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On the largest disk included in a closed curve

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In the present note some extremal properties of a closed curve (not neccesarily convex) in the plane are proved. These properties were proposed in the form of a proposition by A. I. Fyet and proved by G. Pyestov. Subsequently V. Ionin succeeded in simplifying the very complicated initial proof of G. Pyestov. Here is exposed the proof of Pyestov's theorem proposed by V. Ionin.

Theorem 1. If the radius of curvature of a closed nonselfintersecting C^2 curve γ is always no less than R_0, then there is always a ~~disk~~ circle of radius R_0 entirely lying inside the closed domain bounded by γ.

We introduce the idea of an integral ~~disk~~ circle of curvature. We will say that a ~~disk~~ circle $C(Y)$ is an integral ~~disk~~ circle of curvature of the curve γ at point Y, if $C(Y)$ lies entirely inside γ and either the circumference of $C(Y)$ is tangent to γ at at least two distinct points, one of which is Y, or $C(Y)$ is the ordinary ~~disk~~ circle of curvature of the curve γ at the point Y. The radius $R(Y)$ of the ~~disk~~ circle $C(Y)$ we call the integral radius of curvature of the curve γ at the point Y. By $\bar{Y}$ we will designate any other point of tangency of $C(Y)$ with γ, distinct from Y. By $O(Y) = O(\bar{Y})$ we designate the center of the ~~disk~~ circle $C(Y) = C(\bar{Y})$.

We prove a sequence of lemmas, in which we will suppose that we always are proceeding under the conditions of Theorem 1.

Lemma 1. <u>For any point $Y \in \gamma$ there exists an integral ~~disk~~ circle of curvature $C(Y)$ and the radius of this ~~disk~~ circle is different from zero.</u>

Proof.

A. We designate by R_1 the radius of curvature of γ at Y. By the usual methods of differential geometry it can be proved that there exists an arc l such that any ~~disk~~ circle of radius $R = R_1/2$ tangent to γ at Y does not have points in common with l besides Y. Further, let m be the minimum distance from Y to points of γ not belonging to l. Since γ is nonselfintersecting, $m > 0$. It is not hard to see that a disk of radius $R_2 = \min[m/2, R_1/2]$ tangent to γ at Y lies entirely inside γ.

B. We draw through Y the normal n to γ. On the normal n we define a set of points M, enjoying the following property: a point $P \in M$ if the ~~disk~~ circle $C'(P)$ with center P and radius PY lies entirely inside γ. In view of A M is a finite segment YP_0 of nonzero length.

We prove that the ~~disk~~ circle $C'(P_0)$ is an integral disk of curvature. Concerning this, $C'(P_0)$ is tangent to γ at Y and lies entirely inside γ. If $C'(P_0)$ has at ~~most~~ least one point in common with γ, then $C'(P_0)$, by definition, is an integral ~~disk~~ circle of curvature $C(Y)$. Therefore it remains to consider the case where Y is the unique common point of $C'(P_0)$ and γ. In this case we need to prove that $C'(P_0)$ is the ordinary disk of curvature of γ at the point Y.

1171 Let $P_1, P_2, \ldots, P_k, \ldots$ be a sequence of points of n converging to P_0 and not belonging to M, from which it can be assumed that $P_kY \neq R_1$ for any k. Then every ~~disks~~ circle $C'(P_k)$ ~~[illegible]~~ in the point Y is tangent to γ, not intersecting γ in this point. [?] Since $P_kY > P_0Y$, then due to the definition of P_0, the ~~disk~~ circle $C'(P_k)$ has with γ at least two distinct points in common Y_k' and Y_k'' not coinciding with Y.

These points Y_k' and Y_k'' tend to Y as $k \to \infty$, since the ~~disk~~ circle $C'(P_k)$ tends to $C'(P_0)$, but $C'(P_0)$ has only one point Y in common with γ. Whence immediately it follows that $C'(P_0)$ is the ~~disk~~ circle of curvature of γ at Y. The lemma is proved.

[Usually they use "окружность" for circle, "круг" for the "disk"; here the word used is "круг", but they seem to mean circle, the curve.]

Lemma 2. Let $Y_1, Y_2 \in \gamma$, $Y_1 \neq Y_2$. Then if the circles $C(Y_1)$ and $C(Y_2)$ do not have points of tangency with γ in common, the segments $O(Y_2)Y_2$ and $O(Y_2)\bar{Y}_2$ do not have points in common with the segment $O(Y_1)Y_1$ nor with $O(Y_1)\bar{Y}_1$. Here $\bar{Y}_i$ is any point of tangency of $C(Y_i)$ with γ distinct from Y_i $(i=1,2)$.

Proof. For example, let segments $O(Y_2)Y_2$ and $O(Y_1)Y_1$ have a common point P. We suppose, for definiteness, that

$$Y_2P \geq Y_1P. \tag{1}$$

Since circles $C(Y_1)$ and $C(Y_2)$ do not have a common point of tangency, there is valid the inequality

$$O(Y_2)Y_1 > O(Y_2)Y_2. \tag{2}$$

On the otherhand, due to (1) and the triangle inequality, we have:

$$O(Y_2)Y_1 \leq O(Y_2)P + PY_1 \leq O(Y_2)P + PY_2 = O(Y_2)Y_2. \quad (3)$$

Inequality (3) contradicts inequality (2). The lemma is proved.

Lemma 3. Let Y_n be a sequence of points of γ converging to Y_0 (possibly, coinciding). If for the given sequence Y_n there can be obtained a sequence $\bar{Y}_n \neq Y_n$, also converging to Y_0, then the limit circle $C(Y_n)$ is the circle of curvature of γ at Y_0 and, consequently, $R(Y_0) > R_0$.

Proof. We designate by φ_n the angle between the tangents to the curve at points Y_n and $\bar{Y}_n$, by s_n the length of the arc $Y_n\bar{Y}_n$, by σ_n the distance between Y_n and $\bar{Y}_n$, by R_1 the radius of curvature of γ at Y_0. Choosing a subsequence, it can be assumed that $\lim_{n\to\infty} R(Y_n)$ exists. In this notation

$$\sigma_n = 2R(Y_n)\sin\frac{\varphi_n}{2}. \quad (4)$$

Whence we get

$$R_1 = \lim \frac{s_n}{\varphi_n} = \lim\left(\frac{s_n}{\sigma_n}\frac{\sigma_n}{\varphi_n}\right) = \lim(R(Y_n))\cdot\lim\frac{\varphi_n}{2\sin\frac{\varphi_n}{2}}$$

$$= \lim R(Y_n).$$

But evidently $R(Y_0) \geq \lim R(Y_n) = R_1 \geq R_0$, where R_0 is the minimal radius of curvature of γ. The lemma is proved.

Proof of Theorem 1. If there exists on γ a point Y such that $C(Y)$ has only one point in common with γ, then $C(Y)$ is, by definition, the ordinary circle of curvature of γ at Y, and Theorem 1 is evident.

If on γ there exists Y such that $C(Y)$ has an infinite set of points of tangency to γ, then by lemma 3, $C(Y)$ coincides with the circle of curvature of γ in any point of accumulation for the set of tangent points of $C(Y)$ with γ, and Theorem 1 is evident.

1172 Thus, there remains to consider the case where $C(Y)$ for any point Y has a finite number of points of tangency with γ and this number is no less than two. Let $Y_1 \in \gamma$. We designate by $\bar{Y}_1'$ one of the points of tangency of $C(Y_1)$ with γ nearest to Y_1. We take an arc $Y_1\bar{Y}_1'$ of γ, interior points of which ~~is~~ are not points of tangency

of $C(Y_1)$ with γ. Let Y_2 be the midpoint of the arc $Y_1 \bar{Y}_1'$. The circle $C(Y_1)$, due to the choice Y_2? of Y_2, does not have a common point of tangency with circle $C(Y_1)$. Lemma 2 then asserts that the nearest point of tangency $\bar{Y}_2'$ lies interior as well to the arc $Y_1 \bar{Y}_1'$. Further, let Y_3 be the midpoint of arc $Y_2 \bar{Y}_2'$; anew it can be proved that $\bar{Y}_2'$ lies inside $\overline{Y_1 Y_2'}$.
Continuing this process we get ~~a~~ sequences Y_n $\bar{Y}_n'$ for which, as it is not hard to see, the lengths of arcs $Y_n Y_{n+1}$, $\bar{Y}_{n+1}' \bar{Y}_n'$ and $\bar{Y}_n' Y_n$ of the curve γ satisfy relations:

$$\overline{Y_{n+1}' Y_n'} < 2 Y_n Y_{n+1}, \quad \overline{Y_n'} Y_n = 2 Y_n Y_{n+1}, \tag{5}$$
$$Y_n Y_{n+1} \le \frac{1}{2} Y_{n-1} Y_n .$$

From (5) it follows that Y_n and $\bar{Y}_n'$ converge to some point Y_0. But then Lemma 3 can be applied to sequences Y_n and $\bar{Y}_n'$, from which it is found that $R(Y_0) \ge R_0$, which also concludes the ~~[illegible]~~ proof of Theorem 1.

Theorem 1 ~~[illegible]~~ can be generalized. We define on the curve γ a sign of curvature in such a way that the curvature at P should be negative when γ at P is directed convexly to the inside the domain bounded by γ (if the curvature is different from zero at P). Then by verbal changes in the proof of theorem 1 theorem 2 can be proved.

Theorem 2. <u>If the curvature of a closed nonselfintersecting C^2 curve γ is always no greater than k, then there is always a ~~circle~~ circle of radius $1/k$ lying inside the closed region bounded by γ.</u>

From theorem 2 it follows:

Theorem 3. <u>Under the conditions of theorem 2 the length of the curve γ is not less than $2\pi/k$, and the area, bounded by γ, is not less than π/k^2.</u>

For the class of all curves γ, the curvature of which is no greater than k, the bounds specified in theorems 2 and 3 are sharp: the minimum is assumed when γ is a circle of radius $1/k$.