R)$ we designate $S^n_{\varepsilon,\delta}(\alpha, R)$.

6. We introduce notation, with the aid of which we will fix the position in the space E^{n+1} of curves and surfaces constructed by us.

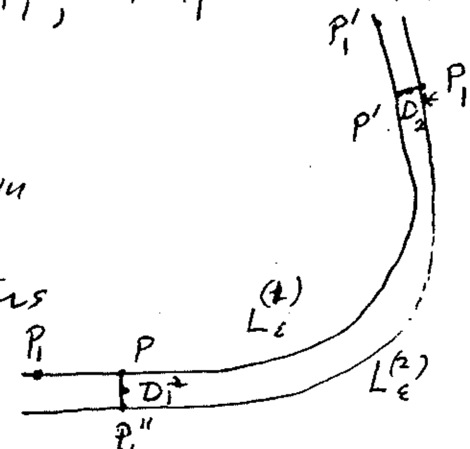
We equip euclidean space E^{n+1} with a rectangular system of coordinates $x^1, \dots, x^{n+1}$, corresponding to an orthonormal frame $\underset{\sim}{e}_1, \dots, \underset{\sim}{e}_{n+1}$.

We consider a pair of tubular surfaces $S^n_{H\theta}(a,b)$, $S^n_{H+\theta}(a,b)$, and the surface $S^n_H(a,b)$ corresponding to them and the curve $L_*(a,b)$ (cf. point 2). We produce a motion of space E^{n+1}, by which $L_*(a,b)$ is moved to arc $\tilde{L}_*$ with ends at points M_1, M_2 and tangent vectors in these points $\underset{\sim}{t}_1, \underset{\sim}{t}_2$, directed to the interior of the arc $\tilde{L}_*$. For the specified motion the surfaces $S^n_{H\pm\theta}(a,b)$ are moved to a pair of surfaces, which we designate by

$$S^n_{H\pm\theta}(a\sqrt{2}, b\sqrt{2}, M_1, M_2, \underset{\sim}{t}_1, \underset{\sim}{t}_2). \qquad (6)$$

We note that with the aid of similarity transformations there can be obtained a pair of surfaces (6) with any positive H and θ, satisfying the inequality $\theta < H$; the numbers a and b can be taken completely arbitrarily.

We consider now the pair of curves $L_\varepsilon^{(1)}(\alpha, R)$, $L_\varepsilon^{(2)}(\alpha, R)$ defined in point 4. We construct segments P_1P_1'', $P_1'P_1'''$, perpendicular respectively to PP_1, $P'P_1'$ and having ends P_1'', P_1''' on $L_\varepsilon^{(2)}(\alpha, R)$.



The midpoints of segments P_1P_1'', $P_1'P_1'''$ we designate by D_1^*, D_2^*. The vectors with origin at points D_1^* and D_2^* perpendicular to P_1P_1'' and $P_1'P_1'''$ and directed interior to the region bounded by the curves $L_2^{(1)}(\alpha, R)$, $L_\varepsilon^{(2)}(\alpha, R)$ and segments P_1P_1'', $P_1'P_1'''$, we designate by $\underset{\sim}{s_1}$ and $\underset{\sim}{s_2}$. Let some motion of E^{n+1} move points D_1^* and D_2^* to points D_1 and D_2 vectors $\underset{\sim}{s_1}$ and $\underset{\sim}{s_2}$ to vectors $\underset{\sim}{t_1}$ and $\underset{\sim}{t_2}$. By the specified motion the pair of curves $L_\varepsilon^{(1)}(\alpha, R)$, $L_\varepsilon^{(2)}(\alpha, R)$ are moved to a pair of curves which we designate by

$$L_\delta(\alpha, R, D_1, D_2, \underset{\sim}{t_1}, \underset{\sim}{t_2}). \qquad (7)$$

We consider the particular case $\alpha = \frac{\pi}{2}$. Let the point D_2 be taken at distance H from axis τ, parallel to vector $\underset{\sim}{t_2}$, and point D_1 at distance h from the same axis τ. The orthogonal projection of points D_1 and D_2 on axis τ we designate by B_1 and B_2. We set $\underset{\sim}{s}_3 = \frac{1}{3}\overline{B_1B_2}$, $\underset{\sim}{s}_4 = \frac{1}{3}\overline{B_2B_1}$. We note that by the underlying choice of segment m (cf. point 4) it can be arranged that h should be equal to any number not less than some h_0.

Rotating the pair of curves (7) around axis τ, we obtain a pair of n-dim'l surfaces. By some motion of E^{n+1}, carrying B_1, B_2, $\underset{\sim}{s}_3$ and $\underset{\sim}{s}_4$ to C_1, C_2, $\underset{\sim}{t}_3$ and $\underset{\sim}{t}_4$, these surfaces are moved to a pair of surfaces which we designate by

$$S^n_{H,S}(h, C_1, C_2, \underset{\sim}{t}_3, \underset{\sim}{t}_4). \tag{8}$$

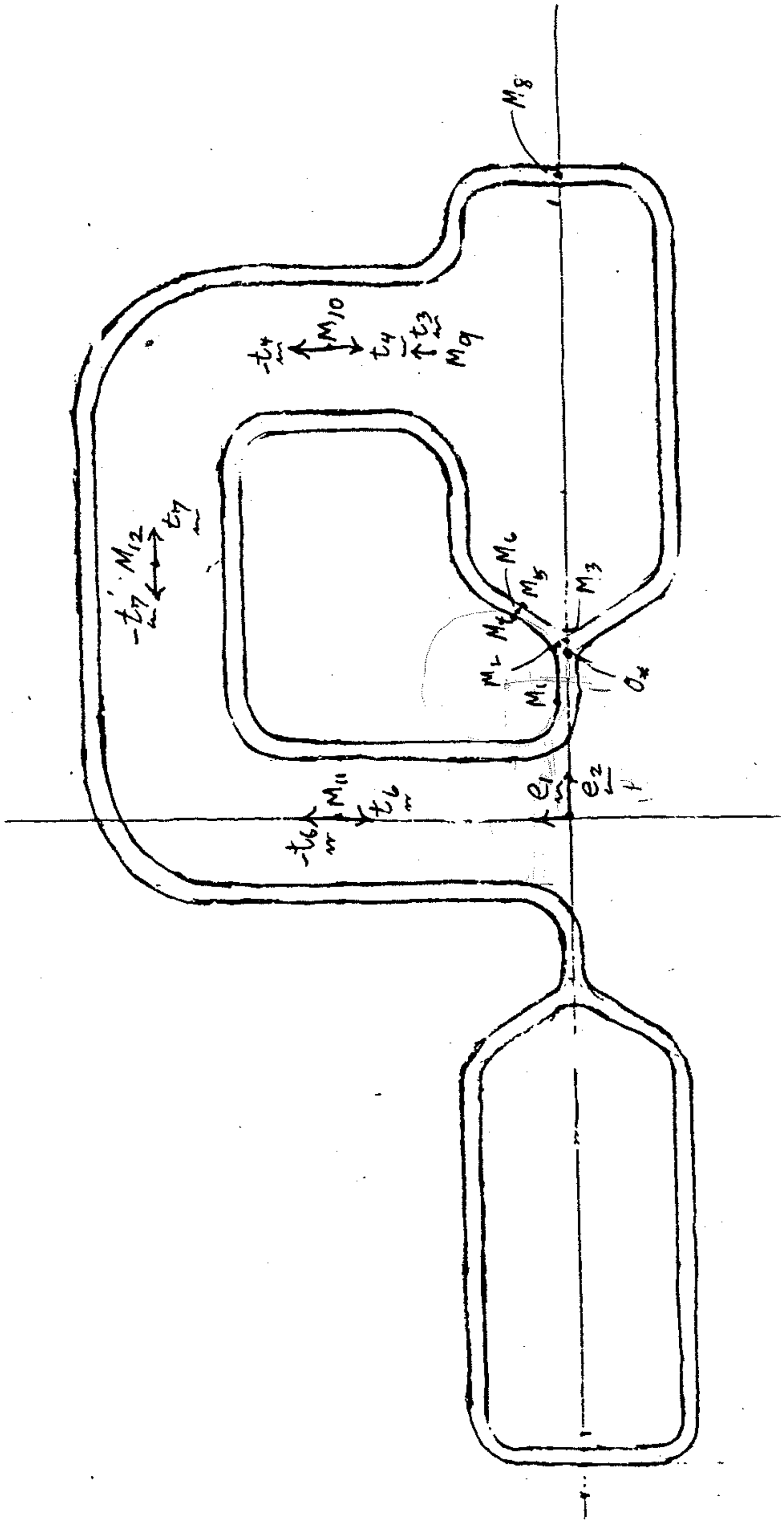
879 From the results of point 5. it is obtained that the surfaces (8) are C^2 and do not have singular points.

7. We construct in the plane E^2 spanned by vectors $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ the following points (cf. diag 2, p 13):

$$M_1(\mu, c),\ M_2(\mu, c+c'),\ M_3(0, c+c'+\mu\sqrt{3}),$$

$$M_4\left(\mu + c'\frac{\sqrt{3}}{2}, c+\frac{3}{2}c'\right),\ M_5\left(c'\frac{\sqrt{3}}{2}, c+\frac{3}{2}c'+\mu\sqrt{3}\right),$$

$$M_6\left(\frac{\mu}{2} + c'\frac{\sqrt{3}}{2}, c+\frac{3}{2}c'+\mu\frac{\sqrt{3}}{2}\right).$$



Diag. 2

The points of the plane E^2, symmetric to the points M_i $(i=1,\dots,6,\ i\neq 3)$, are M_i'. We construct segments M_1M_2, M_2M_2, M_3M_5 and the segments symmetric to them relative to the x^2-axis. Moreover, we construct a pair of curves $L_1 = L_{2\mu}(\frac{2}{3}\pi, R, M_6, M_7, \underset{\sim}{t_1}, \underset{\sim}{t_2})$, where $\underset{\sim}{t_1} = \overline{M_2M_6} + \overline{M_3M_6}$, the direction $\underset{\sim}{t_2}$ and the line containing $\underset{\sim}{t_2}$ are uniquely defined by the given $\underset{\sim}{t_1}$, M_6, and in effect M_7 can be chosen as the point on the line containing the vector $\underset{\sim}{t_2}$ with sufficiently large coordinate x_7^2. We construct the pair of curves

$$L_2 = L_{2\mu}\left(\frac{\pi}{2}, R, M_7, M_8, -\underset{\sim}{t_2}, \underset{\sim}{t_3}\right),$$

where the direction of the vector $\underset{\sim}{t_3}$ and the line containing $\underset{\sim}{t_3}$ are uniquely defined by the given M_7, $\underset{\sim}{t_2}$ and M_8 chosen on the x^2-axis. The pair of curves, symmetric to the two curves L_1 and L_2 relative to the x^2 axis, we designate by L_1' and L_2'. We set

$$L = M_1M_2 \cup M_2M_4 \cup M_1'M_2' \cup M_2'M_4' \cup L_1 \cup L_2 \cup L_2' \cup L_1'. \quad (9)$$

Rotating the curve L around the x^1-axis (cf. point 5), we obtain two surfaces: a closed surface Φ_1^n and a nonclosed surface Φ_2^n with boundary consisting of two spheres S_c^n and $S_c^{n'}$, given by equations

$$x^1 = \pm\mu \text{ and } \sum_{k=2}^{n+1} (x^k)^2 = c^2.$$

The curvature of the curve L in each point, where it exists, by construction does not exceed $1/R$. According to point 5 the principal normal curvatures
890 of the surfaces Φ_1^n, Φ_2^n in every point, where they exist, that is everywhere besides the points of the spheres

$$x^1 = \pm\mu, \quad \sum_{k=6}^{n+1} (x^k)^2 = (c+c')^2,$$

$k = 6$ [sic]
[probably should be 2]

$$x^1 = 0, \quad \sum_{k=2}^{n+1} (x^k)^2 = (c+c' + \mu\sqrt{3})^2,$$

do not exceed the number $\max(1/c, 1/R)$, not depending on μ.

We now choose the number c and coordinate x_7^2 of the point M_7 so large that

$$c > 5R, \quad c > h_0, \quad x_7^2 > 3c + \tfrac{3}{2}c' + 4R,$$

where h_0 is the number defined in point 6. Then the surfaces Φ_1^n, Φ_2^n enjoy the following properties: the principal normal curvatures of Φ_1^n, Φ_2^n in every point, where they exist, do not exceed $1/R$.

Let the point M_q have coordinates $(x_7^1, x_7^2 - h_0, 0, \dots, 0)$ where x_7^1, x_7^2 are coordinates of the point M_7. We construct a finite cylinder $C^{n+1}(M_q)$, the axis of which is parallel to the x^1-axis and runs through the point M_q, radius equal to h_0, center placed at the point M_q, and forming an enclosure between the planes $x^1 = x_7^1 \pm 2\mu$.

It can be assumed that μ is so small that $\mu < x_7^1$. Then $C^{n+1}(M_q)$ intersects Φ_1^n and Φ_2^n in n-dim'l ~~spheres~~ balls

$$x^1 = x_7^1 \pm \mu, \quad (x^2 - x_7^2 + c)^2 + \sum_{k=2}^{n+1} (x^k)^2 \leq c^2.$$

Discarding these balls from the surfaces Φ_1^n, Φ_2^n, we obtain surfaces with boundary Ψ_1^n, Ψ_2^n.

We now construct a pair of surfaces (cf (8))

$$S^n_{H,2\mu}(c,\ M_9,\ M_{10},\ \underset{\sim}{t}_3,\ \underset{\sim}{t}_4), \qquad (10)$$

where $H > R$, vector $\underset{\sim}{t}_3$ is laid out from the point M_9 in the direction of the x'-axis, and the point M_{10} has coordinate $x'_{10} > x'_9 + 3R$.

Then we construct a pair of surfaces

$$S^n_{H,2\mu}(c,\ O,\ \underset{[\uparrow M_{11}\,?]}{N_{11}},\ \underset{\sim}{t}_5,\ \underset{\sim}{t}_6), \qquad (11)$$

~~where $H > R$, vector $\underset{\sim}{t}_3$ is lo~~

where O is the origin of coordinates, H is the same as for surfaces (10), $\underset{\sim}{t}_5 = \underset{\sim}{e}_1$, $x'_{11} = x'_{10}$.

We construct, further, two pairs of surfaces (cf. (6)),

$$S^n_{H\pm\mu}(a', b', M_{10}, M_{12}, -\underset{\sim}{t}_4, \underset{\sim}{t}_7),$$

$$S^n_{H\pm\mu}(a', b'', M_{11}, M_{12}, -\underset{\sim}{t}_6, \underset{\sim}{t}_7),$$

where H is the same as for surfaces (10), (11) and vector $\underset{\sim}{t}_7$ has the direction of the x^2-axis.

We set

$$F^n_{(1)} = \Phi^n_1 \cup \Phi'_2 \cup S^n_{H,2\mu}(c, M_9, M_{10}, \underset{\sim}{t}_3, \underset{\sim}{t}_4)$$
$$\cup\, S^n_{H,2\mu}(c, O, M_{11}, \underset{\sim}{t}_5, \underset{\sim}{t}_6) \cup S^n_{H\pm\mu}(a', b', M_{10}, M_{12}, -\underset{\sim}{t}_4, \underset{\sim}{t}_7)$$
$$\cup\, S^n_{H\pm\mu}(a', b'', M_{11}, M_{12}, -\underset{\sim}{t}_6, \underset{\sim}{t}_7).$$

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We show that $F^n_{(1)}$ is a connected surface.

The surfaces, coming from the pair of surfaces (10), (11) and corresponding to ${}^{(i)}S^n_{\varepsilon}(\alpha, R)$, $i = 1, 2$ (cf. point 5), we designate by

$$ {}^{(i)}S^n_{H,2\mu}(c, M_9, M_{10}, \underset{\sim}{t_3}, \underset{\sim}{t_4}), \quad {}^{(i)}S^n_{H,2\mu}(c, O, M_{11}, \underset{\sim}{t_5}, t_6). $$

It is easy to see that $F^n_{(1)}$ can be obtained by a sequence of joining together one to another the following connected surfaces (cf. diag. 2):

$$ \Psi^n_1, \quad {}^{(2)}S^n_{H,2\mu}(c, M_9, M_{10}, \underset{\sim}{t_3}, t_4), $$

$$ S^n_{H-\mu}(a', b', M_{10}, M_{12}, -\underset{\sim}{t_4}, \underset{\sim}{t_7}), \quad S^n_{H-\mu}(a', b'', M_{11}, M_{12}, -\underset{\sim}{t_6}, -\underset{\sim}{t_7}, $$

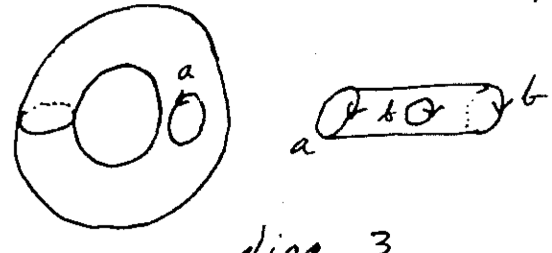
$$ {}^{(2)}S^n_{H,2\mu}(c, O, M_{11}, \underset{\sim}{t_5}, \underset{\sim}{t_6}), \quad \Psi^n_2, \quad {}^{(1)}S^n_{H,2\mu}(c, M_9, M_{10}, \underset{\sim}{t_3}, \underset{\sim}{t_4}), $$

$$ S^n_{H+\eta}(a', b', M_{10}, M_{12}, -\underset{\sim}{t_4}, \underset{\sim}{t_7}), \quad S^n_{H+\mu}(a', b'', M_{11}, M_{12}, -\underset{\sim}{t_6}, -t_7), $$

$$ {}^{(1)}S^n_{H,2\mu}(c, O, M_{11}, \underset{\sim}{t_5}, \underset{\sim}{t_6}), \quad \Psi^n_2. $$

Consequently, the surface $F^n_{(1)}$ is connected.

From only that reduction scheme of the contraction ~~of~~ of $F^n_{(1)}$ it is seen that for $n = 2$ $F^2_{(1)}$ is homeomorphic to a sphere with two handles (cf. diag. 3).



diag. 3

8. We subject the surface $F^n_{(1)}$ to a similarity transformation with center at the origin of coordinates and coefficient $q > 4$ and so large that the surface obtained, $F^n_{(2)}$, should have principal curvatures less than $1/R$ at all points where the curvature exists. We retain for the points of $F_{(2)}$ the same notation that their preimages on $F^n_{(1)}$ had. Henceforth all notation for points, arcs and so forth refer to the surface $F^n_{(2)}$.

We take a curve $L_\varepsilon(\frac{2\pi}{3}, R)$ (cf. point 3) with such a length of segment $PP_1 = P'P_1'$ so that $P_1P_1' = M_1M_2$. We construct three arcs

$$L^{(i)}_\varepsilon\left(\frac{2\pi}{3}, R\right) \quad (i = 1, 2, 3), \tag{12}$$

congruent to $L_\varepsilon(\frac{2\pi}{3}, R)$, tangent to the corresponding sides of the angles

$$\angle M_1M_2M_4, \ \angle M_5M_3M_5', \ \angle M_4'M_2'M_1'$$

and having ends, respectively, in the points

$$M_1, M_4 \ ; \ M_5, M_5' \ , \ M_4', M_1' .$$

Cutting out from L (cf. (9)) the broken line curves $M_1M_2M_4$, $M_5M_3M_5'$; $M_4'M_2'M_1'$ and changing them, respectively, to the arcs (12), we obtain pairs of curves L'.

The part of the surface $F^n_{(2)}$ formed by rotation around the x^1-axis of the broken lines $M_1M_2M_4$, $M_5M_3M_5'$, $M_4M_2'M_1'$, we change, respectively, to the surfaces formed by rotation of the arcs (12) around the x^1-axis. The surface obtained as a result of such changes we designate by F^n. The surface F^n, by construction,

882 is C^2. At all points of the intersection $F^n \cap F^n_{(2)}$ the principal curvatures are less than $1/R$. The set $F^n \setminus F^n_{(2)}$ consists of three surfaces of revolution. Due to the condition $c > 5R$ and the results of point 5, in all points of these surfaces the principal curvatures do not exceed $1/R$. Hence $F^n \in F^n_R$.

9. In the preceding points μ could be chosen arbitrarily small, for which from the previous constructions it is seen that the bounds on principal curvatures do not depend on μ. From the construction of the surface F^n it is also obtained that there exists a function $\nu(\mu) > 0$ such that

$$\lim_{\mu \to 0} \nu(\mu) = 0$$

and that for whatever point $M \in F^n \cap F^n_{(2)}$, the normal to F^n at the point M intersects F^n at a point M' near to M, standing from M not farther than $\nu(\mu)$. We choose and fix μ so that

$$\mu < \frac{\varepsilon}{4}, \quad \nu(\mu) < \frac{1}{2}R\left(\frac{2}{\sqrt{3}}-1\right). \qquad (13)$$

As we have proved F^n is a connected manifold lying in E^{n+1}. By the theorem of Jordan-Brouwer F^n separates E^{n+1} into two components, the exterior $T'(F^n)$ and the interior $T(F^n)$. We take a support n-dim'l plane E^n to the surface F^n at some point $M \in F^n \cap F^n_{(2)}$. One of the half planes defined by E^n (we call it E^{n+1}_*) does not contain points of $T(F^n)$. Therefore the normal $\underset{\sim}{n}(M)$ at the point M, exterior relative to E^{n+1}_* exits at the point M to the interior body $T(F^n)$. Since the extension of the normal, containing $\underset{\sim}{n}(M)$ on the side E^{n+1}_* does not intersect F^n, the nearest point M' to M of the normal belonging to F^n lies in the direction of $\underset{\sim}{n}(M)$ from the point M. Hence the segment MM' belongs to $\overline{T(F^n)}$. From the construction of F^n it is seen that continuously moving M on $F^n \cap F^n_{(2)}$ we obtain a family of segments MM', the ends of which cover all of the surface $F^n \cap F^n_{(2)}$. From the consideration of continuity it is clear that all such segments belong to $T(F^n)$ and cover a domain of the space E^{n+1} which we designate by $T_{(2)}(F^n)$.

Let $K^{n+1}_{r^*}$ be a ball of greatest radius r^* belonging to $T(F^n)$. From the construction of the surface $F^n \setminus F^n_{(2)}$ it is seen that a ball with center O_* and of radius $R(\frac{2}{\sqrt{3}}-1)$ belongs to $T(F^n)$.

This means

$$r^* \geq R\left(\frac{2}{\sqrt{3}}-1\right). \qquad (14)$$

If $O_* \in T_{(2)}(F^n)$, then there is a point $M \in (F^n \cap F^n_{(2)})$ such that O_* belongs to a segment MM' corresponding to M, for which, evidently, M and M' lie outside the ball $K^{n+1}_{r^*}$. Therefore

$$r^* < MM' < \nu(\mu) < \frac{1}{2}R\left(\frac{2}{\sqrt{3}}-1\right) \leq \frac{r^*}{2}$$

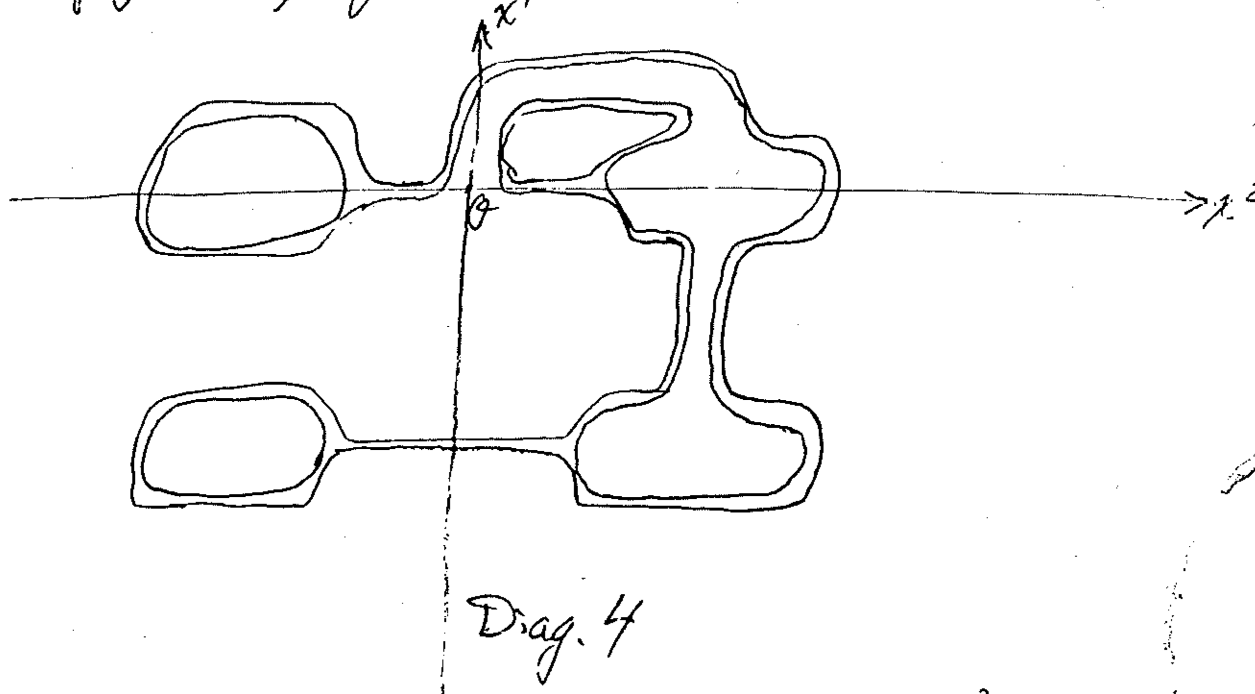
and we have reached a contradiction.

Hence $O_* \in [T(F) \setminus T_{(2)}(F^n)]$. From the definition of $K^{n+1}_{r^*}$ it is seen that $K^{n+1}_{r^*}$ must be tangent to the surface F^n. Let M_0 be a point of tangency. Then according to the definition of the domain $T_{(2)}(F^n)$, $M_0 \in [F^n \setminus F^n_{(2)}]$. Since $F^n \setminus F^n_{(2)}$ is a surface of revolution, obtained by rotating the curve (12) around the x'-axis, r^* equals the radius of the greatest ~~circle~~ disk belonging to ~~the~~ a domain $G \subset E^2$: the boundary of G consists of arcs (12) and segments $M_1'M_1$; M_4M_5; $M_5'M_4'$. The domain G is invariant with respect to rotation around the center O_* of the triangle $M_2M_3M_2'$ by angle $\frac{2}{3}\pi$; the arcs (12) turn convexly to the point O_*. From whence it easily follows that the center of the greatest disk $K^2_{r^*} \subset G$ lies at point O_*. From the construction of the curve $L_\varepsilon(\frac{2\pi}{3}, R)$ (cf. point 3) and the choice $\mu < \frac{\varepsilon}{4}$ (cf. (13)) it is seen that

$$r^* < R\left(\frac{2}{\sqrt{3}} - 1\right) + \varepsilon. \qquad (15)$$

The validity of inequalities (14) and (15) prove that the construction of examples of surfaces F^n with the required properties is finished

10. We offer still some remarks relative to the case $n=2$. From the same pieces of surfaces from which were constructed the surface F^2, can be constructed surfaces $F^2_{(k)}$ of class F^2_R of any genus ≥ 2 (in the construction above the surface F^2 had genus 2). For this one needs to join F^2 sequentially with $k-2$ pairs of surfaces Φ^n_1, Φ^n_2. The scheme of joining for $k=3$ is shown in diag. 4.



Diag. 4

Just as above it is proved that in $T(F^2_{[k]})$ it is impossible to put a ball of radius not less than $R(\frac{2}{\sqrt{3}}-1)+\varepsilon$. Thus the bound for our basic theorem cannot be improved not only for the whole class F^2_R, but also for each subclass of homeomorphic surfaces $F^2_{R,k}$ of genus k ($k \geq 2$) of the class F^2_R.

For the class $F^2_{R,1}$, as was ~~proved~~ shown at a later time by V. I. Diskant, our bound also cannot be improved.

For any dimension n there can be shown by the same examples that our found on the radius of inscribed ball is sharp for any class $F^n_{R,k}$ of surfaces with given one-dimensional Betti number k.

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