QUASIGEODESICS IN MULTIDIMENSIONAL ALEXANDROV SPACES

BY

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Abstract

Here we generalize quasigeodesics to multidimensional Alexandrov space with curvature bounded from below and prove that classical theorems of Alexandrov also hold for this case. Also we develop gradient curves as a tool for studying spaces with curvature bounded from below.

In the second chapter we give some applications of quasigeodesics and gradient curves: the Radius Sphere theorem, the Glueing theorem and the First variation formula for extremal subsets.

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1.

I. GENERAL CONSTRUCTION

§0 INTRODUCTION

0.0. Quasigeodesics are a natural generalization of geodesics to nonsmooth metric spaces. They were introduced in a paper of A.D.Alexandrov [A1] for 2-dimensional convex hypersurfaces in Euclidean space, as curves which turn "right" and "left" simultaneously, and in [A2] for 2−dimensional surfaces with finite integral curvature. The detailed research on these kinds of quasigeodesics can be found in a paper of A.V.Pogorelov [Pog] and in a joint work of A.D.Alexandrov and Yu.D.Burago [AB], respectively. A.D.Milka [M1] considered quasigeodesics for multidimensional polygonal metrics with positive curvature, defining them as broken geodesics satisfying a preentry condition on the directions of exit and entrance at vertices (see below 2.11 and A.1(B)).

For the general case of Alexandrov space these definitions do not work and we shall see that it is much harder to generalize results about quasigeodesics to this case. These proofs require considerable technical background such as the Stratification Theorem of G.Perelman [P] and a construction related to V.A.Sharafutdinov's retraction [Sh, Th.3]. Fortunately, quasigeodesics and gradient curves have many application in the geometry of Alexandrov spaces. They are the main tool in questions about the intrinsic metric of extremal subsets, and the Gluing Theorem (see below). Moreover, quasigeodesics sometimes give new and easier proofs of known results such as the Radius Sphere Theorem. Sometimes they offer a good language to simplify formulas, as in the proof of parallel transport and the second variation formula. Recently they were applied in Perelman's proof that the metric tensor of an Alexandrov space is a tensor of finite variation.

I would like to thank to Stephanie Alexander, Yury Burago and Grisha Perelman who took care to read everything below and help to make this paper more or less possible to read (if you are going to read this then you need to tell them "thanks" too), and also for their interest in this work which was very important for me.

0.1 Notation and conventions

• As in [Bus], by "curve" we mean a continuous map of a real interval into a metric space. Therefore a curve may have self-intersections. For brevity a curve with length parameter will be called a natural curve.

• S_k denotes the k-plane, that is, the sphere, plane or hyperbolic plane with constant curvature k. Let

$$\pi(k) = \left[\begin{array}{ll} \frac{\pi}{\sqrt{k}} & \text{for } k > 0 \\ \infty & \text{for } k \le 0 \end{array} \right. .$$

• M,(Σ) will always denote an Alexandrov space with curvature $\ge k$, (≥ 1), and X, a length-metric space.

• All differential inequalities of the second order will be understood in the barrier sense, i.e.,

$$y''|_{x=x_0} \le C = C(x_0, y)$$

means that in a neighborhood of x_0

$$y \le y(x_0) + B(x - x_0) + C\frac{(x - x_0)^2}{2} + o\big((x - x_0)^2\big)$$

for some B. (It is easy to see that for a smooth function this is equivalent to the usual definition.)

• Σ_p is the space of directions of M at the point p and for $q \neq p$ we denote by q', or more specially q'_p, a direction in Σ_p of some shortest path pq (see [BGP §7]). If H is a subset of M then

$$\Sigma_p(H) = \{\theta \in \Sigma_p;\ \exists q_i \in H, \theta = \lim_{|pq_i| \to 0} (q_i)'_p\}.$$

• C_p is the tangent cone at the point p, where $C_p = C(\Sigma_p)$. C_p can be also defined as a limit $C_p = \lim_{\lambda \to \infty}(\lambda M, p)$ (see [BGP 7.7, 7.8.1]). We set $C_p(H) = C(\Sigma_p(H)) \subset C_p$.

• C will denote any cone with curvature ≥ 0, and o will denote its center. If $x, y \in C$ then

$$|x| \stackrel{\text{def}}{=} |ox|$$

$$\langle x, y\rangle \stackrel{\text{def}}{=} \frac{|x|^2 + |y|^2 - |xy|^2}{2} = |x| \cdot |y| \cos \angle xoy.$$

An element of a cone C will be called a "vector".

• If $C = C(\Sigma)$ then it easy to see that there is a standard embedding $\Sigma \to C$. Thus we may consider Σ as a "unit sphere" around $o \in C$.

• For $a \in C_p$ we say $a = \log_p q$ if there is some shortest path between p and q which starts from p tangent to a and has length $|a| = |pq|$. In this notation $\frac{a}{|a|} = q'_p$. Note that in a space of curvature ≥ 0, $\log_p$ is a noncontracting map.

• f usually denotes a scalar function on M. Let $i_\lambda : \lambda M \to M$ be the canonical map. The limit of $(\lambda M, p)$ for $\lambda \to \infty$ is C_p (see [BGP 7.8.1]). For any function $f\colon M \to \mathbb{R}$ the function $d_p f\colon C_p \to \mathbb{R}$ such that

$$d_p f = \lim_{\lambda \to \infty} \lambda(f \circ i_\lambda - f(p))$$

is called the differential of f at p.

- h_n is n-dimensional Hausdorff measure.

- Let γ be a curve in a metric space, $\tilde{\angle} p\gamma(a)\breve{\gamma}(b)$ or $\tilde{\angle}\gamma(a)\breve{\gamma}(b)p$ is the corresponding angle in the model triangle $\tilde{\Delta} p\gamma(a)\breve{\gamma}(b)$ in S_k with sides $|p\gamma(a)|$, $|p\gamma(b)|$ and $|a-b|$. Note that this is not the same as $\tilde{\angle} p\gamma(a)\gamma(b)$.

- By C_p^k we will mean the k-cone $C_p^k = C^k(\Sigma_p)$, i.e., spherical suspension, cone, or elliptic cone with radial curvature equal to k (see [BGP 4.3.1, 4.3.2]). Mappings for C_p^k will be denoted by upper index k; for example, $\log_p^k : M \to C_p^k$.

- For simplicity sometimes we prove only the case $k=0$, but usually we formulate the results for general k.

Reminder.

Let us say that a point is a 0-dimensional set with boundary and a two-point set is a set without boundary.

Definition. A point $p \in M$ is a boundary point ($p \in \partial M$) if Σ_p has non empty boundary.

Definition. A closed subset $F \subset M$ is an *extremal subset* if for any distance function f any extremal point of f on F is an extremal point of f on M.

The important properties of extremal subset are:

(1) The closure of all points with a fixed topological singularity is an extremal subset, in particular the boundary and the space itself are extremal subsets.

(2) The union of extremal subsets is an extremal subset.

(3) The intersection of extremal subsets is an extremal subset.

(4) The tangent cone of an extremal subset is an extremal subset of the tangent cone of the space.

(5) For any compact subset $K \subset F$ there is an $\epsilon > 0$ such that for any $x, y \in K$

$$|xy|_F \leq \epsilon^{-1}|xy|$$

where $|**|_F$ is the length metric of F.

§1 QUASIGEODESICS

1.0. In this paragraph we give a definition of k-quasigeodesics for an arbitrary length-metric space. We prove that in an Alexandrov space with curvature $\geq k$ the classes of k_1-quasigeodesics and k_2-quasigeodesics coincide if $k_2, k_1 \leq k$. One can find another natural definition in §6.

1.1. Definition. The curve $\tilde{\gamma}(t)$ in the k-plane, (more specifically $\tilde{\gamma}_p^k(t)$) is called the *unfolding* of a curve $\gamma: [a, b] \to X$ with respect to $p \in X$ if

a) t is a natural parameter for $\tilde{\gamma}(t)$

b) there exists $\tilde{p} \in S_k$ such that $|\tilde{\gamma}(t)\tilde{p}| = |\gamma(t)p|$ for every t

c) the direction from $\tilde{p}$ to $\tilde{\gamma}(t)$ turns monotonically with increasing t.

This definition was used by Alexandrov in [A3] for the case of curvature bounded above.

1.2. Lemma. Let $\gamma: [a,b] \to X$ be a 1–Lipschitz curve in X, satisfying $|p\gamma(t)| < \pi(k)$ for every t. Then $\gamma(t)$ has a unique (up to isometry) unfolding $\tilde{\gamma}_p^k(t)$.

Conversely, if γ has an unfolding with respect to every point $p \in X$ then it is a 1–Lipschitz curve.

Proof. Indeed, $x(t) = |p\gamma(t)|$ is a 1–Lipschitz function by the triangle inequality. Therefore we can represent x in the form

$$x(t) = x(t_0) + \int_{t_0}^{t} \varphi(t)dt \text{ where } |\varphi| \leq 1, \varphi \in L_{1,loc}.$$

Let

$$\sigma_k(x) = \sum_{n=0}^{\infty} \frac{(-k)^n}{(2n+1)!} x^{2n+1} = \left[\begin{array}{ll} \frac{1}{\sqrt{k}} \sin(x\sqrt{k}) & \text{if } k > 0 \\ x & \text{if } k = 0 \\ \frac{1}{\sqrt{-k}} \sinh(x\sqrt{-k}) & \text{if } k < 0 \end{array} \right. .$$

It is easy to see that

$$(\#) \qquad \sigma_k(x)'' + k\sigma_k(x) = 0.$$

Let us consider $\tilde{\gamma}_p^k(t)$ in polar coordinates with the center $\tilde{p}$,

$$\tilde{\gamma}_p^k(t) = (x(t), \alpha(t)).$$

Then the formula $dt^2 = \sigma_k^2(x)d\alpha^2 + dx^2$ gives a unique (up to isometry) representation of $\tilde{\gamma}_p^k$:

$$\tilde{\gamma}_p^k(t) = (x(t), \int_{t_0}^{t} \frac{\sqrt{1-\varphi^2(t)}}{\sigma_k(x(t))} dt).$$

(The inequality $|p\gamma(t)| < \pi(k)$ implies that $\sigma_k(x(t)) \neq 0$.)

Now Let us prove that γ is a 1–Lipschitz curve.

Let $|p\gamma(t_1)| < \varepsilon$. Then

$$|\gamma(t_2)\gamma(t_1)| < \varepsilon + |p\gamma(t_2)| = \varepsilon + |\widetilde{p}\widetilde{\gamma}(t_2)| \leq \varepsilon + |\widetilde{p}\widetilde{\gamma}(t_1)| + |\widetilde{\gamma}(t_2)\widetilde{\gamma}(t_1)| < 2\varepsilon + |t_2 - t_1|.$$

Since ε is arbitrary, γ is 1-Lipschitz ♠.

1.3. Definition. The curve $\widetilde{\gamma}$ in S_k is *(locally) convex* at the point $\widetilde{\gamma}(t)$ with respect to $\widetilde{p} \in S_k$ if there exists $\varepsilon > 0$ such that the following "triangle" is convex. The sides of this "triangle" are the curve $\widetilde{\gamma}(t)|_{t-\varepsilon}^{t+\varepsilon}$ and the two shortest paths $\widetilde{\gamma}(t-\varepsilon)\widetilde{p}$ and $\widetilde{\gamma}(t+\varepsilon)\widetilde{p}$.

1.4. Definition. A curve γ in X is called *k-convex* if for all $p \in X$ the curve $\widetilde{\gamma}_p^k$ exists and is a locally convex curve with respect to $\widetilde{p}$ at all $\widetilde{\gamma}(t)$ such that $|\widetilde{p}\widetilde{\gamma}(t)| < \pi(k)$.

Remark.

From Lemma 1.2 we immediately obtain that any convex curve is 1–Lipschitz.

1.5. Definition. A k–convex natural curve $\gamma\colon [a,b] \to X$ is called a *$k-$quasigeodesic.*

This definition was proved by A.D.Milka [M2] for 2-dimensional case as a properety of quasigeodesic in the classical Alexandrov's definition.

1.6. Let

$$\rho_k(x) \stackrel{\text{def}}{=} \int_0^x \sigma_k(y)dy = \sum_{n=1}^{\infty} \frac{(-k)^{n-1}}{(2n)!} x^{2n} = \left[\begin{array}{ll} \frac{1}{k}(1-\cos(x\sqrt{k})) & \text{if } k > 0 \\ x^2/2 & \text{if } k = 0 \\ \frac{1}{k}(1-\cosh(x\sqrt{-k})) & \text{if } k < 0 \end{array} \right. .$$

Note that ρ_k is an increasing function on $[0, \pi(k)]$. Direct calculation shows that if $\gamma(t)$ is a geodesic in the k-plane, then for any point p

$$\text{(\#\#)} \qquad \rho_k(|p\gamma(t)|)'' + k\rho_k(|p\gamma(t)|) = 1.$$

1.7. Theorem. For $\gamma: [a, b] \to X$, the following are equivalent.

A) γ is a k-convex curve.

B) For any point p

$$\rho_k(|\gamma(t)p|)'' \leq 1 - k\rho_k(|\gamma(t)p|).$$

B') For any $p \in X$ and $t_0 \in [a, b]$ the angle $\tilde{\angle} p\gamma(t_0)\gamma(t)$ is nonincreasing in t, $t \geq t_0$ (compare with [BGP 2.5 (A)]).

The proof of Theorem 1.7 will use the following technical results.

1.8. Lemma. Suppose $f''(t) + kf(t) \leq 0$, $f(a) \geq 0$, $f(b) \geq 0$ and $|a - b| < \pi(k)$. Then

$$f(x) \geq 0 \text{ for } x \in [a, b].$$

Proof. Direct calculations.

1.9. Corollary. Let f be a continuous function such that one of the following is true:

A) The inequality $f''(t) + kf(t) \leq 1$ is true for almost all t and everywhere we have $f''(t) + kf(t) < \infty$,

B) $f''(t) + kf(t) \leq 1$ everywhere but $t = t_0$ and at t_0 the function f is convex in the first order, i.e. for upper one-sided derivatives

$$\overline{f'_+} + \overline{f'_-} \leq 0.$$

C) For any $\tilde{f}$ satisfying $\tilde{f}''(t) + k\tilde{f}(t) = 1$ and for $|a - b| < \pi(k)$, the inequalities $f(a) \geq \tilde{f}(a)$, $f(b) \geq \tilde{f}(b)$ imply

$$f(x) \geq \tilde{f}(x) \text{ for any } x \in [a, b].$$

Then we have the inequality $f''(t)+kf(t) \leq 1$ everywhere and moreover for any t_0 there is a function $\tilde{f}$ such that

$$\tilde{f}''(t) + k\tilde{f}(t) = 1,\ \ \tilde{f}(t_0) = f(t_0) \text{ and } \tilde{f}(t) \geq f(t) \text{ for } |t - t_0| \leq \pi(k).$$

Proof. Let f be defined on (a, b) and

$$f_\varepsilon = g_\varepsilon * f,$$

where

$$g_\varepsilon \in C^\infty,\ \ g_\varepsilon = 0 \ \text{ for } |t| > \varepsilon \text{ and } \int_{-\varepsilon}^{\varepsilon} g_\varepsilon dt = 1.$$

Then it is easy to see that

$$f = \lim_{\varepsilon \to 0} f_\varepsilon,\ \ f_\varepsilon \in C^\infty$$

and everywhere in $(a + \varepsilon, b - \varepsilon)$ we have

$$f''_\varepsilon + kf_\varepsilon(t) \leq 1$$

if f satisfies any one of A, B or C. Therefore we only need to prove the second part of the Corollary for the smooth case and we will obtain the general case as a limit.

Let $\tilde{f}''_\varepsilon(t) + k\tilde{f}_\varepsilon(t) = 1$ and $\tilde{f}_\varepsilon(t_0 \pm \varepsilon) = f(t_0 \pm \varepsilon)$. Then from the Lemma we obtain

$$\tilde{f}_\varepsilon(t) \leq f(t) \text{ for } |t - t_0| \leq \varepsilon$$

(we need only apply the Lemma to the function $f(t) - \tilde{f}_\varepsilon(t)$). Moreover,

$$\tilde{f}_\varepsilon(t) \geq f(t) \text{ for } \varepsilon \leq |t - t_0| < \pi(k) + \varepsilon.$$

Indeed, suppose the last statement is wrong. Then there is b such that $f(b) - \tilde{f}_\varepsilon(b) > 0$. Assume $t_0 + \varepsilon < b < t_0 + \varepsilon + \pi(k)$. Then applying the Lemma for interval $[a, b]$ where $|a - b| < \pi(k)$ and $a < t_0 + \varepsilon$, to the function $f(t) - \tilde{f}_\varepsilon(t) + \epsilon\sigma_k(t - a)$, we obtain $f(t_0 + \varepsilon) - \tilde{f}_\varepsilon(t_0 + \varepsilon) + \epsilon\sigma_k(t_0 + \varepsilon - a) \geq 0$. But $f(t_0 + \varepsilon) - \tilde{f}_\varepsilon(t_0 + \varepsilon) = 0$ and this is a contradiction.

Therefore for $\tilde{f} = \lim_{\varepsilon \to \infty} \tilde{f}_\varepsilon$, the conclusion of the Lemma follows ♠.

1.10. Proof of Theorem 1.7. ($A \to B$) and ($B \to A$) This is easy to prove if $x(t) = |\gamma(t)p|$ is a C^2-function. Direct calculations show that if we have representation $\tilde{\gamma}(t) = (x(t), \alpha(t))$ in polar coordinates, then the curvature of $\tilde{\gamma}$ is

$$k(t) = \frac{\rho_k(x)'' + k\rho_k(x) - 1}{\sigma_k(x)\sqrt{1 - (x')^2}}.$$

Therefore $\tilde{\gamma}$ is locally convex if and only if $k(t) \leq 0$ for all t, or equivalently

$$\rho_k(x)'' + k\rho_k(x) \leq 1.$$

The general case can be obtained from the fact that any solution of the differential inequality is a limit of smooth solutions, and any locally convex curve is a limit of smooth locally convex curves ♠.

$(B \to B')$ and $(B' \to B)$. The implication $(B \to B')$ is exactly Lemma 1.8 for the function $\rho_k(|p\gamma(t)|) - \rho_k(\tilde{x}_{t_1,t_2}(t))$ where $\tilde{x}_{t_1,t_2}(t)$ is the distance function between $\tilde{p}$ and a point on the opposite side of a model triangle (we need only to remember (##)). Conversely, let some curve $\gamma(t)$ satisfy B'. Then for any t_1, t_2 the function $\rho_k(|p\gamma(t)|)$ satisfies the condition C of Corollary 1.9 ♠.

1.11. Theorem. Let γ be a natural curve in Alexandrov space M with curvature $\geq k$. Then the following are equivalent:

A) γ is a k-quasigeodesic.

B) For any t we have

$$(*) \qquad \left(\frac{|\gamma(t)p|^2}{2}\right)'' \leq 1 + o(|\gamma(t)p|).$$

1.12. Remark. Let γ be a geodesic (local shortest path) in the Alexandrov space M with a natural parameter. Then from Theorem 1.11 it is easy to see that γ is a k-quasigeodesic. Indeed, let us verify this for $k = 0$. For any point $\gamma(t_0)$ we can find $\varepsilon > 0$ such that $\gamma(t)|_{t_0-\varepsilon}^{t_0+\varepsilon}$ is a shortest path. Let $\alpha = \angle\gamma(t_0+\varepsilon)\gamma(t_0)p$. Then $\pi - \alpha \geq \angle\gamma(t_0-\varepsilon)\gamma(t_0)p$. Using the triangle comparison theorem and the law of cosines we obtain for $|t - t_0| < \varepsilon$

$$\frac{|p\gamma(t)|^2}{2} \leq \frac{|p\gamma(t_0)|^2}{2} - \cos\alpha|p\gamma(t_0)|(t - t_0) + \frac{(t-t_0)^2}{2}.$$

This implies

$$\rho_k(|\gamma(t_0)p|)'' \le 1 - k\rho_k(|\gamma(t_0)p|)$$

for $k = 0$.

Therefore the definition of quasigeodesic is natural.

1.13. Lemma. Let $\gamma\colon [a,b] \to X$ be a natural curve. Then for almost all $t_0 \in [a,b]$

$$\lim_{t \to t_0} \frac{|\gamma(t_0)\gamma(t)|}{|t_0 - t|} = 1$$

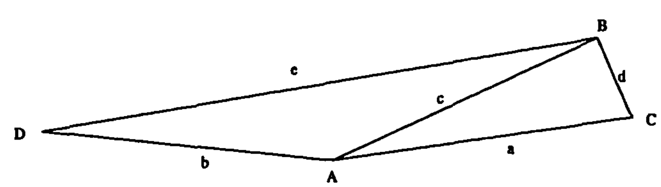
and

$$\left(\frac{|\gamma(t)\gamma(t_0)|^2}{2}\right)''\Bigg|_{t_0} = 1.$$

Proof. Immediately from [Bus 5.14].

1.14. Lemma. Let ABC and ABD be triangles in S_0 with a common side AB, $\angle BAC + \angle BAD \le \pi$ and let a, b, c, d, e be the lengths of sides as illustrated (i.e. $a = |AC|$, $b = |AD|$, $c = |AB|$, $d = |BC|$, $e = |BD|$.) Then

$$e^2 \le \frac{a+b}{a}c^2 - \frac{b}{a}d^2 + b(a+b).$$



Remark. We formulate this Lemma only for S_0 because we prove only case $k = 0$. For general k the formula will be

$$\rho_k(e) \le \frac{\sigma_k(a)+\sigma_k(b)}{\sigma_k(a)}\rho_k(c) + \rho_k(b) + \frac{\sigma_k(b)}{\sigma_k(a)}(\rho_k(a) - \rho_k(d)) - k\rho_k(c)(\rho_k(b) + \frac{\sigma_k(b)}{\sigma_k(a)}\rho_k(a)).$$

Proof. As a corollary of $\angle BAC + \angle BAD \le \pi$ we have the inequality

$$\cos\angle BAC + \cos\angle BAD \ge 0.$$

By the law of cosines for both triangles:

$$2\cos\angle BAC = \frac{a^2+c^2-d^2}{ca}$$

and

$$2\cos\angle BAD = \frac{b^2+c^2-e^2}{cb}.$$

After summing

$$\frac{a^2+c^2-d^2}{ca} + \frac{b^2+c^2-e^2}{cb} \ge 0,$$

which is the desired formula rewritten ♠.

1.15. Proof of Theorem 1.11.

$(B \to A)$ To simplify the formulas we will only prove case $k = 0$. But the general case can be obtained in the same way.

Let p_1 be a point on the shortest path between p and $\gamma(t_0)$. Now let us consider the quadruple $(p_1;\ p, \gamma(t), \gamma(t_0))$. From the definition of Alexandrov space (see [BGP, 2.3]) we obtain

$$\tilde\angle pp_1\gamma(t) + \tilde\angle\gamma(t_0)p_1\gamma(t) \le \pi.$$

Using Lemma 1.14 for the model triangles $\tilde{\Delta} pp_1\gamma(t)$, $\tilde{\Delta}\gamma(t_0)p_1\gamma(t)$ we obtain

$$\frac{|p\gamma(t)|^2}{2} \le \frac{|p\gamma(t_0)|}{|p_1\gamma(t_0)|}\frac{|p_1\gamma(t)|^2}{2} - \frac{|pp_1|}{|p_1\gamma(t_0)|}\frac{|\gamma(t)\gamma(t_0)|^2}{2} + |p_1p|\cdot|p\gamma(t_0)|,$$

where equality holds for $t = t_0$. Therefore taking the second derivative of this inequality at t_0 we obtain

$$\left(\frac{|p\gamma(t)|^2}{2}\right)''\Bigg|_{t=t_0} \le \frac{|p\gamma(t_0)|}{|p_1\gamma(t_0)|}\left(\frac{|p_1\gamma(t)|^2}{2}\right)''\Bigg|_{t=t_0} - \frac{|pp_1|}{|p_1\gamma(t_0)|}\left(\frac{|\gamma(t)\gamma(t_0)|^2}{2}\right)''\Bigg|_{t=t_0}.$$

Now using $(*)$, we obtain

$$\left(\frac{|p\gamma(t)|^2}{2}\right)''\Bigg|_{t=t_0} \le$$

$$\le \frac{|p\gamma(t_0)|}{|p_1\gamma(t_0)|}(1 + o(|\gamma(t)p_1|)) - \frac{|pp_1|}{|p_1\gamma(t_0)|}\left(\frac{|\gamma(t)\gamma(t_0)|^2}{2}\right)''\Bigg|_{t=t_0} < \infty$$

for all t_0. (Indeed $|\gamma(t)\gamma(t_0)|^2 \ge 0$ and $|\gamma(t_0)\gamma(t_0)|^2 = 0$, so $(|\gamma(t)\gamma(t_0)|^2)'' \ge 0$).

From Lemma 1.13 $\left(|\gamma(t)\gamma(t_0)|^2/2\right)''|_{t=t_0} = 1$ for almost all t_0. Therefore

$$\left(\frac{|p\gamma(t)|^2}{2}\right)''\Bigg|_{t=t_0} \le \frac{|p\gamma(t_0)|}{|p_1\gamma(t_0)|}(1 + o(|\gamma(t)p_1|)) - \frac{|pp_1|}{|p_1\gamma(t_0)|} < 1 + \frac{o(|\gamma(t)p_1|)}{|\gamma(t)p_1|}.$$

Therefore, if $p_1 \to p$ we obtain

$$\left(\frac{|p\gamma(t)|^2}{2}\right)''\Bigg|_{t=t_0} \le 1,$$

and Corollary 1.9 completes our proof ♠.

$(A \to B)$ We need only to rewrite inequality

$$\rho_k(|\gamma(t)p|)'' \le 1 - k\rho_k(|\gamma(t)p|)$$

using the series representation of ρ_k, to obtain

$$\text{something positive} + (1 + O(|\gamma(t)p|^2))\left(\frac{|\gamma(t)p|^2}{2}\right)'' \le 1 + O(|\gamma(t)p|^2).$$

Therefore we obtain an even stronger inequality:

$$\left(\frac{|\gamma(t)p|^2}{2}\right)'' \leq 1 + O(|\gamma(t)p|^2) \spadesuit.$$

1.16. Corollary. *(invariance)* If $k_1, k_2 \leq k$ then in an Alexandrov space M with curvature $\geq k$ the classes of k_1-quasigeodesics and k_2-quasigeodesics coincide.

Proof. Using the equivalence $A \Leftrightarrow B$ of Theorem 1.11 we obtain an invariant definition of quasigeodesic $\spadesuit$.

Remark. From now on we can talk about "quasigeodesics" instead of "k-quasigeodesics" in an Alexandrov space.

1.17. Quasigeodesics give a new and very useful definition of Alexandrov space:

Theorem. A length-metric space X is an *Alexandrov space* with curvature $\geq k$ if between any pair of points there is a shortest path in X which is a k-quasigeodesic.

Proof. By (B′), this is only a small generalization of a standard definition [BGP 2.5(A)]. The proof is a repetition of the proof in [BGP 2.7-8]. But we have to say "the shortest path which is a k-quasigeodesic" instead of "geodesic". The only point needing attention is [BGP 2.8.1.] where we have to prove that the limit of "the shortest paths which are k-quasigeodesics" is a "shortest path which is a k-quasigeodesic". Indeed the natural parameter of the limit curve is the limit of natural parameters because they are shortest paths. And the limit of unfoldings will be the unfolding of the limit curve because of local convexity ♠.

§2 REGULARITY and PASSAGE to LIMIT

2.0. Here we prove that a limit of quasigeodesics is also a quasigeodesic and as a corollary we obtain some regularity properties of quasigeodesics.

2.1. Collapse and the Gromov-Hausdorff metric were considered in [GPL].

Definition. The Gromov-Hausdorff metric (on the set of all compact metric spaces) is the metric

$$d_{GH}(X,Y) = \inf_{\{Z;\ X,Y \hookrightarrow Z\}} \{d_H(X,Y)\},$$

where d_H is the standard Hausdorff distance between sets in the metric space Z, and $\hookrightarrow$ denotes a global isometric embedding.

2.2. The Gromov-Hausdorff limit can be defined as a limit in this metric. We will use another definition (it is simple to check equivalence).

Definition. *(for compact spaces)* A sequence of metric spaces (X_n, ρ_n) tends to a compact metric space (X, ρ) $(X_n \xrightarrow{GH} X)$, if there exists a metric d on $\bigsqcup_{n=0}^{\infty} X_n$ such that $d|_{X_n} \equiv \rho_n$ and for every $\varepsilon > 0$ there is a number N, such that for $n > N$

$$X_n \subset B_\varepsilon(X) \text{ and } X \subset B_\varepsilon(X_n)$$

where B_ε is an ε-neighborhood in the d-metric.

Definition. *(for finitely compact spaces)*

A sequence of metric spaces (X_n, p_n, ρ_n) with base points tends to a complete metric space (X, p, ρ) with base point $(X_n \xrightarrow{GH} X)$ if there exists a metric d on $\bigsqcup_{n=0}^{\infty} X_n$ such that for every $R > 0$ the balls $B_R(p_n) \xrightarrow{GH} B_R(p)$ in the sense of the preceding definition.

2.3. It is easy to see that if $\{M_n\}$ is a sequence of Alexandrov spaces with curvature $\geq k$ and $M_n \xrightarrow{GH} M$, then $\dim(M_n) \geq \dim(M)$ for sufficiently big n.

Definition. A sequence $\{M_n\}$ *collapses* to M if in addition to satisfying one of the definitions in 2.2,

$$\dim(M_n) > \dim(M).$$

2.4. Theorem. *(passage to limit)*

Assume $M_n \xrightarrow{GH} M$ without collapse (i.e. $\dim(M_n) = \dim(M)$) and $\gamma_n \in M_n$ is a sequence of quasigeodesics which in the d-metric converges pointwise to γ. Then γ is a quasigeodesic.

(Compare with the 2-dimensional case [A3' Th.7] and [AB Th.7].)

Theorem 2.4 will be proved in paragraph 2.16.

2.5. Definition. The *cutlocus of M with respect to p* is the subset of M given by

$$\mathrm{Cutloc}(M,p) = \{q \in M; \text{ there is no } q_1 \in M \text{ such that } |pq| + |qq_1| = |pq_1| \text{ and } q_1 \neq q\}.$$

Lemma. For any point $p \in M$, the cutlocus of M has vanishing Hausdorff n-measure:

$$h_n \mathrm{Cutloc}(M,p) = 0.$$

Proof. Let us consider $B = \mathrm{Im}(\log_p)$, which is a star-shaped body in C_p with center o. Let

$$\widetilde{\partial} B = \{x \in B;\ \text{there is no } x_1 \in B \text{ such that } x \text{ lies in the relative interior of } ox_1\}.$$

It is easy to see that $\log(\mathrm{Cutloc}(M,p) \subset \widetilde{\partial} B$. But

$$h_n \widetilde{\partial} B \leq \lim_{\varepsilon \to 0} h_n (1+\varepsilon) B \backslash (1-\varepsilon) B = 0.$$

Therefore since $\log_p$ is noncontracting,

$$h_n \mathrm{Cutloc}(M,p) \leq h_n \log_p(\mathrm{Cutloc}(M,p)) \leq h_n \widetilde{\partial} B = 0 \ \spadesuit.$$

Corollary. For a given point p, the vectors $\log_p(q)$ and $\log_q(p)$ are uniquely defined for almost all q.

Proof. Suppose $\log_p(q)$ or $\log_q(p)$ is not uniquely defined. Then $q \in \mathrm{Cutloc}(M,p)$. Thus we need only apply the preceding Lemma.

2.6. Lemma. Let C be a cone with curvature ≥ 0, $a, b \in C$, $\dim C = n$ and suppose there is a subset $K \subset C$ such that for any $x \in K$

$$\langle a, x \rangle = \langle b, x \rangle.$$

Then $a = b$ or $h_n \mathrm{clos}(K) = 0$.

Proof. Suppose $a \neq b$. By continuity, for any $x \in \mathrm{clos}(K)$ we have

$$\langle a, x\rangle = \langle b, x\rangle.$$

Let us start with the case $C = \mathbb{R}^n$. The function $f(x) = \langle a, x\rangle - \langle b, x\rangle$ is linear and therefore the set of zeros is a hyperplane and has zero measure.

Now turn to the general case. Using the preceding Corollary we obtain that $\log_x a$ and $\log_x b$ are uniquely defined for almost all x. Direct calculation shows that in this case

$$d_x f(y) = \langle \log_x a, y\rangle - \langle \log_x b, y\rangle,$$

for f defined as above. Since $\log_x$ is a noncontracting map, $\log_x a \neq \log_x b$ if $a \neq b$. From above the density function $\sigma(x)$ of $\mathrm{clos}(K)$ is zero at almost all smooth points (points where $C_x = \mathbb{R}^n$). Thus from [F 2.9.11, 2.9.12]

$$h_n \mathrm{clos}(K) = \int_C \sigma(x) dV = 0$$

because as a corollary of [BGP 10.6], almost all points of an Alexandrov space are smooth ♠.

2.7. Definition. Two vectors a and b in C are *antipodal* (we write $a + b = 0$) if for any vector $x \in C$,

$$\langle a, x\rangle + \langle b, x\rangle = 0.$$

2.8. Lemma. The following properties of two vectors a, b in a cone C with curvature ≥ 0 are equivalent.

1. C can be represented in the form $C = \mathbb{R} \times C'$ such that $a = (x, o)$ and $b = (-x, o)$ for some x or $a = b = o$

2. $\langle a, x\rangle + \langle b, x\rangle = 0$ for any vector x of some everywhere dense subset.

2$'$. a and b are antipodal ($a + b = 0$).

3. $|ab| = 2|a| = 2|b|$

Proof. The implications ($1 \to 2 \to 2'$) are trivial.

($2' \to 3$). Using $2'$ for $x = a$ and $x = b$ we obtain

$$\langle a, a\rangle + \langle b, a\rangle = 0 \text{ and } \langle a, b\rangle + \langle b, b\rangle = 0.$$

After summing,

$$-2\langle b, a\rangle = |a|^2 + |b|^2$$

By the definition of scalar product (see above)

$$|ab|^2 = 2|a|^2 + 2|b|^2$$

and as corollary we obtain 3.

($3 \to 1$). By 3 the union of the two rays in directions a and b is an infinite shortest path if $|a| = |b| \neq 0$. From Toponogov's splitting Theorem (see [BGP 7.15]) we obtain that $C = \mathbb{R} \times C'$ where $\mathbb{R} \times o$ corresponds to the union of the two rays. The equation $|a| = |b|$ completes our proof ♠.

2.9. Definition. An element $\gamma^{+(-)} \in C_p$, $p = \gamma(t)$ is called the right (left) derivative of the curve $\gamma\colon [a,b] \to M$ if there is a sequence $t_i \to t$, $t_i > t$ $(t_i < t)$ such that

$$\gamma^{+(-)}(t) = \lim_{t_i \to t} \frac{\log_p \gamma(t_i)}{|t_i - t|}.$$

Remark. If M is a Riemannian manifold then this definition gives the right derivative in the usual sense but the left derivative has a different sign.

2.10. Definition. A curve $\gamma\colon [a,b] \to M$ is differentiable at t if the left and right derivatives $(\gamma^+(t), \gamma^-(t))$ are unique and antipodal.

2.11. Definition. Two vectors a and b of C are called *polar* if for any $x \in C$

$$\langle a, x\rangle + \langle b, x\rangle \geq 0.$$

2.12. Lemma.

A. Any Lipschitz curve $\gamma\colon [a,b] \to M$ is differentiable almost everywhere.

B. Any $k-$convex curve $\gamma\colon [a,b] \to M$ has unique one side derivatives, which are polar.

Proof.

A. Let us consider a countable everywhere dense subset $Q = \{r_i\} \subset M$. From the the triangle inequality we obtain that functions $f_i(t) \stackrel{\text{def}}{=} |r_i\gamma(t)|$ are Lipschitz and so differentiable almost everywhere. The set Q is countable and therefore all functions f_i are differentiable almost everywhere simultaneously. Now let all f_i be differentiable for $t = t_0$ and γ_1^+ and γ_2^+ be two right derivatives of γ at t_0. Let

$$K = \bigcup_{\lambda>0} \lambda \log_{\gamma(t_0)}(Q).$$

Then using the first variation formula we obtain that for any element x of $K \subset C_{\gamma(t_0)}$ we have

$$\langle \gamma_1^+, x \rangle = |x| f_i'(t_0) = \langle \gamma_2^+, x \rangle.$$

K is easily a dense subset of $C_{\gamma(t_0)}$, therefore using Lemma 2.6 we obtain uniqueness of the right (left) tangent vector.

From the same idea we obtain that if $x \in K$ then

$$\langle \gamma^+, x \rangle + \langle \gamma^-, x \rangle = 0$$

and using Lemma 2.8 we obtain $\gamma^+ + \gamma^- = 0$.

B. By Theorem 1.7(B) and series representation of ρ_k (see 1.6) the definition of k-convex curve may be rewritten in the following form:

$$(f^2)'' \leq \alpha(f^2)$$

where α is a C^∞function and $f = \mathrm{dist}_p \circ \gamma$. This implies that the function

$$f^2(t) + \int_{t_0}^{t} (x - t)\alpha(f^2(x))dx$$

is concave and therefore has unique one-sided derivatives. Now $\int_{t_0}^{t}(x - t)\alpha(f^2(x))dx$ is easily a C^2-function. Hence $f^2(t)$ has one-sided derivatives everywhere, and from the same idea as in A we obtain uniqueness of one-sided derivatives. Now $(f^2)'_- + (f^2)'_+ \leq 0$. Using first variation formula we obtain that for any $x \in K$ (see A)

$$\langle x, \gamma^- \rangle + \langle x, \gamma^+ \rangle \geq 0.$$

After passing to limit we obtain that γ^- and γ^+ are polar ♠.

2.13. Corollary. Let $\gamma: [a, b] \to M$ be a Lipschitz curve and s be a natural parameter of γ. Then

$$s(t) = s(a) + \int_a^t |\gamma^{\pm}(x)| dx.$$

Proof. Let $\gamma_s = \gamma \circ s^{-1}$ be a natural curve, $\gamma_s: [s(a), s(b)] \to M$. By the definition γ_s is also a Lipschitz curve. Using 1.13 and the definition of one-sided derivative we obtain that almost everywhere $|\gamma_s^{\pm}| = 1$. On the other hand, $s(t)$ is easily a nondecreasing Lipschitz function. Therefore

$$s(t) = s(a) + \int_a^t s'(x)$$

where $s' \in L^{\infty}$ and $0 \leq s' \leq L$, letting L be the Lipschitz constant of curve γ. Immediately from the definition of derivative we obtain that if s' is defined for t then $|\gamma^{\pm}| = s'|\gamma_s^{\pm}|$. Therefore almost everywhere $|\gamma^{\pm}| = s'$ and

$$s(t) = s(a) + \int_a^t |\gamma^{\pm}(x)| dx. ♠$$

2.14. Key Lemma. Assume $M_n \xrightarrow{GH} M$ without collapse (i.e. $\dim(M_n) = \dim(M)$). Let $\gamma_n: [a, b] \to M_n$ be a sequence of k-convex curves which in the d-metric converges pointwise to γ. Then γ is a $k-$convex curve. Moreover if γ is differentiable at t then

$$|\gamma^{\pm}(t)| = \lim_{n\to\infty} |\gamma_n^{\pm}(t)|.$$

Proof. We consider the case $k = 0$. Let us fix t such that γ is differentiable at t. Let $p = \gamma(t)$ and $p_n = \gamma_n(t)$. If the Lemma is wrong then we can pass to a subsequence $\{\gamma_n\}$ such that $\big||\gamma^{\pm}(t)| - |\gamma_n^{+}(t)|\big| > \varepsilon > 0$. Using Gromov's Compactness Theorem (see [BGP 8.5]) we can pass to a subsequence $\{\gamma_n\}$ such that

$$C_{p_n} \overset{GH}{\to} C,\ \gamma_n^{\pm}(t) \to \omega^{\pm},$$

for $\gamma_n^{\pm}(t) \in C_{p_n}$, $\omega^{\pm} \in C$. Let us consider maps

$$\log_{p_n}: M_n \to C_{p_n}.$$

Take in M a countable everywhere dense subset $Q \subset M$ and for any $q \in Q$ find a sequence $q_n \in M_n$ such that $q_n \to q$. Now pass to a subsequence $\{M_n\}$ such that for any $q \in Q$ there is a limit

$$l(q) = \lim_{n\to\infty} \log_{p_n}(q_n).$$

Therefore we have constructed a mapping

$$l: Q \to C.$$

Note that

$$|l(q)| = \lim_{n\to\infty} |\log_{p_n}(q_n)| = \lim_{n\to\infty} |p_n q_n| = |pq|.$$

Therefore

$$(*) \qquad |l(q)| \equiv |pq|.$$

Now

$$\langle \omega^{\pm}, l(q)\rangle = \lim_{n\to\infty} \langle \gamma_n^{\pm}, \log_{p_n}(q_n)\rangle$$

And using property of k-convex curves (see Theorem 1.7 B and Corollary 1.9) it is easy to obtain that

$$\frac{|\gamma_n((\tau))q_n|^2}{2} \leq \frac{|p_n q_n|^2}{2} \mp \langle \gamma_n^{\pm}, \log_{p_n}(q_n)\rangle(\tau - t) + (\tau - t)^2.$$

Taking the limit as $n \to \infty$,

$$(\#) \qquad \frac{|\gamma((\tau))q|^2}{2} \leq \frac{|\gamma((t))q|^2}{2} \mp \langle \omega^{\pm}, l(q)\rangle(\tau - t) + (\tau - t)^2.$$

Therefore γ is a convex curve. We can easily assume that for any $q \in Q$ there is a unique shortest path between p and q. From the first variation formula and differentiability of γ at t we obtain that $\frac{|\gamma((\tau))q|^2}{2}$ has a derivative at t we obtain

$$\langle \omega^{\pm}, l(q)\rangle = \mp(\frac{|\gamma((t))q|^2}{2})'.$$

In particular

$$\langle \omega^{+}, l(q)\rangle + \langle \omega^{-}, l(q)\rangle = 0$$

In addition let us note that l is a noncontracting map since it is a limit of noncontracting maps (because we consider the case of curvature ≥ 0).

Take for any $x \in M \backslash Q$ a sequence of points $q_i \in Q_f$ such that $q_i \to x$ and there is a limit of the sequence $\{l(q_i)\}$. Now extend l to $\tilde{l} : M \to C$, such that

$$\tilde{l}(x) = \lim_{i\to\infty} l(q_i).$$

It is easy to see that $\tilde{l}$ is noncontracting as well as l and $(*)$ also hold for $\tilde{l}$.

Now let us consider the map $\tilde{l}\circ\gamma : [a,b] \to C$. From (*) we obtain $|\tilde{l}\circ\gamma(\tau)| = |\gamma(t)\gamma(\tau)|$. Since $\tilde{l}$ is a noncontracting map we obtain for any τ and q

$$|\gamma(\tau)q| \le |\tilde{l}\circ\gamma(\tau)\, l(q)|.$$

Therefore using first variation formula and linearity of scalar product (i.e. $\lambda\langle a,b\rangle = \langle \lambda a, b\rangle$) we obtain that if

$$\nu^{\pm} = \lim_{\epsilon_i \to 0+} \frac{\tilde{l}\circ\gamma(t \pm \epsilon_i)}{\epsilon_i}$$

then

$$\langle \nu^{\pm}, l(q)\rangle = \lim_{\epsilon_i\to 0+} \frac{1}{\epsilon_i}\langle \tilde{l}\circ\gamma(t\pm\epsilon_i), l(q)\rangle \le$$

$$\le \lim_{\epsilon_i\to 0+}\frac{1}{\epsilon_i}(|\tilde{l}\circ\gamma(t\pm\epsilon_i))|^2 + |l(q)|^2 - |q\gamma(t\pm\epsilon_i)|^2)/2 = \mp(\frac{|\gamma((t))q|^2}{2})' = \langle \omega^{\pm}, l(q)\rangle.$$

But it is easy that

$$|\nu^+\nu^-| = \lim_{\epsilon_i\to 0,}\frac{|l\circ\gamma(t+\epsilon_i)l\circ\gamma(t-\epsilon_i)|}{\epsilon_i} \ge \lim_{\epsilon_i\to 0,}\frac{|\gamma(t+\epsilon_i)\gamma(t-\epsilon_i)|}{\epsilon_i} = |\gamma^+\gamma^-|$$

and

$$|\nu^{\pm}| = \lim_{\epsilon_i\to 0,}|\frac{l\circ\gamma(t\pm\epsilon_i)}{\epsilon_i}| = \lim_{\epsilon_i\to 0,}|\frac{\gamma(t\pm\epsilon_i)}{\epsilon_i}| = |\gamma^{\pm}|.$$

Hence $|\nu^+\nu^-| = 2|\nu^{\pm}| = 2|\gamma^{\pm}| = |\gamma^+\gamma^-|$. From Lemma 2.8, for any $x \in C$

$$\langle \nu^+, x\rangle + \langle \nu^-, x\rangle = 0.$$

Therefore

$$0 = \langle \omega^+, l(q)\rangle + \langle \omega^-, l(q)\rangle \ge \langle \nu^+, l(q)\rangle + \langle \nu^-, l(q)\rangle = 0.$$

Hence

$$\langle \omega^{\pm}, l(q)\rangle = \langle \nu^{\pm}, l(q)\rangle.$$

And using Lemma 2.6 we obtain

$$\omega^{\pm} = \nu^{\pm},$$

because l is a noncontracting map and $h_n \mathrm{clos}(l(Q)) \geq h_n M > 0$. In particular

$$|\gamma^{\pm}| = |\nu^{\pm}| = |\omega^{\pm}| = \lim_{n \to \infty} |\gamma_n^{\pm}|,$$

a contradiction ♠.

2.15. Corollary. For any convex curve γ the functions $|\gamma^{\pm}|$ are one-sided continuous, i.e.

$$|\gamma^{+}(t)| = \lim_{t_n \to t^{+}} |\gamma^{\pm}(t_n)|$$

and

$$|\gamma^{-}(t)| = \lim_{t_n \to t^{-}} |\gamma^{\pm}(t_n)|.$$

In particular, if γ is a quasigeodesic then $|\gamma^{\pm}(t)| \equiv 1$.

(Compare with [Pog2 Th 5].)

Proof. Let $t_n \to t^{+}$. Let us consider the limit $(M/(t_n - t), p) \overset{GH}{\to} C_p$, where $p = \gamma_0(t)$. Recall that $i_{1/(t_n - t)} \colon M \to M/(t_n - t)$ is the canonical map. Let $\gamma_n(\tau) = i_{1/(t_n - t)} \circ \gamma(t + (t_n - t)\tau)$ and $\gamma_\infty(\tau) = \lim_{n \to \infty} \gamma_n(\tau)$.

Using existence of one-sided derivatives we obtain that

$$\gamma_\infty(\tau) = \left[\begin{array}{ll} |\tau|\gamma^{+} & \text{for } \tau \geq 0, \\ |\tau|\gamma^{-} & \text{for } \tau \leq 0. \end{array} \right.$$

Using the Key Lemma (2.14) for $\tau = 1$ we obtain

$$|\gamma^+(t)| = |\gamma^{\pm}_{\infty}(1)| = \lim_{n\to\infty} \gamma^{\pm}_n(1) = \lim_{n\to\infty} \gamma^{\pm}(t_n)) \spadesuit.$$

2.16. Proof of Theorem 2.4 We can assume that all $\{\gamma_n(t)\}$ are defined on a segment $[a,b]$. The curve γ is the pointwise limit of $\{\gamma_n\}$, and as a limit it is a Lipschitz map. Using Lemma 2.12(A) we obtain that almost all points of γ are smooth. Therefore using Lemma 2.14 we obtain

$$|\gamma^{\pm}(t)| = \lim_{n\to\infty} |\gamma^{\pm}_n(t)|$$

for almost all t. Using Corollary 2.13

$$s(t) = s(a) + \int_a^t |\gamma^{\pm}(x)|dx = s(a) + \lim_{n\to\infty} \int_a^t |\gamma^{\pm}_n(x)|dx.$$

Since t is a natural parameter of γ_n, then $|\gamma^{\pm}_n(t)| = 1$ almost everywhere by 1.13 and

$$s(t) = s(a) + \lim_{n\to\infty} \int_a^t |\gamma^{\pm}_n(x)|dx = \{s(a) - a\} + t.$$

Therefore t is a natural parameter of γ.

Now γ is a convex natural curve and by definition it is a quasigeodesic $\spadesuit$.

2.17. Theorem. *(regularity)* For any point of a quasigeodesic there are uniquely defined right and left derivatives which are polar and have absolute value equal to 1.

(Compare with 2-dimensional case [Pog2 Th.5])

Proof. See Corollary 2.15 and Lemma 2.12(B) $\spadesuit$.

2.18. Theorem. *(gluing)* Suppose $\gamma_1\colon [a,b] \to M$ and $\gamma_2\colon [bc] \to M$ be convex curves such that $\gamma_1(b) = \gamma_2(b)$ and $\gamma_1^-(b)$ is polar to $\gamma_2^+(b)$. Then

$$\gamma\colon [ac] \to M,\ \gamma(t) = \begin{bmatrix} \gamma_1(t) & \text{for } t \le b \\ \gamma_2(t) & \text{for } t \ge b \end{bmatrix}$$

is a convex curve.

Proof. Using Theorem 1.7 we need only check the inequality

$$\rho_k(|\gamma(t)p|)'' \le 1 - k\rho_k(|\gamma(t)p|)$$

for any point p and $t = b$. This follows from Corollary 1.9(B) ♠.

§3 GRADIENT CURVES.

3.0. Here we start preparation for construction of quasigeodesics in any direction. Namely we define gradient curves for a general semiconvex function in an Alexandrov space. This construction works as well in any metric space with first variation formula and complete tangent cone at any point. As was shown by Perelman, this construction can be generalized to Alexandrov space with infinite Hausdorff dimension, and it also completes Plaut's argument that Alexandrov space with finite topological dimension has finite Hausdorff dimension. Our construction is very similar to the construction of Sharafutdinov retraction ([Sh Th.3]). In the next paragraph, by applying this theory to distance functions we will obtain a very useful class of proper curves.

3.1. Definition. *(for a space without boundary)* Let M be an Alexandrov space without boundary. A function $f\colon U \subset M \to \mathbb{R}$ is $\lambda-$*concave* if for any shortest path $\gamma \subset U$ with a natural parameter,

$$f \circ \gamma(t) - \lambda(t^2/2)$$

is concave.

From [P2 5.2] we obtain that if M is an Alexandrov space with boundary, then its double $\widetilde{M}$ is also an Alexandrov space. Let $\mathrm{p} : \widetilde{M} \to M$ be the canonical map. Then for any subset $U \subset M$ we set $\widetilde{U} = \mathrm{p}^{-1}(U)$. If f is a function on U, then we set $\tilde{f} = f \circ \mathrm{p}$.

Definition. *(for a space with boundary)* Let M be an Alexandrov space with boundary. A function $f\colon U \subset M \to \mathbb{R}$ is $\lambda-$*concave* if for any shortest path $\gamma \subset \widetilde{U} \in \widetilde{M}$ with a natural parameter, the function

$$\tilde{f} \circ \gamma(t) - \lambda(t^2/2)$$

is concave.

Remark A. Note that the restriction of a linear function on $\mathbb{R}^n$ to a convex subset need not be 0-concave in this sense.

Definition. A function $f\colon U \subset M \to \mathbb{R}$ is *semiconcave* if for any point $p \in U$ there is a neighborhood $U_p \ni p$ such that the function $f|_{U_p}$ is $\lambda-$concave for some λ.

Remark B. It is easy to see that the differential $d_p f$ is defined everywhere for semiconcave functions and is a 0-concave function on the tangent cone at p.

3.2. Definition. A vector field g is called a gradient of the concave function $f: U \subset M \to \mathbb{R}$ (or $g = \mathrm{grad} f$) if for any $p \in U$

$$d_p f(x) \leq \langle g, x \rangle \text{ and } d_p f(g) = \langle g, g \rangle.$$

3.3. Definition. A curve $\gamma: [a, b] \to U$ is called an *integral curve* for a vector field v on $U \subset M$ if for any t

$$\gamma^+(t) = v(\gamma(t)).$$

3.4. Definition. A curve $\gamma: [a, b] \to U$ is called a *gradient curve* for the semiconcave function $f: U \subset M \to \mathbb{R}$ if it is an integral curve for a gradient vector field of f.

We claim that semiconcave functions have uniquely defined gradient curves (see below Theorem 3.8). A similar construction was given by V.A.Sharafutdinov for convex functions in a smooth space with nonnegative curvature ([Sh Th.3]) and by Perelman for convex functions in an Alexandrov space with curvature ≥ 0 [P2 6.3].

We begin by developing the basic properties of gradient fields.

3.5. Lemma. Let $f: U \subset M \to \mathbb{R}$ be a semiconcave function.

A. Then f has a uniquely defined gradient vector field.

B. If $d_pf \leq 0$ then $\operatorname{grad} f(p) = o$; otherwise,

$$\operatorname{grad} f(p) = d_pf(\xi)\xi$$

where ξ is the (necessarily unique) maximum unit vector for the function d_pf.

C. If F is an extremal subset of C_p then $\operatorname{grad} f(p) \in F$.

Proof.

A & B. If $d_pf \leq 0$ we easily obtain $\operatorname{grad} f(p) = o$.

Now assume that $d_pf(y) > 0$ for some $y \in C_p$. Direct calculation shows that $\check{d}_pf = d_pf|_{\Sigma_p}$ is *spherically concave*, i.e. if $\gamma(t)$ is a shortest path with a natural parameter in Σ_p then

$$(\check{d}_pf \circ \gamma)''(t) + \check{d}_pf \circ \gamma(t) \leq 0.$$

Therefore for any $\zeta, \xi \in \Sigma_p$,

$$d_pf(\zeta) \leq (d_\xi \check{d}_pf)(\zeta'_\xi)\sin(|\xi\zeta|) + d_pf(\xi)\cos(|\xi\zeta|),$$

(where by ζ'_ξ we understand a tangent vector to Σ_p).

Let ξ be a maximum unit vector for d_pf. Since $d_\xi\check{d}_pf \leq 0$, for any ζ we obtain

$$d_pf(\zeta) \leq d_pf(\xi)\cos(|\xi\zeta|).$$

or for any $x \in C_p,\ x = \lambda\zeta,\ \lambda \in \mathbb{R}_+$ such that $|u| = 1$

$$d_pf(u) \leq d_pf(\xi)\langle u, \xi\rangle.$$

Since $d_pf(y) > 0$ for some y, we obtain $d_pf(\xi) \geq d_pf(y/|y|) > 0$ and therefore C_p contains the vector $d_pf(\xi)\xi$. Hence for any $x \in C_p$

$$d_pf(x) \leq \langle d_pf(\xi)\xi, x\rangle$$

and

$$d_pf(d_pf(\xi)\xi) = (d_pf(\xi))^2 = \langle d_pf(\xi)\xi, d_pf(\xi)\xi\rangle.$$

Therefore $d_pf(\xi)\xi$ is a gradient of f at p.

Now assume g_1 and g_2 are two gradient vectors at the point p. Then

$$\langle g_1, g_1\rangle = d_pf(g_1) \leq \langle g_2, g_1\rangle \geq d_pf(g_2) = \langle g_2, g_2\rangle$$

and from the definition of scalar product $|g_1g_2|^2 \leq 0$ and therefore $g_1 = g_2$.

C. Now assume that F is an extremal subset of C_p. It is easy to see that F has the same cone structure as C_p, i.e., if $x \in F$ then $\lambda x \in F$ for any $\lambda \geq 0$ and in particular $o \in F$.

The proof is by induction on the dimension of F.

Assume $\dim F = 0$. Then it is easy to see that $F = o$ and from the definition of extremal subset $\mathrm{diam}\Sigma_p \leq \pi/2$. Now let ζ be a minimum unit vector for d_pf on Σ_p and ξ be the unit vector, given above. From above

$$d_pf(\zeta) \leq d_pf(\xi)\cos(|\xi\zeta|),$$

and so $\xi = \zeta$ only if $\Sigma = \{\xi\}$. In this case $\mathrm{grad}f(p) = o$ from Definition 3.1(for a space with boundary).

Now assume $\xi \neq \zeta$. From the same idea as above we obtain:

$$d_pf(\xi) \leq d_pf(\zeta)\cos(|\xi\zeta|).$$

Therefore

$$d_p f(\xi) \le d_p f(\xi) \cos^2(|\xi\zeta|).$$

Therefore $d_p f(\xi) \le 0$ and hence $d_p f \le 0$ and $\mathrm{grad} f(p) = o \in F$.

Now suppose the claim is true if $\dim F < n$ and let $\dim F = n$.

Let $F' = F \bigcap \Sigma_p \ne \emptyset$. In order to prove that $\mathrm{grad} f \in F$ we need to prove that $\xi \in F$. Let $\xi^\circ \in F'$ be the closest point of F' to ξ (in length metric of Σ_p). From [PP1 1.4] F' is an extremal subset and from the definition of extremal subset (see [PP 1.1]) $|\xi\xi^\circ| \le \pi/2$. Now

$$d_p f(\xi) \le d_{\xi^\circ} d_p f(\xi'_{\xi^\circ}) \sin(|\xi\xi^\circ|) + d_p f(\xi^\circ) \cos(|\xi\xi^\circ|) =$$

$$= d_{\xi^\circ} d_p f(\xi'_{\xi^\circ}) \sin(|\xi\xi^\circ|) + d_p f(\xi^\circ) \cos(|\xi\xi^\circ|) \le$$

$$\le \langle (\mathrm{grad} d_p f|_{\Sigma_p})(\xi^\circ), \xi'_{\xi^\circ} \rangle \sin(|\xi\xi^\circ|) + d_p f(\xi^\circ) \cos(|\xi\xi^\circ|)$$

(where by ξ'_{ξ° we understand a tangent vector to Σ_p). Thus $d_p f|_{\Sigma_p}$ is a semiconcave function. Using the induction assumption we obtain $\mathrm{grad} d_p f|_{\Sigma_p}(\xi^\circ) \in C_{\xi^\circ}(F')$. Since $|\xi\xi^\circ|$ realizes the distance between ξ and F' we obtain that

$$\langle (\mathrm{grad} d_p f|_{\Sigma_p})(\xi^\circ), \xi'_{\xi^\circ} \rangle \le 0.$$

Hence

$$d_p f(\xi) \le d_p f(\xi^\circ) \cos(|\xi\xi^\circ|).$$

We know that $d_p f(\xi) > 0$ and $|\xi\xi^\circ| \le \pi/2$. Therefore if $\xi \ne \xi^\circ$ then

$$d_p f(\xi) < d_p f(\xi^\circ),$$

a contradiction ♠.

3.6. It is easy to see that for a cone with curvature ≥ 0 the function $-\langle a, *\rangle$ is concave for any $a \in C$. Set $a^* = \operatorname{grad}(-\langle a, *\rangle)(o)$ for any $a \in C_p$.

Corollary. a and a^* are polar vectors and $|a^*| \leq |a|$.

3.7. Lemma. If f is a semiconcave function, then the function $|\operatorname{grad} f| \colon U \to \mathbb{R}_+$ is semicontinuous, i.e.

$$\liminf_{p_i \to p} |\operatorname{grad} f(p_i)| \geq |\operatorname{grad} f(p)|.$$

Proof. If $|\operatorname{grad} f(p)| = 0$ then it is easy. Otherwise for any $\varepsilon > 0$ we can find a point p_ε such that $f(p_\varepsilon) - f(p) > (|\operatorname{grad} f(p)| - \varepsilon)|p_\varepsilon p|$ and $|p_\varepsilon p| < \varepsilon$. On the other hand, using λ-concavity we have

$$f(p_\varepsilon) \leq f(p_i) + d_{p_i} f(\log_{p_i} p_\varepsilon) + \lambda |p_i p_\varepsilon|^2/2.$$

Therefore

$$\lim_{p_i \to p} |\operatorname{grad} f(p_i)| \geq \lim_{p_i \to p} d_{p_i} f(\log_{p_i} p_\varepsilon)/|p_i p_\varepsilon| \geq$$

$$\geq \lim_{p_i \to p} (f(p_\varepsilon) - f(p_i))/|p_i p_\varepsilon| - \lambda |p_i p_\varepsilon|/2 \geq |\operatorname{grad} f(p)| - (1 + \lambda/2)\varepsilon.$$

When $\varepsilon \to 0$ we obtain the conclusion of the Lemma ♠.

3.8. Theorem. For any semiconcave function $f \colon M \to \mathbb{R}$ and point p there is a unique gradient curve $\gamma \colon [0, \infty) \to M$ such that $\gamma(0) = p$. Moreover if F is an extremal subset of M, and $\gamma(t_0) \in F$ for some t_0, then for any $t > t_0$ we have $\gamma(t) \in F$.

3.9. Proof. Let us construct an integral curve for the field

$$u = \begin{cases} o & \text{if } |\mathrm{grad} f| = 0, \\ \mathrm{grad} f/|\mathrm{grad} f| & \text{if } |\mathrm{grad} f| \neq 0. \end{cases}$$

If $u(p) = o$ then it is the trivial curve $\gamma(t) = p$. Otherwise we can find a ball neighborhood $B_\varepsilon(p)$ such that in this neighborhood, the function f is λ-concave for some λ. Using Lemma 3.7 we can also assume that $|\mathrm{grad} f| > \delta > 0$ everywhere in $B_\varepsilon(p)$. Now using Lemma 3.5(B) we obtain that the curve $\gamma\colon [0,\varepsilon] \to B_\varepsilon(p)$ is an integral curve for u if and only if for any t

$$(f \circ \gamma(t))'_+ = |\mathrm{grad} f(\gamma(t))|.$$

For any $\delta > 0$ and point p we can easily find a point q, such that $|pq| < \delta$ and $f(q) - f(p) > (|\mathrm{grad} f| - \delta)|pq|$. Using the standard open-closed arguments, for any $\delta > 0$ we can construct a curve $\gamma_\delta : [0, \varepsilon) \to B_\varepsilon(p)$, $\gamma_\delta(0) = p$, which can be divided into half open geodesics $[p_i, q_i)$ with the same property as above. Note that if p_i lies on some extremal subset F, then using Lemma 3.5 C we can find q_i on F. Now let $\gamma = \lim_{\delta \to 0} \gamma_\delta$. From the semicontinuity of $|\mathrm{grad} f|$ we obtain

$$(f \circ \gamma(t))'_+ \geq |\mathrm{grad} f(\gamma(t))|.$$

The reverse inequality is obvious because the limit of natural curves is 1-Lipschitz. Therefore γ is an integral curve for u.

Let $d\tau = |\mathrm{grad} f(\gamma(t))| dt$ and γ be an integral curve for field u. Then $\gamma \circ \tau^{-1}$ is easily a gradient curve for f.

Now let $a = a(t)$ and $b = b(t)$ be two gradient curves from $[0, \varepsilon]$ into $B_\varepsilon(p)$ and $a(0) =$

$b(0)$. From the first variation formula and the definition of gradient

$$\frac{d}{dt}|ab| = -\langle \mathrm{grad} f(a), (b)'_a \rangle - \langle \mathrm{grad} f(b), (a)'_b \rangle \leq -d_a f(b'_a) - d_b f(a'_b).$$

By the definition of λ-concave function we obtain that for any shortest path γ between points a and b the function

$$f \circ \gamma(t) - \lambda t^2/2$$

is concave. Therefore

$$d_a f(b'_a) + d_b f(a'_b) + \lambda |ab| \geq 0.$$

and therefore

$$d|ab| \leq \lambda |ab| dt.$$

Therefore we may obtain the uniqueness of gradient curves by the standard method of differential equations ♠.

§4 PROPERTIES of GRADIENT CURVES

4.0. Here we apply results of the previous paragraph to the case of distance functions and construct for such a gradient curve a "proper" parameterization. These "proper gradient curves" yield in particular a nonexpanding exponential map for Alexandrov spaces. They also make it possible to prove the Lieberman Lemma using Lieberman's original construction (see [L] and below II 1.1) for the case of Alexandrov spaces. In the next section we use proper curves in order to construct convex curves and pre-quasigeodesics which in turn leads to the construction of quasigeodesics.

4.1. Definition. Let $\gamma\colon (0,\infty) \to M\backslash p$ be a gradient curve for the function $\mathrm{dist}_p(x)$, with $\gamma(0) = p$ (where $\mathrm{dist}_p(x) \overset{\mathrm{def}}{=} |px|$). Let

$$\tau(0) = 0,\ d\tau = \left(\frac{\sigma_k(\tau)}{\sigma_k(|p(\gamma(t)|)}\right) dt.$$

τ will be called a proper parameterization of a *k-gradient curve* and $\alpha^k = \gamma \circ \tau^{-1}$ will be called a *k-proper gradient curve.*

Remark A. Note that the notion of *k-proper gradient curve* for the function $\mathrm{dist}_p(x)$ coincides with that of geodesic natural parameterization from p everywhere outside of $\mathrm{Cutloc}(M, p)$.

Remark B. From Theorem 3.8 we can find the unique gradient curve which starts at any point $q \neq p$. We can extend this curve to p using any shortest path between p and q. Therefore we obtain a gradient curve $\gamma_q\colon (0,\infty) \to M$ for dist_p such that $\gamma_q(0) = p$, $\gamma_q(|pq|) = q$. Set $\alpha_q^k = \gamma_q \circ \tau^{-1}$ as above. Therefore we have that for any point q there is a proper curve α_q^k such that $\alpha_q^k(0) = p$ and $\alpha_q^k(|pq|) = q$.

4.2. Definition. A curve $\gamma\colon [a, b] \to M$ is called *k−monotonic* if $|\gamma^+| \leq 1$ for all t and for any point $q \in M$ the comparison angle $\tilde{\angle} q\gamma(a)\gamma(t)$ in S_k is nonincreasing.

Remark. From Theorem 1.7 it is easy to see that the curve $\gamma\colon [a, b] \to M$ is convex if and only if for any $a' \in [a, b]$ the curve $\gamma|_{[a',b]}$ is monotonic.

4.3. Assume

$$k_\circ = \begin{cases} 0 & \text{if } k > 0, \\ k & \text{if } k \le 0. \end{cases}$$

Theorem. Let M be an Alexandrov space with curvature $\ge k$. Then any $k_\circ$-proper gradient curve for dist_p is $k_\circ$-monotonic.

Remark. The main reason for using $k_\circ$ instead of k is: $\pi(k_\circ)/2 = \infty$ and therefore $\pi(k_\circ)/2 \ge \mathrm{diam}(M)$. Furthemore, even if we consider a neighborhood of p with diameter $\le \pi(k)/2$, for positive k we will have different parameterizations with the needed properties (see this Theorem and Theorem 4.4), while for nonpositive bound we have one proper parameterization which meets all our needs.

Proof. As usual we prove only for $k_\circ = 0$. It is enough to prove that $\cos \tilde{\angle} q\alpha(0)\alpha(t)$ is a nondecreasing function. From the law of cosines

$$\cos \tilde{\angle} q\alpha(0)\alpha(t) = \frac{t^2 + |pq|^2 - |q\alpha(t)|^2}{2(t|pq|)}.$$

Now by differenting this formula we have

$$(\cos \tilde{\angle} q\alpha(0)\alpha(t))' \ge 0$$

is equivalent to

$$|q\alpha(t)|' \le \cos \tilde{\angle} \alpha(0)\alpha(t)q = \frac{|q\alpha(t)|^2 + t^2 - |pq|^2}{2t|q\alpha(t)|}.$$

Now using the first variation formula and the definition of gradient for $f = \mathrm{dist}_p$,

$$|q\alpha(t)|' = -\langle \alpha^+(t), q'_{\alpha(t)} \rangle = -\frac{|p\alpha(t)|}{t} \langle \mathrm{grad} f, q'_{\alpha(t)} \rangle \le -\frac{|p\alpha(t)|}{t} (d_{\alpha(t)} f)(q') =$$

$$= \frac{|p\alpha(t)|}{t}\cos\angle(q'_{\alpha(t)}, p'_{\alpha(t)}) \le \frac{|p\alpha(t)|}{t}\cos\tilde\angle q\alpha(t)p =$$

$$= \frac{|p\alpha(t)|}{t}\left\{\frac{|p\alpha(t)|^2 + |q\alpha(t)|^2 - |pq|^2}{2|p\alpha(t)||q\alpha(t)|}\right\} =$$

$$= \left\{\frac{|q\alpha(t)|^2 + t^2 - |pq|^2}{2t|q\alpha(t)|}\right\} - \frac{t^2 - |p\alpha(t)|^2}{2t|q\alpha(t)|} \le \cos\tilde\angle\alpha(0)\alpha(t)q \spadesuit.$$

4.4. Theorem. Let α_{q_1} and α_{q_2} be two $k_\circ$-proper curves for dist_p, passing through q_1 and q_2, respectively, and containing the shortest paths pq_i (see Remark 4.1 B). Suppose $t_i \ge |pq_i|$, and let $\tilde p\tilde q_1^{t_1}\tilde q_2^{t_2}$ be the triangle in $S_{k_\circ}$ such that $\tilde p\tilde q_1^{t_1} = t_1$, $\tilde p\tilde q_2^{t_2} = t_2$ and $\angle\tilde q_1^{t_1}\tilde p\tilde q_2^{t_2} = \tilde\angle q_1pq_2$. Then

$$|\alpha_{q_1}(t_1)\alpha_{q_2}(t_2)| \le |\tilde q_1^{t_1}\tilde q_2^{t_2}|.$$

Proof. Again we prove this only for $k_\circ = 0$. In an algebraic form it means that

$$\cos\tilde\angle q_1pq_2 \le \frac{t_1^2 + t_2^2 - |\alpha_{q_1}(t_1)\alpha_{q_2}(t_2)|^2}{2t_1t_2}.$$

Assume $q_1 \notin \mathrm{Cutloc}(p)$ (see 2.5). Then there is a point $q_1^\circ \in \mathrm{Cutloc}(p)$ such that $q_1 \in pq_1^\circ$. Now if $t_1 \le |pq_1^\circ|$ then Theorem follows from Theorem 4.3. Assume $t_1 > |pq_1^\circ|$. It is easy that $\alpha_{q_1} = \alpha_{q_1^\circ}$ and $\tilde\angle q_1pq_2 \ge \tilde\angle q_1^\circ pq_2$. Therefore assume that $q_1 := q_1^\circ \in \mathrm{Cutloc}(p)$. Analogously $q_2 \in \mathrm{Cutloc}(p)$. Now let $\delta_i(t) = t^2 - |p\alpha_{q_i}(t)|^2$. The function δ_i is easily positive increasing on $(|pq_i|, \infty)$. Now direct calculation shows that

$$\left(\frac{\tau_1^2 + \tau_2^2 - |\alpha_{q_1}(\tau_1)\alpha_{q_2}(\tau_2)|^2}{2\tau_1\tau_2}\right)_{\tau_1} \ge 0$$

if

$$|\alpha_{q_1}(\tau_1)\alpha_{q_2}(\tau_2)|_{\tau_1} \le \left(\frac{\tau_1^2 + |\alpha_{q_1}(\tau_1)\alpha_{q_2}(\tau_2)|^2 - \tau_2^2}{2\tau_1|\alpha_{q_1}(\tau_1)\alpha_{q_2}(\tau_2)|}\right),$$

but

$$|\alpha_{q_1}(\tau_1)\alpha_{q_2}(\tau_2)|_{\tau_1} = -\frac{|p\alpha_{q_1}(\tau_1)|}{\tau_1}\langle \mathrm{grad}(\mathrm{dist}_p), \alpha_{q_2}(\tau_2)'_{\alpha_{q_1}(\tau_1)}\rangle =$$

from the same idea as above

$$= \left(\frac{\tau_1^2 + |\alpha_{q_1}(\tau_1)\alpha_{q_2}(\tau_2)|^2 - \tau_2^2}{2\tau_1|\alpha_{q_1}(\tau_1)\alpha_{q_2}(\tau_2)|}\right) - \frac{\delta_1(\tau_1) - \delta_2(\tau_2)}{2\tau_1|\alpha_{q_1}(\tau_1)\alpha_{q_2}(\tau_2)|} \leq \left(\frac{\tau_1^2 + |\alpha_{q_1}(\tau_1)\alpha_{q_2}(\tau_2)|^2 - \tau_2^2}{2\tau_1|\alpha_{q_1}(\tau_1)\alpha_{q_2}(\tau_2)|}\right)$$

if $\delta_1(\tau_1) \geq \delta_2(\tau_2)$.

Analogously

$$\left(\frac{\tau_1^2 + \tau_2^2 - |\alpha_{q_1}(\tau_1)\alpha_{q_2}(\tau_2)|^2}{2\tau_1\tau_2}\right)_{\tau_2} \geq 0$$

if $\delta_1(\tau_1) \leq \delta_2(\tau_2)$.

Let us consider two functions $\tau_i(x)$ such that

$$\delta_i(\tau_i(x)) = \min\{x, \delta_i(t_i)\}.$$

Using previous calculations we obtain that

$$f(x) = \frac{\tau_1^2(x) + \tau_2^2(x) - |\alpha_{q_1}(\tau_1(x))\alpha_{q_2}(\tau_2(x))|^2}{\tau_1(x)\tau_2(x)}$$

is a nondecreasing function because

$$\begin{array}{ll} \delta_1(\tau_1(x)) = \delta_2(\tau_2(x)) & \text{if } \tau_1'(x) > 0 \text{ and } \tau_2'(x) > 0, \\ \delta_1(\tau_1(x)) \leq \delta_2(\tau_2(x)) & \quad\quad\quad\text{if } \tau_1'(x) > 0, \\ \delta_1(\tau_1(x)) \geq \delta_2(\tau_2(x)) & \quad\quad\quad\text{if } \tau_2'(x) > 0. \end{array}$$

But it is easy that

$$f(0) = \cos \tilde{\angle} q_1 p q_2$$

and

$$f(\infty) = \frac{t_1^2 + t_2^2 - |\alpha_{q_1}(t_1)\alpha_{q_2}(t_2)|^2}{2t_1t_2} \spadesuit.$$

4.5. Lemma. For any direction $\xi \in \Sigma_p$ there is unique $k_\circ$-proper gradient curve $\alpha_\xi\colon [0,\infty) \to M$ for dist_p such that $\alpha_\xi^+(0) = \xi$.

Proof. Let $\{q_i\}$ is a sequence such that $q_i' \to \xi$. Consider curves $\{\alpha_{q_i}\}$ (see Remark 4.1 B). We can pass to a subsequence $\{q_i\}$ such that for any t the following limit exists

$$\alpha_\xi(t) = \lim_{i\to\infty} \alpha_{q_i}(t).$$

$\alpha_\xi(t)$ is easily a proper curve. Uniqueness is an immediate corollary of Theorem 4.4 ♠.

4.6. Assume $\varpi_p^k : C_p^k \to M$ is the map defined by

$$\varpi_p^k(\lambda\xi) = \alpha_\xi^k(\lambda).$$

Theorem. For an Alexandrov space with curvature $\geq k$ the map $\varpi_p^{k_\circ}$ is a nonexpanding map and is a left inverse for $\log_p^{k_\circ}$, i.e.

$$\varpi_p^{k_\circ} \circ \log_p^{k_\circ} = \mathrm{id}.$$

Moreover if p lies in an extremal subset F then $\varpi_p^{k_\circ}(C_p(F)) \subset F$.

Proof. Immediately from 3.8, 4.4, 4.5 ♠.

§5 PRE-QUASIGEODESICS

5.0. In this paragraph we use proper curves in order to construct convex curves and pre-quasigeodesics which in turn lead to the construction of quasigeodesics. The first proof of the following theorem as well as the definition of pre-quasigeodesics was given by G.Perelman. Perelman's proof uses some topological arguments. Here we give another proof which is simpler and uses only geometric ideas.

5.1 Definition. A convex curve $\gamma[a, b] \to M$ is called a *pre-quasigeodesic* if for any $s \in [a, b]$ such that $|\gamma^+(s)| > 0$, the curve γ^s such that

$$\gamma^s(t) = \gamma\left(s + \frac{t}{|\gamma^+(s)|}\right)$$

is convex for $t \geq 0$ and if $|\gamma^+(s)| = 0$ then $\gamma^+(t) = \gamma^+(s)$ for any $t \geq s$.

5.2. Theorem. For any vector $v \in C_p$ such that $|v| \leq 1$ there is a pre-quasigeodesic $\gamma\colon [0, \infty) \to M$ such that $\gamma(0) = p$ and $\gamma^+(0) = v$.

We begin by developing the basic properties of convexity and monotonicity.

5.3. Lemma. Let $\gamma\colon [0, \infty) \to M$ be k_o-monotonic. Then for any $\lambda \leq 1$ the curve $\gamma'(t) = \gamma(\lambda^{-1} t)$ is k_o-monotonic.

Proof. We need only to prove that for any q function

$$\cos \tilde{\angle} q\gamma'(0)\gamma'(t) = \lambda \frac{(\lambda^{-1}t)^2 + |pq|^2 - |q\gamma(\lambda^{-1}t)|^2}{t|pq|}$$

is a nondecreasing function. Indeed

$$\lambda\frac{(\lambda^{-1}t)^2+|pq|^2-|q\gamma(\lambda^{-1}t)|^2}{t|pq|}=\lambda\frac{(t)^2+|pq|^2-|q\gamma(\lambda^{-1}t)|^2}{t|pq|}+(\lambda^{-1}-\lambda)t.$$

The first term is nondecreasing by the definition of monotonic curve. Therefore, as a sum of two nondecreasing functions, it is a nondecreasing function ♠.

5.4. Corollary. Let $\gamma\colon[0,\infty)\to M$ be $k_\circ$-convex. Then for any $\lambda\leq 1$ the curve $\gamma'(t)=\gamma(\lambda^{-1}t)$ is $k_\circ$-convex.

Proof. See Remark 4.2.

5.5. Lemma. Let $\gamma_1\colon[0,\infty)\to M$ and $\gamma_2\colon[0,\infty)\to M$ be two monotonic curves such that $\gamma_1(a)=\gamma_2(0)$ and $\gamma_1^+(a)=\gamma_2^+(0)$. Then the curve

$$\gamma\colon[0,\infty]\to M,\quad \gamma(t)=\left[\begin{array}{cc}\gamma_1(t) & \text{for } 0\leq t\leq a\\ \gamma_2(t-a) & \text{for } t\geq a\end{array}\right.$$

is monotonic.

Proof. We need only to prove that for any q the function

$$\tilde{\angle}q\gamma(0)\gamma(t)$$

is nonincreasing. As above we obtain that this is equivalent to

$$|q\gamma(t)|'\leq\cos\tilde{\angle}\gamma(0)\gamma(t)q.$$

We know this for the interval $[0,a]$; therefore we need to prove it only for (a,∞). From the monotonicity of γ_2 we obtain that

$$|q\gamma(t)|' \leq \cos \tilde{\angle}\gamma(a)\breve{\gamma}(t)q.$$

Therefore we need only prove that

$$\tilde{\angle}\gamma(a)\breve{\gamma}(t)q \geq \tilde{\angle}\gamma(0)\breve{\gamma}(t)q.$$

Using Lemma ([BGP 2.5]) we obtain that this is equivalent to

$$\tilde{\angle}q\,\gamma(\breve{a)}\gamma(t) + \tilde{\angle}\gamma(0)\breve{\gamma}(a)q \leq \pi.$$

The last inequality follows immediately from condition $\gamma_1^+(a) = \gamma_2^+(0)$ and monotonicity.

5.6. Lemma. Let $\gamma_i\colon [0,\infty) \to M$ be a sequence of monotonic curves such such that $\gamma_i(0) = p$, $\gamma_i^+(0) = \xi$, and $\gamma = \lim_{i\to\infty}\gamma_i$. Then γ is also monotonic and $\gamma^+(0) = \xi$.

Proof. Assume $\beta(t) = \angle((\gamma_1(t))'_p, \xi)$. Then we have $\beta(t)t = o(t)$. From the definition of monotonic curve we have $|\gamma_i(t)\gamma_1(t)| \leq \beta(t)t$, hence $|\gamma(t)\gamma_1(t)| \leq \beta(t)t = o(t)$. This means that γ and γ_1 go in the same direction ξ ♠.

5.7. Lemma. For any direction $\xi \in \Sigma_p$ there is a convex curve $\gamma\colon [0,\infty) \to M$ such that $\gamma(0) = p$, and $\gamma^+(0) = \xi$.

Proof. Take some $\varepsilon > 0$. From Lemmas 4.3, 4.5, 5.3 we can construct a monotonic map $\alpha(0) = p$ and $\alpha^+(0) = v$ such that $|v| \leq 1$. Now assume α_0 is a gradient curve with proper parametrization such that $\alpha_0(0) = p$ and $\alpha_0^+(0) = \xi$. Consider gradient curves α_n with proper parametrization such that $\alpha_n(0) = \alpha_{n-1}(\varepsilon)$ and $\alpha_n^+(0) = \alpha_{n-1}^+(\varepsilon)$. Let $\gamma^\varepsilon\colon [0,\infty) \to M$ be defined by

$$\gamma^\varepsilon = \alpha_n(t - n\varepsilon) \text{ for } t \in [n\varepsilon, (n+1)\varepsilon].$$

From Lemma 5.4 for any $n \in N$ the curve $\gamma^\varepsilon[n\varepsilon, \infty) \to M$ is monotonic. We can find a sequence $\{\varepsilon_n\}$ such that the following limit exists

$$\gamma = \lim_{\varepsilon_n \to 0} \gamma^{\varepsilon_n}$$

and it is easy to see that γ is a convex curve (see Remark 4.2). Lemma 5.6 completes the proof. ♠.

5.8. Proof of Theorem 5.2. First of all let us construct a convex curve $\gamma_\varepsilon\colon [0,\infty) \to M$ such that there is a representation of $[0,\infty)$ as a disjoint union of half-open intervals $[a_i, \bar{a}_i)$, such that for any a_i,

$$\gamma_\varepsilon^+(a_i) \geq \gamma_\varepsilon^+(t) \geq (1-\varepsilon)\gamma_\varepsilon^+(a_i)$$

for $t \in [a_i, \bar{a}_i)$ and $|\bar{a}_i - a_i| \leq \varepsilon$. Moreover the curve

$$\gamma_\varepsilon^{a_i}(t) = \gamma_\varepsilon\left(a_i + \frac{t}{|\gamma_\varepsilon^+(a_i)|}\right)$$

will be convex too.

Let us denote by γ_ξ a convex curve $[0,\infty) \to M$ such that $\gamma_\xi^+(0) = \xi$, $\gamma_\xi(0) = p$ where $\xi \in \Sigma_p$. The curve γ_ξ is not uniquely defined but for the rest of the proof we fix one for every p and $\xi \in \Sigma_p$. From Corollary 5.3 we have that for any $\lambda \leq 1$, $\gamma'_\xi(t) = \gamma_\xi(\lambda^{-1}t)$ is also convex. Therefore, in this way we can construct a convex curve for all initial data $x \in C_p$ such that $|x| \leq 1$. The resulting curves we will denote by $\gamma_x = \gamma_\xi(\lambda t)$ where $x = \lambda\xi$. It is easy that for any $\varepsilon > 0$ we can find $\varepsilon > \delta_\xi > 0$ such that $|\gamma_\xi^+(t)| \geq 1 - \varepsilon$ for $t \in [0, \delta_\xi]$. Assume γ_ε is defined on $[0, a)$. Then set

$$\gamma_\varepsilon(a) = \lim_{t \to a} \gamma_\varepsilon(t).$$

Assume $v = \bar{\gamma}_\varepsilon^-(a)$. Assume that for $t > a$

$$\gamma_\varepsilon(t) = \gamma_{v^*}(t - a) \text{ for } t \in [a_i, a_i + \delta_\xi/|v^*|),$$

(v^* is defined in 3.6).

Using the standard open-closed argument we obtain desired curve $\gamma_\varepsilon\colon [0, \infty) \to M$.

Now we can find a sequence $\varepsilon_n \to 0$ such that the following limit exists for all $t \geq 0$:

$$\gamma(t) = \lim_{n \to \infty} \gamma_{\varepsilon_n}(t).$$

From Lemma 2.14 and Corollary 2.15 γ is a convex curve and

$$\gamma^a(t) = \gamma\left(a + \frac{t}{|\gamma^+(a)|}\right)$$

for any $a \geq 0$ is also convex. Lemma 5.6 completes the proof ♠.

§6 GENERAL REMARKS and an OPEN QUESTION

6.1 One can consider the following definition of quasigeodesics:

Definition. A natural curve γ in an Alexandrov space is called a quasigeodesic if for any λ-concave function f (see 3.1) the function

$$f \circ \gamma - \lambda t^2/2$$

is concave.

This definition can be generalized on any metric space. It would be intreresting to develop quasigeodesics for spaces which have "a lot" of concave functions (these would be a multidimensional analog of the spaces considered in [AB]). Here we prove equivalence of this and the previous definition.

6.1 Theorem. The Definition 6.1 is equivalent to 1.5, or:

Let M be an Alexandrov space, $\gamma\colon \mathbb{R} \to M$ be a quasigeodesic. Then for any λ-concave function f the function $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is λ-concave.

The Theorems 3 and 4 of [Pog2] can be understood as a very partial case of this one.

For the proof we need the following

Definition. A function f is called a *strong λ-concave* function if for any quasigeodesic γ the function $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is λ-concave.

Proof. Using the last definition our Theorem can be reformulated as

"Any λ-concave function is strongly λ-concave."

It is easy to see that limit of strongly λ-concave functions is strongly λ-concave. Therefore it is enough to prove that for any $\varepsilon > 0$ the function f_ε defined by

$$f_\varepsilon(x) = \min_{y \in M} f(y) + \varepsilon^{-1}|xy|$$

is strongly λ-concave.

The rest of the proof is only a repetition of Perelman's proof [P2 6] (the distance to the boundary of an Alexandrov space with curvature ≥ 0 is a convex function).

5.6 Conjecture. Any quasigeodesic is a curve with bounded variation of turn.

From Lieberman's Theorem ([L]) and Theorem 6.1 this is true for quasigeodesics in an Alexandrov space which admits an embedding in $\mathbb{R}^n$ as a convex surface.

APPENDIX

CONSTRUCTION of QUASIGEODESICS

A.0. Here we finish the construction of quasigeodesics in every direction. All of the arguments of this appendix are due to G.Perelman. We include this section only for completeness of proofs.

A.1. Theorem. *(existence)* Let M be an n-dimmensional Alexandrov space and $p \in M$.

A. For any direction $\xi \in \Sigma_p$ there is a quasigeodesic $\gamma: [0, \infty) \to M$ such that $\gamma(0) = p$ and $\gamma^+(0) = \xi$.

A$'$. Let $F \ni p$ be an extremal subset and $\xi \in \Sigma_p(F)$. Then there is a quasigeodesic $\gamma: [0, \infty) \to F$ such that $\gamma(0) = p$ and $\gamma^+(0) = \xi$.

B. For any vector $v \in C$ (a cone with curvature ≥ 0) there is a polar vector $v^* \in C$, such that $|v^*| = |v|$.

B$'$. Let $C \supset F$ be an extremal subset and $F \ni v$. Then there is a polar vector $v^* \in F$, such that $|v^*| = |v|$.

(Compare with 2-dimensional cases [Pog Th.11] and [AB Th.4].)

We will prove $\mathrm{A}_{n-1} \to \mathrm{B}_n \to \mathrm{A}_n$ and $\mathrm{A}'_{n-1} \to \mathrm{B}'_n \to \mathrm{A}'_n$.

A.2. Proof of implications $\mathbf{A}_{n-1} \to \mathbf{B}_n$ ($\mathbf{A}'_{n-1} \to \mathbf{B}'_n$). B, ($\mathrm{B}'_n$) is trivial if $\dim C = 1$ ($\dim F = 1$) (for B$'$ see [PP1 1.4, 1.1]). Now assume $C = C(\Sigma)$ and $\dim C > 1$ ($\dim F > 1$). Assume $v = \lambda \xi$ where $\xi \in \Sigma$ and $\lambda \in \mathbb{R}_+$. Let us consider any quasigeodesic $\gamma: [0, \pi] \to \Sigma$ which starts at ξ (for B$'$ we need to consider $\gamma: [0, \pi] \to F' = F \bigcap \Sigma$; as was shown in [PP1 1.4], F' is an extremal subset of Σ). Assume $\xi^* = \gamma(\pi)$. From Theorem 1.7 we have that for any $\eta \in \Sigma$

$$\rho_1(|\gamma(t)\eta|)'' \leq 1 - \rho_1(|\gamma(t)\eta|).$$

or

$$\cos(|\gamma(t)\eta|)'' \geq \cos(|\gamma(t)\eta|).$$

Therefore for some c and all t, $0 < t \leq \pi$

$$\cos(|\gamma(t)\eta|) \geq \cos(|\gamma(0)\eta|) \cos(t) + c \sin(t).$$

Hence

$$\cos(|\gamma(\pi)\eta|) \geq \; -\cos(|\gamma(0)\eta|).$$

Therefore for any $\eta \in \Sigma$

$$\cos|\xi\eta| + \cos|\xi^*\eta| \geq 0.$$

Hence if $v^* = \lambda\xi^*$ then for any $x \in C$ we obtain

$$\langle v^*, x\rangle + \langle v, x\rangle \geq 0 \; \spadesuit.$$

A.3. Definition. Let $\gamma : [0, \infty) \to M$ be a pre-quasigeodesic. Let μ defined by

$$\mu(a, b) = \ln|\gamma^+(a)| - \ln|\gamma^-(b)|$$

be the *entropy* of γ.

Remark. It is easy to see that μ is a real measure because the function $|\gamma^\pm(t)|$ is monotonically nonincreasing (see Definition 5.1 and Remark 1.4). From one-sided continuity (see Corollary 2.15) we obtain that

$$\mu\{a\} = \ln|\gamma^-(a)| - \ln|\gamma^+(a)|.$$

In addition it is easy that a pre-quasigeodesic γ is a quasigeodesic if and only if μ is vanishing and $|\gamma^+(0)| = 1$.

A.4. Lemma. *(gluing)*

Let $\gamma_1: [0, l_1] \to M$ and $\gamma_2: [l_1, l_2] \to M$ be two pre-quasigeodesics and assume $\gamma_1(l_1) = \gamma_2(l_1)$, $\gamma_1^-(l_1)$ is polar to $\gamma_2^+(l_1)$ and $|\gamma_1^-(l_1)| \geq |\gamma_2^+(l_1)|$. Then the curve

$$\gamma(t) = \begin{bmatrix} \gamma_1(t) & \text{for } t \leq l_1 \\ \gamma_2(t) & \text{for } t \geq l_1 \end{bmatrix}$$

is a pre-quasigeodesic. Moreover if μ_1, μ_2 and μ are corresponding entropies for γ_1, γ_2 and γ, then

$$\mu(A \backslash l_1) = \mu_1(A \cap (0, l_1)) + \mu_2(A \cap (l_1, l_2))$$

and

$$\mu(l_1) = \ln(|\gamma_1^-(l_1)|) - \ln(|\gamma_2^+(l_1)|).$$

Proof. From Lemma 2.18 and definition of pre-quasigeodesic (see 5.1) we obtain the first conclusion. The equations for measures we obtain immediately from the definition of entropy (see A.3).

A.5. Lemma. *(passage to limit)* Let $\{\gamma_n\}$ be a sequence of pre-quasigeodesics $[a, b] \to M$ and μ_n be the sequence of corresponding entropies on $[a, b]$. Let γ be the limit curve for this sequence and μ be the weak limit of measures $\{\mu_n\}$. Then γ is a pre-quasigeodesic and μ is the entropy of γ.

Proof. Immediately from 2.15, 2.14 and definition of entropy (see A.3) ♠.

A.6. Lemma. Let $\gamma: [0, l] \to M$ be a pre-quasigeodesic, $p = \gamma(a)$. Let $\eta \in \Sigma_p$ denote the direction from p to some point q and $\xi \in \Sigma_p$ denote the direction of exit of γ at p.

Then

$$\left|\gamma^-\left(a+\frac{|pq|}{2|\gamma^+(a)|}\right)\right| \geq |\gamma^+(a)|(1-2\angle(\xi,\eta)^2).$$

If $\angle(\xi,\eta) \leq 1/2$ then

$$\mu(a, a+\frac{|pq|}{2|\gamma^+(a)|}) \leq 4\angle(\xi,\eta)^2.$$

Proof. We can assume that $a = 0$ and $|\gamma(0)| = 1$. In other case we only need to consider a curve γ^a instead of γ (see the definition of pre-quasigeodesic 5.1).

Now we need to prove that

$$|\gamma^-(|pq|/2)| \geq (1-2\angle(\xi,\eta)^2).$$

Indeed from the law of cosines

$$|\gamma(|pq|/2)q| \leq \frac{|pq|}{2}(1+2(\angle(\xi,\eta))^2).$$

On the other hand,

$$|pq| = |\gamma(0)q| \leq |\gamma(|pq|/2)q| + \frac{|pq|}{2}|\gamma^-(|pq|/2)|.$$

Therefore

$$|\gamma^-(|pq|/2)| \geq (1-2(\angle(\xi,\eta))^2).$$

The inequality for entropy is a trivial corollary of the last one ♠.

A.7. Proposition. Let the functions $h, g : \mathbb{R}_+ \to \mathbb{R}_+$ be such that for sufficiently small s,

$$h(s/3) \leq Cg^2(s),\ s \leq g(s) \text{ and } \lim_{s\to 0} g(s) = 0.$$

Then for any $\varepsilon > 0$ there is $s > 0$ such that

$$h(s) < 10Cg^2(s) \text{ and } g(s) \leq \varepsilon.$$

Proof. We can easily find some sufficiently small $s_o > 0$ such that for any $s \leq s_o$ $g(s) \leq \varepsilon$. Assume for any $n \in N$

$$10Cg^2(s_o/3^n) \leq h(s_o/3^n).$$

Then

$$10g^2(s_o/3^n) \leq g^2(s_o/3^{n-1})$$

and so

$$10^{1/2}g(s_o/3^n) \leq g(s_o/3^{n-1}).$$

Hence

$$\varepsilon \geq g(s_o) \geq 10^{n/2}g(s_o/3^n) \geq (10/9)^{n/2}s_o.$$

The right side goes to infinity when $n \to \infty$, a contradiction ♠.

A.8. Lemma. Let $\gamma\colon [0,l] \to M$ be a pre-quasigeodesic and $a \in [0,l]$. Then for any $\varepsilon > 0$ there is $\bar{a}$ such that if $p = \gamma(a), q = \gamma(\bar{a}), \eta \in \Sigma_p$ denotes the direction of $\overline{pq}$ and $\xi \in \Sigma_p$ denotes the direction of exit of γ at p, then there is some C ($C = 40$ will do) such that

$$\mu(a,\bar{a}) \leq C\left(\angle(\eta,\xi) + \bar{a} - a\right)^2 \text{ and } \angle(\eta,\xi) + \bar{a} - a \leq \varepsilon.$$

Proof. We can apply Proposition A.7 to functions $h(\bar{a}-a)=\mu(a,\bar{a})$ and $g(\bar{a}-a)=\angle(\eta,\xi)+\bar{a}-a$. Indeed it is easy that $\bar{a}-a\le g(\bar{a}-a)$ and we only need to prove that for sufficiently small $\bar{a}-a$,

$$h((\bar{a}-a)/3)\le 4g^2(\bar{a}-a).$$

Using Lemma A.6 we obtain that $h(\frac{|pq|}{2|\gamma^+(a)|})\le 4g^2(\bar{a}-a)$, h is a nondecreasing function and for sufficiently small $\bar{a}-a$ we have $(\bar{a}-a)/3<\frac{|pq|}{2|\gamma^+(a)|}$. Therefore $h((\bar{a}-a)/3)\le 4g^2(\bar{a}-a)$ ♠.

A.9. Proof of implications $\mathbf{B}_n\to\mathbf{A}_n$ and $\mathbf{B}'_n\to\mathbf{A}'_n$.

Definition. Let $U\subset M$ be an open subset. A pre-quasigeodesic γ is called a U-quasigeodesic if $\mu(\gamma^{-1}(U))=0$.

Let V be an open subset of M. For $0<A\le 1$ we denote by V_A the set of points $p\in V$ such that $\mathrm{Vol}\,(\Sigma_p)>A\mathrm{Vol}\,(S^{n-1})$. This set is open for any A (see [BGP 7.14]). Assume

$$A_0=\frac{1}{2}\inf_{x\in V}\mathrm{Vol}\,\Sigma_x/\mathrm{Vol}\,S^n.$$

Then it is easy to see that $V_{A_0}=V$ and if V is bounded then $A_0>0$. Let

$$V_A(\delta)=\{x\in M:\mathrm{Vol}\,B_\delta(x)>A\delta^n\,\mathrm{Vol}\,B^n,\ \bar{B}_\delta(x)\subset V\}.$$

Clearly $V_A(\delta)$ is an open subset of V_A and $V_A=\lim_{\delta\to 0}V_A(\delta)$.

We are going to prove that for any $0<A\le 1$ and ξ there exists an M_A-quasigeodesic $\gamma\colon[0,\infty)\to M$ with any initial date $\gamma(0)=p$, $\gamma^+(0)=\xi$. We proceed "by induction",

namely, we prove that if this assertion is true for $A(1+\epsilon)$ then it is true for A, where $\epsilon > 0$ is a fixed small number determined in A.11. Since our assertion is trivial for $A = 1$ (since $M_{A_0} = M$ and a U-quasigeodesic which is contained in U is a quasigeodesic (see A.3)) this inductive argument proves the existence theorem.

Now in order to prove existence of M_A-quasigeodesics we need construct them only locally, i.e statements A and A$'$ can be reduced to the local statements:

A$_{loc}$. For any $p \in M$ and $A > 0$ there is some $l > 0$ such that for any direction $\xi \in \Sigma_p$ there is an M_A-quasigeodesic $\gamma: [0, l] \to M$ such that $\gamma(0) = p$ and $\gamma^+(0) = \xi$.

A$'_{loc}$. In addition to A$_{loc}$, let $F \ni p$ be an extremal subset and $\xi \in \Sigma_p(F)$. Then there is an M_A-quasigeodesic $\gamma: [0, l] \to F$ such that $\gamma(0) = p$ and $\gamma^+(0) = \xi$.

Indeed let l_{max} be the maximum value such that there is an M_A-quasigeodesic

$$\gamma: [0, l_{max}] \to M \ (F)$$

with initial date p, ξ. Then using B (B$'$) we can find a polar vector $\gamma^+(l_{max})$ to $\gamma^-(l_{max})$ with the same absolute value and using A$_{loc}$ (A$'_{loc}$) we can find an M_A-quasigeodesic $\gamma_1: [l_{max}, l_{max} + l] \to M \ (F)$ with initial data $\gamma(l_{max}), \gamma^+(l_{max})$. Using the gluing Lemma (A.4) we can glue this M_A-quasigeodesic and obtain an M_A-quasigeodesic $\gamma: [0, l_{max} + l] \to M \ (F)$. Therefore l_{max} is not a maximal value, a contradiction.

A.10. Now we start to prove A$_{loc}$ and A$'_{loc}$ with assumption that for any initial dates there is a complete $M_{A(1+\epsilon)}$-quasigeodesic. Using Lemma A.6 we can find for any p some

$c > 0$ and $l > 0$ such that for any pre-quasigeodesic $\gamma\colon [0, l] \to M$ such that $\gamma(0) = p$ and $|\gamma^+(0)| = 1$ we have $|\gamma^-(l)| > c$.

For the rest we fix p and assume $V = B_l(p)$. It is easy that any V_A-quasigeodesic $\gamma\colon [0, l] \to M$ such that $\gamma(0) = p$ is an M_A-quasigeodesic (see Remark 1.4).

We will construct this V_A-quasigeodesic as a limit of $V_A(\delta)$-quasigeodesics for $\delta \to 0$ and each $V_A(\delta)$-quasigeodesic as a limit of $M_{A(1+\varepsilon)}$-quasigeodesics $\gamma_j\colon [0, l] \to M$, with the same initial data, such that $\mu_j(\gamma_j^{-1}(V_A(\delta))) \to 0$ when $j \to \infty$. From the convexity property and Lemma about passage to limit (A.5) it is easy to see that the limit of such a sequence is a V_A-quasigeodesic with required initial data.

Thus, the only remaining problem is to construct a sequence of $M_{A(1+\epsilon)}$-quasigeodesic $\gamma_j\colon [0, l] \to M$ with given initial data p, ξ such that $\mu_j(\gamma_j^{-1}(V_A(\delta))) \to 0$ for given $\delta > 0$.

Let $H_A(\delta) = V_A(\delta) \backslash M_{A(1+\epsilon)}$

Take some sequence $\varepsilon_j \to 0$.

We will look for the $M_{A(1+\epsilon)}$-quasigeodesic $\gamma_j\colon [0, l] \to M$ such that there is a representation of $[0, l]$ as a disjoint union of half-open intervals $[a_i, \bar{a}_i)$ and $\{l\}$ such that $\mu_j\{a_i\} = 0$, and any arc $\gamma_j|_{[a_i, \bar{a}_i)}$ is one of the following kinds:

1. $i \in I$, $V_A(\delta)$-quasigeodesic $\gamma_j|_{[a_i, \bar{a}_i)}$.

2. $i \in J$, $M_{A(1+\epsilon)}$-quasigeodesic $\gamma_j|_{[a_i, \bar{a}_i)}$, $p_i = \gamma_j(a_i)$, $q_i = \gamma_j(\bar{a}_i)$, $\eta_i \in \Sigma_{p_i}$ is the direction of $\overline{p_i q_i}$ and $\xi_i \in \Sigma_{p_i}$ is the direction of exit of γ_j. Such that

$$\mu_j(a_i, \bar{a}_i) \leq C(\angle(\eta_i, \xi_i) + |\bar{a}_i - a_i|)^2,$$

$$p_i \in H_A(\delta),\ \xi_i \in \Sigma_{p_i}(H_A) \text{ and } \angle(\eta_i, \xi_i) + |\bar{a}_i - a_i| \le \varepsilon_i.$$

We start with $a = 0$. Take any $M_{A(1+\epsilon)}$-quasigeodesic $\gamma\colon [0, l] \to M$ with initial data p, ξ. Assume there is some subarc $\gamma|_{[0,\bar{a})}$ which is a $V_A(\delta)$-quasigeodesic. Then set $\gamma_j|_{[0,\bar{a})} = \gamma|_{[0,\bar{a})}$. If there is no such arc then it is easy to see that $p \in H_A(\delta)$ and $\xi \in \Sigma_p(H_A)$ and therefore using Lemma A.8 we obtain existence of a subarc $\gamma|_{[0,\bar{a})}$ of the second kind and assume that $\gamma_j|_{[0,\bar{a})} = \gamma|_{[0,\bar{a})}$.

Similarly assume we already have constructed γ_j on $[0, a)$. Assume $\gamma_j(a) = \lim_{t \to a} \gamma_j(t)$. Using B (or B$'$) we can find a vector polar to $\gamma_j^-(a)$ and having the same absolute value. Denote this vector by $\gamma_j^+(a)$. Using the assumption we can construct an $M_{A(1+\epsilon)}$-quasigeodesic $\gamma\colon [a, l] \to M$ with this initial data. As before assume there is some subarc $\gamma|_{[a,\bar{a})}$ which is a $V_A(\delta)$-quasigeodesic. Then set $\gamma_j|_{[a,\bar{a})} = \gamma|_{[a,\bar{a})}$. If there is no such arc then it is easy to see that $p = \gamma(a) \in H_A(\delta)$ and $\gamma^+(a) \in C_p(H_A(\delta))$ and therefore using Lemma A.6 we obtain existence of a subarc $\gamma|_{[a,\bar{a})}$ of the second kind and assume that $\gamma_j|_{[a,\bar{a})} = \gamma|_{[a,\bar{a})}$. After gluing we obtain a suitable $M_{A(1+\epsilon)}$-quasigeodesic $\gamma_j : [0, \bar{a}) \to M$. Now using open-closed argument we obtain needed $M_{A(1+\epsilon)}$-quasigeodesic $\gamma_j : [0, l] \to M$.

By the definition of the first kind of arcs, it has zero measure on $V_A(\delta)$, therefore we need just to estimate entropy on the second class.

$$\mu_j(\gamma_j^{-1}(V_A(\delta))) =$$

$$= \sum_{I \cup J} \mu_j\left(\{a_i\} \cap \gamma_j^{-1}(V_A(\delta))\right) + \sum_I \mu_j\left((a_i, \bar{a}_i) \cap \gamma_j^{-1}(V_A(\delta))\right) +$$

$$+\sum_J \mu_j \left((a_i, \bar{a}_i) \cap \gamma_j^{-1}(V_A(\delta))\right) \leq$$

$$\leq 0 + 0 + C\sum_J (\angle(\eta_i, \xi_i) + |\bar{a}_i - a_i|)^2 \leq$$

$$\leq C\varepsilon_j \sum_J (\angle(\eta_i, \xi_i + |\bar{a}_i - a_i|) \leq Cl\varepsilon_j + C\varepsilon_j \sum_J \angle(\eta_i, \xi_i).$$

Therefore in order to prove that $\mu_j(\gamma_j^{-1}(V_A(\delta))) \to 0$ we need only to prove

Key Lemma.

There is a constant K which does not depend on ε_j, $(K = K(c, \delta, \epsilon, l, n))$ such that

$$\sum_J \angle(\eta_i, \xi_i) \leq K.$$

In order to prove this we need the following Lemmas.

Set $f_x = \mathrm{dist}_x^2/2$.

A.11. Lemma. There exists $\epsilon > 0$ (depending on n and A_0) such that if $p \in H_A(\delta)$, $\xi \in \Sigma_p(H_A(\delta))$, then for any $\eta \in B_\xi(\pi/4) \subset \Sigma_p$ there is a subset $v(\eta, \xi) \subset B_\delta(p)$ such that:

$$\mathrm{Vol}\, v(\eta, \xi) \geq \epsilon\, \delta^n \,\mathrm{Vol}\, B^n$$

and for any $x \in v(\eta, \xi)$

$$df_x(\xi) - \epsilon\delta\angle(\xi, \eta) \geq df_x(\eta) \geq 0.$$

Proof. Let us construct a noncontracting map $B_\delta(p) \to B_\delta(o) \subset C(S(\Sigma_\xi(\Sigma_p)))$

The map $\log_p$ acting from $B_\delta(p)$ to $B_\delta(o) \subset C_p$ is a noncontracting map.

Now the map $\log_\xi : \Sigma_p \to S(\Sigma_\xi)$ is noncontracting. Let us define map $h : B_\delta(o) \subset C_p \to B_\delta(o) \subset C(S(\Sigma_\xi(\Sigma_p)))$ by

$$h(\lambda\theta) = \lambda \log_\xi(\theta).$$

It is a noncontracting map too. Therefore map $h \circ \log_p$ is a noncontracting map $B_\delta(p) \to B_\delta(o) \subset C(S(\Sigma_\xi(\Sigma_p)))$.

Let $\hat{\xi}$ be a pole of $S(\Sigma_\xi(\Sigma_p))$ and $\hat{\eta} = \log_\xi(\eta)$. Assume

$$U(\eta, \xi) = \{x \in B_\delta(o) \subset C(S(\Sigma_\xi(\Sigma_p)));$$

$$x = \lambda\theta,\ \delta/2 \le \lambda \le \delta\ \angle\theta\xi\eta \le \pi/4 \text{ and } 3\pi/4 \le \theta\xi \le 4\pi/5\}.$$

Then it is easy that there is some $c > 0$ such that

$$\operatorname{Vol} U(\eta,\xi) \ge c \operatorname{Vol}(B_\delta(o) \subset C(S(\Sigma_\xi(\Sigma_p)))),$$

and for any $x \in B_\delta(p)$ such that $h \circ \log_p(x) \in U(\eta,\xi)$

$$df_x(\xi) - c\delta\angle(\xi,\eta) \ge df_x(\eta) \ge 0.$$

Assume $u(\eta,\xi)$ is the inverse image of $U(\eta,\xi)$ and $\operatorname{Vol} u(\eta,\xi) < \epsilon\delta^n \operatorname{Vol} B^n$. Then using monotonicity of h we obtain

$$\operatorname{Vol} B_\delta(p) \le \operatorname{Vol} u(\eta,\xi) + \operatorname{Vol}(B_\delta(o) \backslash U(\eta,\xi) \subset C(S(\Sigma_\xi(\Sigma_p))) \le$$

$$\le \epsilon\delta^n \operatorname{Vol} B^n + (1-c)\operatorname{Vol}(B_\delta(o) \subset C(S(\Sigma_\xi(\Sigma_p))).$$

From $\xi \in \Sigma_p(H_A(\delta))$ it is easy to see that

$$\operatorname{Vol}(B_\delta(o) \subset C(S(\Sigma_\xi(\Sigma_p))) \le A(1+\epsilon)\delta^n \operatorname{Vol} B^n$$

and from the definition of $H_A(\delta)$ we obtain that

$$\operatorname{Vol} B_\delta(p) > A\delta^n \operatorname{Vol} B^n.$$

Therefore

$$A\delta^n \operatorname{Vol} B^n < (\epsilon + (1-c)A(1+\epsilon))\delta^n \operatorname{Vol} B^n$$

and this is impossible for sufficiently small ϵ ♠.

A.12. Lemma. Let γ be a pre-quasigeodesic, $p = \gamma(a), q = \gamma(\bar{a}), \eta \in \Sigma_p$ denote the direction of $\overline{pq}$, $\xi \in \Sigma_p$ denote the direction of exit of γ at p ($\xi = \frac{\gamma^+(a)}{|\gamma^+(a)|}$) and $f_x = \mathrm{dist}_x^2/2$. Assume that

$$0 \le df_x(\eta) \le df_x(\xi).$$

Then

$$df_x(\xi) - df_x(\eta) \le |\gamma^+(a)|^{-1} \left(df_x(\gamma^+(a)) + df_x(\gamma^-(\bar{a})) + \bar{a} - a\right).$$

Proof. Clearly from Theorem 1.7

$$f_x(q) - f_x(p) \le df_x(\eta)|pq| + |pq|^2/2,$$

$$f_x(p) - f_x(q) \le df_x(\gamma^-(\bar{a}))(\bar{a} - a) + (\bar{a} - a)^2/2.$$

After summing

$$0 \le df_x(\eta)|pq| + df_x(\gamma^-(\bar{a}))(\bar{a} - a) + |pq|^2/2 + (\bar{a} - a)^2/2 \le$$

$$\le (df_x(\eta)|\gamma^+(a)| + df_x(\gamma^-(\bar{a})))(\bar{a} - a) + (\bar{a} - a)^2 \le$$

$$\leq \left(df_x(\xi)|\gamma^+(a)| + df_x(\gamma^-(\bar{a})) - (df_x(\xi) - df_x(\eta))|\gamma^+(a)| + (\bar{a} - a)\right)(\bar{a} - a) =$$

$$= \left(\{df_x(\gamma^+(a)) + df_x(\gamma^-(\bar{a}))\} - \{df_x(\gamma^+(a))/|\gamma^+(a)| - df_x(\eta)\}|\gamma^+(a)| + \{\bar{a} - a\}\right)(\bar{a}-a).$$

which is the desired formula rewritten ♠.

A.13. Proof of Key Lemma (A.10). Let $f_x = \mathrm{dist}_x^2/2$. From definition of prequasigeodesic for any $x \in M$ function $f_x \circ \gamma(t) - t^2/2$ is concave and therefore

$$\left(df(\gamma^+(a)) + df(\gamma^-(\bar{a})) + \bar{a} - a\right) \geq 0.$$

Assume $\bar{f} = \text{mean value}_{x \in V} f_x$. Using Lemma A.11 and Lemma A.12 we obtain

$$\angle(\eta_i, \gamma^+(a_i)) \leq \frac{1}{\epsilon\delta \mathrm{Vol}\, v(\eta,\xi)} \int_{v(\eta,\xi)} (df_x(\xi) - df_x(\eta)) dh_n \leq$$

$$\leq \frac{1}{\epsilon^2\delta^{n+1}|\gamma^+(a)|} \int_{v(\eta,\xi)} \left(df_x(\gamma^+(a)) + df_x(\gamma^-(\bar{a})) + \bar{a} - a\right) dh_n \leq$$

$$\leq \frac{\mathrm{Vol}\, V}{\epsilon^2\delta^{n+1} c\, \mathrm{Vol}\, v(\eta,\xi)} \left(d\bar{f}(\gamma^+(a)) + d\bar{f}(\gamma^-(\bar{a})) + \bar{a} - a\right) \leq$$

$$\leq \frac{l^n}{\epsilon^2\delta^{2n+1}c} \left(d\bar{f}(\gamma^+(a)) + d\bar{f}(\gamma^-(\bar{a})) + \bar{a} - a\right).$$

And therefore

$$\sum_J \angle(\eta_i, \xi_i) \leq \frac{l^n}{\epsilon^2\delta^{2n+1}c} \sum_J \left(d\bar{f}(\gamma^+(a)) + d\bar{f}(\gamma^-(\bar{a})) + \bar{a} - a\right) \leq$$

$$\leq \frac{l^n}{\epsilon^2\delta^{2n+1}c} \left(d\bar{f}(\gamma^+(0)) + d\bar{f}(\gamma^-(l)) + l\right) \leq$$

$$\leq \frac{5l^{n+1}}{\epsilon^2\delta^{2n+1}c} = K(c,\delta,\epsilon,l,n)$$

and we obtain needed estimate ♠♠.

2.

II. APPLICATIONS of QUASIGEODESICS and GRADIENT CURVES

Here we have gathered together some applications of quasigeodesic and gradient curves. The first section considers extremal subsets; in the second section we prove the Gluing Theorem for multidimensional Alexandrov spaces; in the third we give another proof of the Radius Sphere Theorem of Grove and Petersen.

• Let H be a subset of M and $p, q \in H$. By $|pq|_H$ we will denote the distance between p and q in the intrinsic metric of H.

• Let X be a metric space with metric ρ. By X/c we will denote the space X with metric ρ/c. Where no confusion will arise, we may use the same notation for points in X and their images in X/c.

§1 INTRINSIC METRIC of EXTREMAL SUBSETS

Наташе

1.0. The notion of extremal subset was introduced in [PP1 1.1] and has turned out to be very important for the geometry of Alexandrov spaces. It gives a natural stratification of Alexandrov space into open topological manifolds. Also, as is shown in recent results of G.Perelman, extremal subsets in some sense account for the singular behavior of collapse. Therefore the intrinsic metric of such subsets turns out to be important. Moreover, there is hope that extremal subsets with intrinsic metric will give a way to approach the idea of multidimensional generalized spaces with bounded integral curvature.

In this section we give a new proof of the generalized Lieberman lemma, prove a kind of "stability" property for extremal subsets and prove the First Variation formula for the intrinsic metric of extremal subsets. The Lieberman lemma can be understood as a totally quasigeodesic property of extremal subsets and therefore offers some hope that extremal subsets with the intrinsic metric might be Alexandrov spaces with the same curvature bound; at the end of this section we give a counterexample to this conjecture for extremal subsets with codimension ≥ 3. Therefore this question is still open for codimension one (i.e for a boundary) and for codimension two.

1.1. Theorem. *(generalized Lieberman lemma)* Any shortest path in the intrinsic metric of an extremal subset $F \subset M$ is a quasigeodesic in M.

The first proof of this Theorem was given in [PP1 5.3]

Proof. Assume γ is a shortest path in the length metric of some extremal subset F. Suppose γ is not a quasigeodesic. Then there is a point p such that the development $\tilde{\gamma}(t)$ from p is not convex in every neighborhood of some t_0. Now for every $\varepsilon > 0$ it is easy to find a "rounded" curve $\tilde{\delta}(t)$ such that $\tilde{\delta}(t) = \tilde{\gamma}(t)$ if $|t - t_0| > \varepsilon$, $length(\tilde{\delta}) < length(\tilde{\gamma}) = length(\gamma)$ and for every t the points $\tilde{p}$, $\tilde{\gamma}(t)$, $\tilde{\delta}(t)$ are collinear in the same order. Now let us consider the curve in M given by

$$\delta(t) = \alpha_{\gamma(t)}(|\tilde{p}\tilde{\delta}(t)|),$$

where $\alpha_{\gamma(t)} : [0, \infty) \to M$ is the dist_p-gradient curve which goes through $\gamma(t)$ such that

$\alpha_{\gamma(t)}|_{[0,|p\gamma(t)|]}$ is a shortest path, and reparameterized as in [I 4.1].

By Theorem [I 3.8](which states that if such a gradient curve starts at a point of an extremal subset F then it is contained in F) we obtain that $\delta \subset F$. From [I 4.4](expansion along gradient curves is not more than in the model space)

$$length(\delta) \leq length(\widetilde{\delta}) < length(\widetilde{\gamma}) = length(\gamma).$$

Therefore γ is not a shortest path in F ♠.

1.2 Theorem. Let $M_n \overset{GH}{\to} M$ without collapse (i.e. $\dim M_n \equiv \dim M$) and $F_n \subset M_n$ be extremal subsets. Assume $F_n \to F \subset M$ as subsets. Then $F_n \overset{GH}{\to} F$ as length metric spaces with intrinsic metrics induced from M_n and M.

Proof. Let x and y lie in an extremal subset G. By the equivalence of the intrinsic metric of an extremal subset and the metric of the ambient space [PP1 3.2(2)], we have for every open subset U in M an $\epsilon = \epsilon(\mathrm{Vol}_n(U), \mathrm{Diam}(U)) > 0$ such that $|xy|_G \leq \epsilon^{-1}|xy|$ if $x, y \in U$. (The dependence on $\mathrm{Vol}_n(U)$ and $\mathrm{Diam}(U)$ can be easily obtained from the proof).

Consider $p, q \in F$ and $p_n, q_n \in F_n$ such that $p_n \to p$ and $q_n \to q$. It is easy to see that $|pq|_F \leq \liminf_{n\to\infty} |p_n q_n|_{F_n}$. Therefore we need to show only that $|pq|_F \geq \limsup_{n\to\infty} |p_n q_n|_{F_n}$. Set $\|pq\| = \limsup_{n\to\infty} |p_n q_n|_{F_n}$; it is easily a metric. From the previous paragraph $\|pq\|$ does not depend on the choice of sequences $\{p_n\}$, $\{q_n\}$ and we have $\|pq\| < \epsilon^{-1}|pq|$, because from above ϵ can be found uniformly for all M_n in the absence of collapse.

Let $\gamma: [a,b] \to F$ be a shortest path in F between p and q parameterized by arclength. Assume $|pq|_F < \|pq\|$. Then from [Bus, 5.14] for some $t_0 \in [a,b]$ and $\varepsilon > 0$ there is a sequence $t_i \to t_0 \pm$ such that

$$\|\gamma(t_0)\gamma(t_i)\| \geq (1+\varepsilon)|t_i - t_0|.$$

Setting $r = \gamma(t_0)$ and $s = \gamma(t_i)$, take sequences $r_n, s_n \in F_n$ such that $r_n \to r$ and $s_n \to s$. Let γ_i in F be the limit curve to the shortest paths between r_n and s_n in F_n. By [I 2.4] and the generalized Lieberman Lemma, γ_i is a quasigeodesic between $\gamma(t_0)$ and $\gamma(t_i)$. From above $length(\gamma_i) \geq (1+\varepsilon)|t_i - t_0|$. Now let us consider the limit $(M/|t_0 - t_i|, r) \to (C_r, o)$. Consider the curve in C_r given by

$$\gamma_*(t) = \lim_{i\to\infty}(\frac{\gamma_i}{|t_0 - t_i|})(t \cdot |t_0 - t_i|) \in M/|t_0 - t_i|,$$

where $(\gamma_i/|t_0 - t_i|)$ denotes the image of γ_i in $M/|t_0 - t_i|$. Then γ_* is a quasigeodesic between o and the tangent vector $\gamma^{\pm}(t_0)$ which has length not less then $1+\varepsilon$. This is a contradiction since $|\gamma^{\pm}(t_0)| = 1$ by [I 2.15]♠.

Remark. The author does not know a counterexample for the following conjecture:

Let $M_n \overset{GH}{\to} M$, dim$M_n \leq N$ and $F_n \subset M_n$ be extremal subsets. Assume $F_n \to F \subset M$ as subsets and $F_n \overset{GH}{\to} \bar{F}$. Then there is a discrete group of isometries G on $\bar{F}$ such that $F = \bar{F}/G$.

As an example, consider a collapse of spaces with boundary $M_i \overset{GH}{\to} M$ such that dim$M$ =dim$M_i - 1$. Then $\partial M_i \to M$ as subsets and $\partial M_i \overset{GH}{\to} \widetilde{M}$ where $\widetilde{M}$ is the double of M.

1.3.

• Let M be an Alexandrov space and $F \subset M$ be an extremal subset. From the generalized Lieberman Lemma every shortest path in the length metric of F is a quasigeodesic as a curve in M and every quasigeodesic at every point has directions of exit and entrance (see [I 2.12 and 2.15]). Thus if p and q lie in F we can define $q^\circ(= q_p^\circ)$ as the set of all directions of entrance in $\Sigma_p(F)$ of shortest paths between p and q in the length metric of F. It is easy to see that q° is compact.

Theorem. *(The first variation formula)* Let F be an extremal subset of the Alexandrov space M. Let $p, q \in F$ and $\xi(t)$ be a curve in F starting from p in direction $\xi_o' \in \Sigma_p(F)$. Assume $|p\,\xi(t)| = t + o(t)$. Then

$$|\xi(t)\,q|_F = |pq|_F - \cos|\xi_o' q^\circ|_{\Sigma_p(F)} \cdot t + o(t).$$

Proof. To prove this we have to prove two inequalities:

$$\begin{array}{ll} \textbf{(i)} & |\xi(t)q|_F \leq |pq|_F - \cos|\xi_o' q^\circ|_{\Sigma_p(F)} \cdot t + o(t) \\ \textbf{(ii)} & |\xi(t)q|_F \geq |pq|_F - \cos|\xi_o' q^\circ|_{\Sigma_p(F)} \cdot t + o(t) \end{array}$$

and we will prove them separately.

1.4. Proof of (i). Take some $R \gg 1$. Set $\alpha = |\xi_o' q^\circ|_{\Sigma_p(F)}$ and $|pq|_F = l$. Take $\eta \in q^\circ$ such that $\alpha = |\xi_o' q^\circ|_{\Sigma_p(F)} = |\xi_o' \eta|$ and let $\gamma\colon [0, l] \to F$ be a shortest path between p and q

in F such that $\gamma(0) = p$ and $\gamma^+(0) = \eta$. Then by the triangle inequality

$$|\xi(t)\, q|_F \leq l - Rt + |\xi(t)\, \gamma(Rt)|_F.$$

The cosine rule gives us that

$$|\xi'_o\, R\eta|_{C_p(F)} = \sqrt{R^2 + 1 - 2R\cos\alpha}.$$

Now using Theorem 1.2 for limit $(M/t, p) \to C_p$, we obtain

$$\lim_{t \to 0} |\xi(t)\gamma(Rt)|_F / t = |\xi'_o\, R\eta|_{C_p(F)}.$$

Therefore

$$|\xi(t)\, q|_F \leq l - Rt + t\sqrt{R^2 + 1 - 2R\cos\alpha} + o(t) \leq$$

$$\leq l - \cos\alpha \cdot t + \frac{t}{R-1} + o(t).$$

When $R \to \infty$ we obtain

$$|\xi(t) q|_F \leq |pq|_F - \cos|\xi'_o q^\circ|_{\Sigma_p(F)} \cdot t + o(t) \; \spadesuit.$$

1.5. To prove (ii), we shall use the following

Lemma. Let $C = C(\Sigma)$ be a cone with curvature ≥ 0 (so curvature of $\Sigma \geq 1$). Let γ be a quasigeodesic in C not passing through the vertex o. Then the projection of γ on Σ parameterized by the arclength is a quasigeodesic in Σ and the development of γ in the plane with respect to the vertex of C is a straight line.

Proof. To prove the second part of this lemma we have to prove that

$$(|\gamma(t)|^2)'' = 2.$$

In order to prove

$$(|\gamma(t)|^2)'' \leq 2$$

it is enough to consider the development of γ with respect to the vertex o of the cone.

Let us prove that

$$(|\gamma(t)|^2)'' \geq 2.$$

Consider the Buseman function for $\theta \in \Sigma$,

$$f_\theta = \lim_{\lambda \to \infty} (\mathrm{dist}_{\lambda \cdot \theta} - \lambda).$$

The condition of convexity of the development with respect to $\lambda \cdot \theta$ gives concavity of the function $f_\theta \circ \gamma(t)$ for every quasigeodesic γ in C. Using this for $\theta = \frac{\gamma(t)}{|\gamma(t)|}$ gives the needed inequality.

Therefore if γ^* is the projection of γ on Σ, then we can choose a unique arclength parameter x on γ^* such that the following will be true:

$$\mathrm{pr}\big(\gamma(c \tan x + d)\big) = \gamma^*(x)$$

for some constants $c > 0$ and d; without loss of generality we can set $d = 0$.

Now we have to prove that the development of γ^* in a standard sphere with respect to every $\theta \in \Sigma$ is convex, i.e. $\cos(|\theta\gamma^*(x)|)'' + \cos(|\theta\gamma^*(x)|) \geq 0$. By [I 1.11] it is enough to prove only for $|\theta\gamma^*(x)| < \pi/2$. It is easy to see that

$$\cos(|\theta\gamma^*(x)|) = -f_\theta(\gamma(c\tan x))/|\gamma(c\tan x)|.$$

Then direct calculation gives what we need, because $f_\theta \circ \gamma$ is convex and

$$|\gamma(c\tan x)| = c/\cos x \;\spadesuit.$$

1.6. Proof of (ii). Assume that (ii) is false. Then one can find a sequence $\{t_i\}$ ($t_i \to 0^+$) such that

$$|\xi(t_i)q|_F < |pq|_F - \cos|\xi_o' q^\circ|_{\Sigma_p(F)} \cdot t_i - \varepsilon \cdot t_i$$

for some fixed $\varepsilon > 0$.

Assume $|pq|_F = l$ and $|\xi(t_i)q|_F = l_i$. Let $\gamma_i: [0, l_i] \to F$ be the shortest paths between $\xi(t_i)$ and q in F such that $\gamma(0) = \xi(t_i)$. We can pass to a subsequence of $\{\gamma_i\}$ such that the shortest paths γ_i approach some shortest path $\gamma: [0, l] \to F$ between q and p. Let $\eta \in q^\circ$ be the direction of this shortest path γ. By Theorem 1.1 γ_i and γ are quasigeodesics.

Now let us consider the Gromov-Hausdorff limit

$$(M/t_i, p) \overset{GH}{\to} C_p$$

and pass to a subsequence again so that there exists $\hat\gamma : [0, \infty) \to C_p$ satisfying

$$\hat\gamma(t) = \lim_{i\to\infty} (\frac{\gamma_i}{t_i})(tt_i)(\in M/t_i),$$

where (γ_i/t_i) denotes the image of γ_i in M/t_i.

By [I 2.4] $\hat\gamma$ is a quasigeodesic in C_p and it is easy to see that $\hat\gamma(0) \in \Sigma_p \subset C_p$.

1.7. Continuing, we define the direction at infinity of the curve $\hat{\gamma}$ in C_p by

$$\lim_{t\to\infty} \frac{\hat{\gamma}(t)}{|o\hat{\gamma}(t)|}.$$

By Lemma 1.5 this is well defined for quasigeodesics.

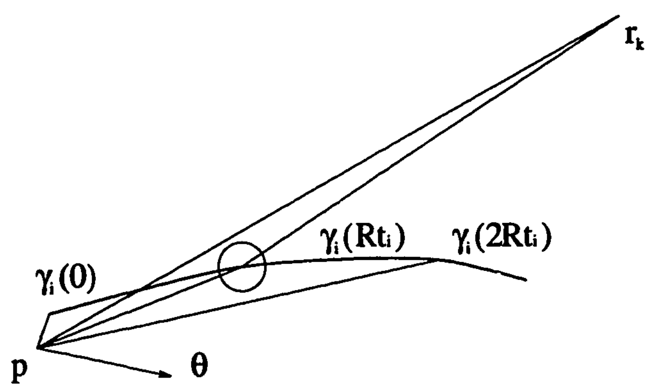
We claim that the direction at infinity of $\hat{\gamma}$ is η. Indeed let θ be the direction of $\hat{\gamma}$ at infinity.

By the cosine rule for $R \gg 1$ we obtain

$$|\hat{\gamma}(2R)|^2 = \lim_{i\to\infty} (|p\,\gamma_i(2Rt_i)|/t_i)^2 \leq$$

$$\leq \lim_{i\to\infty} \left(|p\,\gamma_i(Rt_i)|^2 + (Rt_i)^2 - 2Rt_i|p\,\gamma_i(Rt_i)| \cos \angle(\gamma_i^+(Rt_i), p'_{\gamma_i(Rt_i)})\right)/t_i^2 =$$

$$= |\hat{\gamma}(R)|^2 + R^2 - 2R|\hat{\gamma}(R)| \lim_{i\to\infty} \cos \angle(\gamma_i^+(Rt_i), p'_{\gamma_i(Rt_i)}).$$



Now by Lemma 1.5, for some β

$$\lim_{i\to\infty} \angle(\gamma_i^+(Rt_i), p'_{\gamma_i(Rt_i)}) \geq \arccos(|\hat{\gamma}(R)|^2 + R^2 - |\hat{\gamma}(2R)|^2)/2R|\hat{\gamma}(R)|) =$$

$$= \arccos\left[\frac{(\{R^2+1-2R\cos\beta\} + R^2 - \{4R^2+1-4R\cos\beta\})}{2R\sqrt{R^2+1-2R\cos\beta}}\right] =$$

$$= \arccos\left[-\sqrt{\frac{R^2-2R\cos\beta+\cos^2\beta}{R^2-2R\cos\beta+1}}\right] \geq \arccos\left[-\sqrt{1-\frac{1}{(R-1)^2}}\right] \geq$$

$$\geq \arccos\left[-1+1/(R-1)^2\right] > \pi(1-1/R).$$

Taking $r_k \to p$, such that $(r_k)'_p \to \theta$,

$$\lim_{i\to\infty} \tilde{\angle} p\,\gamma_i(Rt_i)\,r_k \geq \pi - \angle(\hat{\gamma}(R),(r_k)'_p) > \pi - \pi/R - \angle(\theta,(r_k)'_p).$$

The latter inequality is a corollary of Lemma 1.5 ($\sin\angle(\hat{\gamma}(R),\theta) \leq 1/R$). Therefore, since the perimeter of every triangle in the space of directions is at most 2π,

$$\lim_{i\to\infty} \angle(\gamma_i^+(Rt_i),\,(r_k)'_{\gamma_i(Rt_i)}) \leq$$

$$\leq 2\pi - \lim_{i\to\infty} \tilde{\angle} p\,\gamma_i(Rt_i)\,r_k - \lim_{i\to\infty} \angle\left(\gamma_i^+(Rt_i),p'_{\gamma_i(Rt_i)}\right) \leq \pi/R + \angle(\theta,(r_k)'_p) + \pi/R.$$

Using [I 1.7 B′] for γ_i with respect to the points r_k and starting at $\gamma_i(Rt_i)$, we obtain the estimates

$$|r_k\gamma(|pr_k|)| = \lim_{i\to\infty}\left|r_k\,\gamma_i\big(Rt_i+|\gamma_i(Rt_i)r_k|\big)\right| \leq$$

$$\leq \lim_{i\to\infty} |\gamma_i(Rt_i)r_k| \cdot \lim_{i\to\infty} \angle(\gamma_i^+(Rt_i),\,(r_k)'_{\gamma_i(Rt_i)}) \leq |pr_k|\cdot(2\pi/R+\angle(\theta,(r_k)'_p)).$$

This means that η is $2\pi/R$-close to θ. Sending R to infinity we obtain $\theta=\eta$.

1.8. Now let us fix $R \gg 1$ and divide γ_i into two pieces using a parameter value $x_i \in [0,l_i)$ such that $|p\gamma_i(x_i)| = Rt_i$. We estimate the length of each part separately.

By Theorem 1.2 the length of the first part $|q\gamma_i(x_i)|_F$ is possible to estimate from the triangle inequality:

$$|q\gamma_i(x_i)|_F \geq |pq|_F - |p\gamma_i(x_i)|_F = |pq|_F - Rt_i + o(t_i)$$

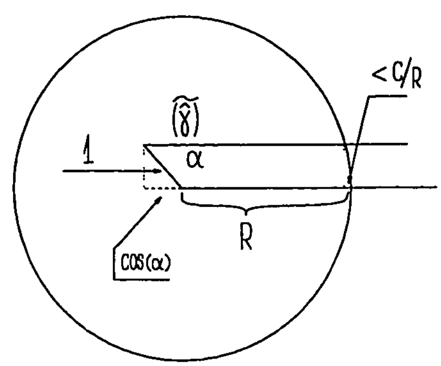
The length of the second part is estimated using the fact that the limit of lengths of quasigeodesics is the length of the limit quasigeodesic ([I 2.4]). Therefore

$$|\xi(t_i)\gamma_i(x_i)|_F/t_i \to length(\hat{\gamma} \cap B_R(o) \subset C_p).$$

By Lemma 1.5 the last expression can be estimated from below as

$$R - \cos\angle(q^\circ, \xi_o') - C/R.$$

This estimate is easily deduced from the following diagram in the plane of the development $\widetilde{(\hat{\gamma})}$ of $\hat{\gamma}$ from o. Here α is the angle at $\tilde{o}$ subtended by $\widetilde{(\hat{\gamma})}$. Clearly α is not less than $\angle(q^\circ, \xi_o')$.



By these two estimates we obtain

$$|\xi(t_i)q|_F \geq |pq|_F - \cos|\xi_o' q^\circ|_{\Sigma_p(F)} \cdot t_i - C/R \cdot t_i + o(t_i),$$

a contradiction with the assumption 1.6 for $C/R < \varepsilon$. This completes the proof of the first variation formula ♠.

1.9. Counterexample. In [PP1 6.1] we formulated a conjecture that the intrinsic metric of a primitive extremal subset has curvature bounded from below. Here we show a counterexample to this conjecture for $\operatorname{codim} F \geq 3$. Therefore this question is still open for $\operatorname{codim} F = 1$ (i.e for a boundary) and $\operatorname{codim} F = 2$. Sergey Buyalo [Buy] has settled the first of these questions affirmatively for "smooth" Alexandrov space, i.e. for a convex subset in a Riemanian manifold with curvature bounded from below.

Let us consider a right simplex $\operatorname{conv}\{a_1 a_2 a_3 a_4 a_5\}$ in a standard S^4 such that $|a_i a_j| = \pi/2$ for $i \neq j$. Assume $a_5 = a_0$, take some $\varepsilon > 0$ and consider the closed broken geodesic $F = a_0^+ a_1^- a_1^+ a_2^- a_2^+ a_3^- a_3^+ a_4^- a_4^+ a_5^- a_0^+$ where $a_i^\pm$ is the point on the geodesic $a_i a_{i\pm 1}$ such that $|a_i a_i^\pm| = \varepsilon$. Let $\Sigma = \operatorname{conv}\{F\}$. Then direct calculation shows that F is a primitive extremal subset of Σ and for ε sufficiently small, $length(F) > 2\pi$. In particular $C(F)$ is an extremal subset of $C(\Sigma)$ which has a singular point of negative curvature.

§2 GLUING THEOREM

Моей тёще

2.0. The Gluing Theorem is due to A.D.Alexandrov for the two dimensional case (see for example [Pog §11). Later Perelman [P2 5.2] proved the Doubling Theorem for multidimensional Alexandrov spaces; this is a special case of the theorem formulated below. The original Alexandrov's Theorem had a lot of applications for bending of convex surfaces with boundary, which are currently impossible to generalize to the multidimensional case, because they are supported by the Theorem about convex embeddings [Pog §6-7]. Formally the following theorem gives some new examples of Alexandrov spaces, but unfortunately we have not too many examples of Alexandrov spaces with isometric boundaries.

2.1. Theorem. Let M_1 and M_2 be Alexandrov spaces with nonempty boundary and curvature $\geq k$. Let there be an isometry

$$\mathrm{is}\colon \partial M_1 \to \partial M_2$$

(here ∂M_1 and ∂M_2 are considered as length-metric spaces with induced metric from M_1 and M_2.)

Then the glued space $X = M_1 \cup_{\mathrm{is}(x)=x} M_2$ is an Alexandrov space with curvature $\geq k$.

2.2. Lemma. Let $p \in \partial M$ and $\eta \in \partial\Sigma_p$. Then there exists a shortest path in ∂M which starts at p in a direction arbitrarily close to η.

Proof. Let $N = \partial M$. The boundary is an extremal subset and therefore we can use notation $q^\circ (= q_p^\circ)$ for the set of all directions of entrance in $\Sigma_p(N)$ of shortest paths between p and q in the length metric of N.

Let us choose a sequence of point $q_n \in N$ such that $q_n \to p$ and $\angle(q_n', \eta) \to 0$ (where $q_n' = (q_n)_p'$ is the set directions at p of the shortest paths between p and q). Assume that for all n, $\angle(\eta q_n^\circ) \geq \varepsilon$. Let us pass to a subsequence such that $\lim_{n\to\infty} \angle(\theta\, q_n^\circ) \to 0$ for some direction θ.

Find a point $r \in M$ such that $\angle(r', \theta) < \varepsilon/6$. Let $\{r_n\}$ be points on the shortest path pr such that $|pr_n| = |pq_n|$. Since the shortest path from p to q_n in N is a quasigeodesic (see 1.1) we obtain by [I 1.7(B′)] that for n sufficiently large $|r_n q_n| < \varepsilon/5 \cdot |pq_n|$, hence for $\varepsilon \le \pi/4$

$$\lim_{n\to\infty} \angle(q_n', r') < \varepsilon/3.$$

Therefore

$$\lim_{n\to\infty} \angle(q_n', \theta) < \varepsilon/2.$$

We obtain a contradiction because

$$\lim_{n\to\infty} q_n' = \eta \text{ and } \angle(\eta, \theta) \ge \varepsilon \spadesuit.$$

2.3. Preparation for Proof of 2.1. The proof is by induction on dimension.

Let $N = M_1 \cap M_2 = \partial M_i \subset X$.

Definitions. The *m-predistance* $|pq|_m$ between points p and q in X is the minimal length of broken geodesics with vertices $p = p_0, p_1, ..., p_{k+1} = q$ where $k \le m$, $p_l p_{l\pm1}$ is a shortest path which lies completely in one of M_i for every $l \in \{1, 2, ..., k\}$ and p_l lies in N. A broken geodesic which realizes this minimum is called an *m-shortest path.*

It is easy to see that

$$(*) \qquad \begin{gathered} |pq|_m \ge |pq|_{m+1} \ge |pq| \\ \lim_{m\to\infty} |pq|_m = |pq| \end{gathered}$$

$$(**) \qquad \begin{aligned} |pq|_m + |qr|_l &\ge |pr|_{m+l} \quad \text{if } q \in X \backslash N \\ |pq|_m + |qr|_l &\ge |pr|_{m+l+1} \quad \text{if } q \in N \end{aligned}$$

For every interior vertex $p = p_l$, $l \in \{1, 2, ..., k\}$, of an m-shortest path we can define directions of exit and entrance ξ_i as directions in $\Sigma_p(M_i)$ of shortest paths in M_i.

By Theorem 1.2 the isometry $\mathrm{is}\colon \partial M_1 \to \partial M_2 = N$ gives an isometry $\mathrm{is}'_p\colon \partial\Sigma_p(M_1) \to \partial\Sigma_p(M_2) = \Sigma_p(N)$ and $\mathrm{is}_p\colon \partial C_p(M_1) \to \partial C_p(M_2) = C_p(N)$. Set

$$\Sigma_p^{\#}(X) \stackrel{\mathrm{def}}{=} \Sigma_p(M_1) \cup_{\mathrm{is}'_p(x)=x} \Sigma_p(M_2)$$

and

$$C_p^{\#}(X) \stackrel{\mathrm{def}}{=} C(\Sigma_p^{\#}(X)) = C_p(M_1) \cup_{\mathrm{is}_p(x)=x} C_p(M_2).$$

From the induction hypothesis, $\Sigma_p^{\#}(X)$ will be an Alexandrov space with curvature ≥ 1 and therefore $C_p^{\#}(X)$ will be a cone with curvature ≥ 0.

Notation. If K_1 and K_2 are two compact metric spaces, we say that $K_1 \leq K_2$ if there is a noncontracting map $m : K_1 \to K_2$. If (L_1, p_1) and (L_2, p_2) are two locally compact metric spaces with base points, we say that $(L_1, p_1) \leq (L_2, p_2)$ if for every $R > 0$ there is a noncontracting map $m : B_R(p_1) \to B_R(p_2)$.

We will write $\limsup_{i\to\infty} K_i \leq K$ if for every Hausdorff subsequence $K_{i_k} \stackrel{GH}{\to} K'$ we have $K' \leq K$. Similarly one can write $\liminf_{i\to\infty} K_i \geq K$. We write $\limsup_{i\to\infty}(L_i, p_i) \leq (L, p)$ if for every $R > 0$ we have $\limsup_{i\to\infty} B_R(p_i) \leq B_R(p)$ (compare with [BGP 7.13]).

2.4. Proof of the Theorem 2.1. The rest of §2 will be devoted to this proof.

As a base we can take the classical Gluing Theorem of A.D. Alexandrov (dim = 2) (see [Pog §11]). Assume we have already proved Theorem 2.1 for dim $< n$.

Claim.

A. For every point $p \in N$

$$\limsup_{\delta \to 0}(X/\delta, p) \le (C_p^{\#}(X)), o).$$

B. The directions of exit and entrance (ξ_i) of every m-shortest path at every interior vertex $p = p_l$, $l \in \{1, 2, ..., k(\le m)\}$ (see 2.3), are opposite in $C_p^{\#}(X))$ (i.e. $|\xi_1\xi_2| = 2|\xi_1| = 2|\xi_2|$ see [I 2.7,2.8]).

Remark. It is easy to see that as a corollary of the Theorem we will have equality in (A) instead of the inequality.

Proof of A. Consider the gradient-exponential maps: $\varpi_1 : C_p(M_1) \to M_1$ and $\varpi_2 : C_p(M_2) \to M_2$ (see [I 4.6]). By [I 4.6], $\varpi_i(C_p(N)) \subset N$. Let us construct an exponential map $\exp : C_p^{\#}(X)) \to X$ by

$$\exp(v) = \begin{bmatrix} \varpi_1(v) & \text{for} & v \in C_p(M_1) \subset C_p^{\#}(X)) \\ \varpi_2(v) & \text{for} & v \notin C_p(M_1) \end{bmatrix}.$$

Define $\exp_\delta : C_p^{\#}(X)) \to X/\delta$ by $\exp_\delta(v) = i_\delta \circ \exp \circ (v\delta)$, where $i_\delta : X \to X/\delta$ is the canonical mapping.

Let $x = x_0, x_1, ..., x_k, x_{k+1} = y$ be vertices of an m-shortest path in $C_p^{\#}(X))$. Then it is easy to see that $|x_l x_{l+1}| \ge |\exp_\delta(x_l)\exp_\delta(x_{l+1})| + o(\delta)/\delta$. Therefore for the m-predistance in $C_p^{\#}(X))$ we have $|xy|_m \ge |\exp_\delta(x)\exp_\delta(y)| + o(\delta)/\delta$. Now $|xy| = \lim_{m\to\infty}|xy|_m$ for every $x, y \in C_p^{\#}(X))$. Hence

$$\lim_{\delta\to 0}|\exp_\delta(x)\exp_\delta(y)| \le \lim_{m\to\infty}|xy|_m = |xy|.$$

Now in order to complete the proof we need to verify that for every $R > 0$

$$\lim_{\delta\to 0} \exp_\delta^{-1}(B_R(p) \subset X/\delta) \subset B_R(o) \subset C_p^\#(X)).$$

Equivalently, for every $x \in C_p^\#(X))$

$$\lim_{\delta\to 0} |p\exp_\delta(x)| \geq |x|.$$

Assume otherwise. Therefore we can find $x \in C_p^\#(X))$ and a sequence $\delta_n \to 0$ such that for some $\varepsilon > 0$ we have

$$|p\exp_{\delta_n}(x)| \leq (1-\varepsilon)|x|.$$

Let us consider shortest paths $p\exp_{\delta_n}(x) \subset X/\delta_n$ for all n. No subsequence lies completely in M_i/δ_n for fixed i. Let $y_n \in N/\delta_n \subset X/\delta_n$ be the closest point of N/δ_n to $\exp_{\delta_n}(x)$ on $p\exp_{\delta_n}(x)$. Pass to a subsequence of $\{\delta_n\}$ such that $\exp_{\delta_n}^{-1}(y_n) \to x^*$. By [P2 4.7] $x^* \in C(\Sigma_p(N)) = C(\partial M_i)$ and $\lim_{\delta_n\to 0} |\exp_{\delta_n}(x)\exp_{\delta_n}(x^*)| = |xx^*|$ (because a shortest path $\exp_{\delta_n}(x)y_n$ completely lies in one of the M_i and $|y_n\exp_{\delta_n}(x^*)| = o(\delta_n)/\delta_n$). Therefore for n sufficiently large

$$|p\exp_{\delta_n}(x^*)| \leq (1-\varepsilon)|x^*|.$$

By Lemma 1.5 a limit of shortest paths in N/δ_n between p and $\exp_{\delta_n}(x^*)$ (which is a quasigeodesic by Lieberman Lemma 1.1) is a shortest path ox^* in $C_p(M_i)$. By the fact that limits preserve lengths of quasigeodesics ([I 2.4]),

$$\lim_{n\to\infty} |p\exp_{\delta_n}(x^*)|_{N/\delta} = |x^*|.$$

Hence for n sufficiently large

$$|p\exp_{\delta_n}(x^*)| \leq (1-\varepsilon)|p\exp_{\delta_n}(x^*)|_{N/\delta}.$$

Therefore we can find a segment $s_n r_n$ on a shortest path $p \exp_{\delta_n}(x^*)$ which completely lies in one of the M_i/δ_n, such that $s_n, r_n \in N/\delta_n$ and

$$(\#) \qquad |s_n r_n|_{M_i} \le (1-\varepsilon)(|pr_n|_N - |ps_n|_N)$$

(here we employ our convention of using the same notation for points in N and N/δ).

We can easily pass to a subsequence such that for some $0 \le c \le 1$,

$$\lim_{n\to\infty} \frac{|ps_n|_N}{|pr_n|_N} = c.$$

Now let us consider two cases, $c \ne 1$ and $c = 1$.

1) Suppose $c \ne 1$. Let us consider limit $(M_i/|pr_n|_N, p) \stackrel{GH}{\to} C_p(M_i)$. Pass to a subsequence such that $s_n \to s$ and $r_n \to r$. The boundary N is an extremal subset; therefore by Theorem 1.2, $(N/|pr_n|_N, p) \stackrel{GH}{\to} C_p(N)$ as length-metric spaces. Hence

$$\lim_{n\to\infty} \frac{|s_n r_n|_{M_i}}{|pr_n|_N} = |sr| \ge |r| - |s| = |r|_{C(N)} - |s|_{C(N)} = 1 - \lim_{n\to\infty} \frac{|ps_n|_N}{|pr_n|_N},$$

a contradiction to $(\#)$.

2) Suppose $c = 1$. Pass to a subsequence such that there exists a limit

$$(M_i/|s_n r_n|_{M_i}, s_n) \stackrel{GH}{\to} (M_s, s).$$

We remark that M_s need not be the tangent cone. Set $N_s = \partial M_s$. By Theorem 1.2 we have

$$(N/|s_n r_n|_{M_i}, s_n) \stackrel{GH}{\to} (N_s, s).$$

Let $f_n : N/|s_n r_n|_{M_i} \to R$ be functions defined by

$$f_n(x) = |px|_{N/|s_n r_n|_{M_i}} - |ps_n|_{N/|s_n r_n|_{M_i}}.$$

Pass to a subsequence such that there exists a limit $f : N_s \to R$, $f = \lim_{n\to\infty} f_n$.

It is easy to see that M_s can be represented as a product $R\times M_s'$ such that $f(x) \le pr_R(x)$ where pr_R is the projection $M_s \to R$. Indeed a sequence of quasigeodesics which prolong shortest paths ps_n in N easily goes to a straight line in M_s, so by the Toponogov splitting theorem we have such a representation. Therefore N_s is split as well, $N_s = R \times N_s'$.

Let σ_n be a shortest path in N between p and s_n parameterized by distance from s_n and σ be a limit of $\{\sigma_n/|r_n s_n|_{M_i}\}$. By the triangle inequality, for every $T > 0$ we have $|xp|_N - |s_n p| \le |x\sigma_n(|s_n r_n|T)| - |s_n r_n|T$. As a limit we obtain that $f(x) \le |x\sigma(T)| - T$. For $T \to \infty$ the right side goes to the Buseman function of σ which coincides with pr_R.

Pass to a subsequence such that there is a limit as $r_n \to r$. We obtain that

$$1 = |rs| \ge pr_R(r) \ge f(r) = \lim_{n\to\infty} (|pr_n|_N - |ps_n|_N)/|r_n s_n|_{M_i},$$

a contradiction to (#).

Proof of B. Let $\xi_i \in \Sigma_p(M_i)$ be directions of exit/entrance of the m-shortest path at the interior vertex p (see 2.3). Let us first prove that for every $\nu \in \Sigma_p(N) \subset \Sigma_p^{\#}(X)$

$$|\xi_1\nu|_0 + |\xi_2\nu|_0 = \pi.$$

Here the left side is the sum of two 0-distances in the glued space $\Sigma_p^{\#}(X)$, each of which by the definition 2.3 is measured in one of the $\Sigma_p(M_i)$. Assume we have proved Claim B for dim$< n$ and let $\dim\Sigma_p^{\#}(X) = n$. From the first variation formula we obtain for every $\nu \in \Sigma_p(N)$

$$f(\nu) \overset{\text{def}}{=} |\xi_1\nu|_0 + |\nu\xi_2|_0 \ge \pi.$$

Assume $\bar{\nu}$ is the minimum point in $\Sigma_p(N)$ of the last function. Thus, $\xi_1\bar{\nu}\xi_2$ is a 1-shortest path. Let γ be a shortest path in $\Sigma_p(N)$ such that $\gamma(0) = \bar{\nu}$ with arbitrary initial data $\gamma^+(0) = \eta$. Assume $f(\bar{\nu}) > \pi$. By the induction assumption $|(\xi_1)'_{\bar{\nu}}\eta|_0 + |\eta(\xi_2)'_{\bar{\nu}}|_0 = \pi$. By the generalized Lieberman Lemma (see 1.1) γ is a quasigeodesic as a curve in $\Sigma_p(M_1)$ and $\Sigma_p(M_2)$. By [I 1.7(B′)] from $f(\bar{\nu}) > \pi$ we obtain $(f \circ \gamma)(x) < (f \circ \gamma)(0) = f(\bar{\nu})$ for sufficiently small x. This contradicts the assumption that f has a minimum at $\bar{\nu}$.

Therefore $f(\bar{\nu}) = \pi$. Take every shortest path γ in $\Sigma_p(N)$ such that $\gamma(0) = \bar{\nu}$. Then γ is a quasigeodesic for $\Sigma_p(M_1)$ and $\Sigma_p(M_2)$. Set

$$g(\nu) \stackrel{\text{def}}{=} \cos|\xi_1\nu|_0 + cos|\nu\xi_2|_0$$

for $\nu \in \Sigma_p(N)$. From above $g(\bar{\nu}) = g \circ \gamma(0) = 0$, $(g \circ \gamma)'(0) = 0$ and $g \circ \gamma \leq 0$. By [I 17(B)] $(g \circ \gamma)'' + g \circ \gamma \geq 0$. Therefore $(g \circ \gamma)'' \geq 0$ and so $g \circ \gamma \equiv 0$; in particular for every ν, $g(\nu) = 0$. Therefore $f \equiv \pi$, i.e. $|\xi_1\nu|_0 + |\xi_2\nu|_0 = \pi$ as claimed.

In order to prove that ξ_i are opposite it is enough to show that $2|\xi_1| = 2|\xi_2| = |\xi_1\xi_2|$ holds in $C_p^\#(X)$ or equivalently, $|\xi_1\xi_2| = \pi$ holds in $\Sigma_p^\#(X)$. If this is false, then there is m such that $|\xi_1\xi_2|_m < \pi$ in $\Sigma_p^\#(X)$. Let θ be the closest vertex to ξ_1 of the m-shortest path $\xi_1\xi_2$. From above there is a 1-shortest path through θ of length π. Therefore we have two distinct directions at θ which are opposite to $(\xi_1)'_\theta$, a contradiction to the fact that $\Sigma_p^\#$ is an Alexandrov space. ♠.

2.5. Corollary. Let $\xi_i \in \Sigma_p(M)$ be directions of exit/entrance of an m-shortest path at an interior vertex. For every $\eta \in \Sigma_p(M_i)$ there is a unique $\eta^* \in \Sigma_p(N)$ such that

$$|\xi_1\eta|_0 + |\eta\eta^*|_0 + |\eta^*\xi_2|_0 = \pi$$

or

$$|\xi_1\eta^*|_0 + |\eta^*\eta|_0 + |\eta\xi_2|_0 = \pi.$$

Proof. Suppose $\eta \in \Sigma_p(M_1)$. Consider the 1-shortest path $\eta\xi_2$. From the preceding claim applied to $\Sigma_p^{\#}(X)$ the directions at the vertex are opposite and therefore this 1-shortest path is a part of a 1-shortest path $\xi_1\xi_2$ ♠.

2.6. Claim. Let $\gamma\colon [a,b] \to X$ be a quasigeodesic in one of the $\mathrm{int} M_i$ or a shortest path in the length metric of N. Then

$$\rho_k(|p\gamma(t)|_m)'' + k\rho_k(|p\gamma(t)|_m) \leq 1$$

for every $p \in X$ (see [I 1.6,1.7(B)]).

Proof. We consider the case $k = 0$; we must show $(|p\gamma(t)|_m^2)'' \leq 2$.

It is true for $m = 0$ because

$$|pq|_0 = \left[\begin{array}{cc} |pq|_{M_i} & \text{if } p \in M_i, q \in \mathrm{int} M_i \text{ or } q \in M_i, p \in \mathrm{int} M_i \\ \min_i |pq|_{M_i} & \text{if } p, q \in N \\ \infty & \text{in the other cases} \end{array}\right.$$

(Recall that a shortest path in N is a quasigeodesic in both M_i by the generalized Lieberman Lemma (1.1)).

Let it be true for all $l < m$ and false for m. Then the standard idea shows that in this case there exists $t_0 \in (a,b)$ and $\varepsilon > 0$ such that for $|t - t_0| < \varepsilon$

$$|p\gamma(t)|_m^2 \geq |p\gamma(t_0)|_m^2 - A(t - t_0) + (t - t_0)^2 + \varepsilon(t - t_0)^2,$$

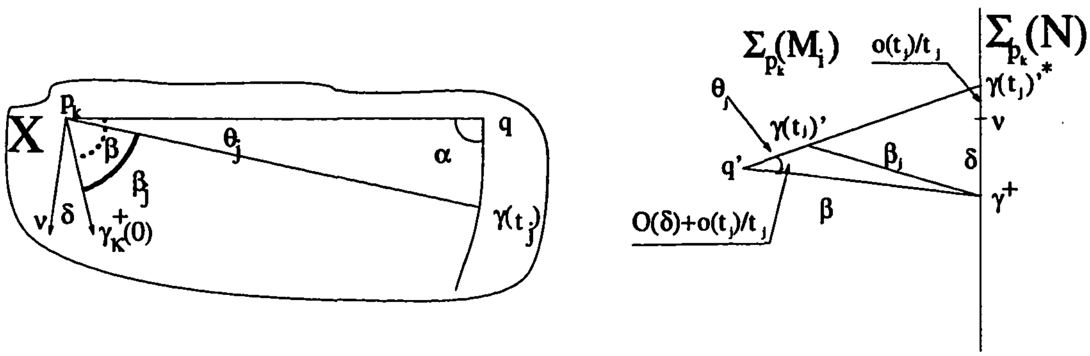
for some constant A.

Let us assume that $t_0 = 0$. Set $q = \gamma(0)$ and let $p = p_0 p_1 \ldots p_k p_{k+1} = q$ be an m-shortest path. Take a sequence $t_j \to 0$ such that the sequence $((\gamma(t_j)'_{p_k})^*$ (as in Corollary 2.5) goes to some direction $\nu \in \Sigma_{p_k}(N)$. Using Lemma 2.2 we can find a shortest path γ_k in N which goes from p_k in a direction arbitrarily close to ν.

In the following proof one might get lost in calculations and lose the main idea. If we assume that all $((\gamma(t_j)'_{p_k})^*$ coincide with ν and there is a shortest path (in the intrinsic metric of N) which goes in this direction then one can ignore the residue terms below.

Assume

$$\begin{gathered}\alpha = \angle((p_k)'_q, \gamma^+(0)) \\ \beta = \angle(q'_{p_k}, \gamma_k^+(0)) \\ \beta_j = \angle(\gamma_k^+(0)\,(\gamma(t_j))'_{p_k}) \\ \theta_j = \angle((\gamma(t_j))'_{p_k}\, q'_{p_k}) \\ \delta = \angle(\gamma_k^+, \nu)\end{gathered}$$



It is easy to see that

$$\theta_j \geq \frac{t_j \sin \alpha}{|p_k q|_0} + o(t_j).$$

We can assume that $q'_{p_k} \notin \Sigma_{p_k}(N)$, otherwise our m-shortest path lies completely in N. By the cosine rule applied to the triangle $\Delta q'_{p_k}(\gamma(t_j))'_{p_k}\gamma_k^+(0)$

$$\beta - \beta_j \geq (1 + o(\delta) + o(t_j)/t_j)\theta_j \geq t_j\left(\frac{\sin \alpha}{|p_k q|_0} + o(\delta)\right) + o(t_j).$$

Hence

$$\cos(\beta - \beta_j) \le 1 - \frac{t_j^2 \sin^2 \alpha}{2|pq|_0^2} + o(\delta)t_j^2 + o(t_j^2).$$

From the induction assumption and Claim 2.4(B) we have

$$|p\,\gamma_k(\tau)|_{m-1}^2 \le |p\,p_k|_{m-1}^2 + 2\tau|p\,p_k|_{m-1}\cos\beta + \tau^2.$$

Because γ_k is a quasigeodesic for both of the M_i,

$$|\gamma(t_j)\gamma_k(\tau)|_0^2 \le |\gamma(t_j)\,p_k|_0^2 - 2\cos\beta_j \cdot \tau|\gamma(t_j)\,p_k|_0 + \tau^2,$$

(where these distances are measured in a fixed M_i).

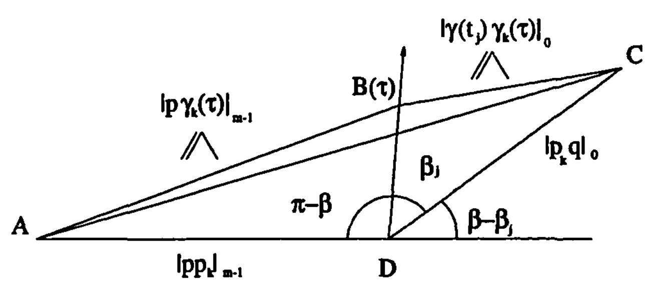
Therefore using (**) (see 2.3) and the previous two inequalities

$$|p\,\gamma(t_j)|_m^2 \le \min_\tau(|p\,\gamma_k(\tau)|_{m-1} + |\gamma(t_j)\gamma_k(\tau)|_0)^2 \le$$

$$\le \min_\tau(|AB(\tau)| + |B(\tau)C|)^2 = |AC|^2 =$$

$$= |p\,p_k|_{m-1}^2 + |\gamma(t_j)\,p_k|_0^2 + 2|p\,p_k|_{m-1}|\gamma(t_j)\,p_k|_0\cos(\beta - \beta_j).$$

Here $A, B(\tau)$ and C are as shown in the following diagram in the plane



Because γ is either a quasigeodesic in one of the M_i, or a shortest path in N and therefore a quasigeodesic in both of the M_i (see the generalized Lieberman Lemma, 1.1)

$$|p_k\,\gamma(t_j)|_0^2 \le |p_k q|_0^2 + t_j^2 - 2t_j|p_k q|_0\cos\alpha$$

and so

$$|p_k\,\gamma(t_j)|_0 \le |p_kq|_0 - t_j\cos\alpha + \frac{t_j^2\sin^2\alpha}{2|p_kq|_0} + o(t_j^2).$$

Hence

$$|p\,\gamma(t_j)|_m^2 \le |p\,p_k|_{m-1}^2 + |q\,p_k|_0^2 + t_j^2 - 2t_j|q\,p_k|_0\cos\alpha +$$

$$+2|p\,p_k|_{m-1}(|p_kq|_0 - t_j\cos\alpha + \frac{t_j^2\sin^2\alpha}{2|p_kq|_0} + o(t_j^2))(1 - \frac{t_j^2\sin^2\alpha}{2|p_kq|_0^2} + t_j^2o(\delta) + o(t_j^2)) \le$$

$$\le (|pp_k|_{m-1} + |p_kq|_0)^2 - 2t_j(|pp_k|_{m-1} + |p_kq|_0)\cos\alpha + t_j^2 + t_j^2o(\delta) + o(t_j^2) =$$

$$= |p\,q|_m^2 - 2t_j|pq|_m\cos\alpha + t_j^2 + t_j^2o(\delta) + o(t_j^2) \quad .$$

This inequality for two sequences $t_j \to 0^+$ and $t_j \to 0^-$ contradicts our assumption for sufficiently small δ ♠.

2.7 Now let us prove that every m-shortest path is a k-quasigeodesic. Indeed using [I 1.7(B)] we only need to verify that $\rho_k(|\gamma(t)p|)'' \le 1 - k\rho_k(|\gamma(t)p|)$. Now $|\gamma(t)p| = \lim_{n\to\infty}|\gamma(t)p|_n$ and using Claim 2.6 we obtain the needed inequality for all $t \ne t_l$ (where $\gamma(t_l) = p_l$).

Let σ be a shortest path between an arbitrary point x and $\gamma(t_l)$, parameterized by distance from $\gamma(t_l)$. By Claim 2.4 we obtain that for fixed ε

$$|\sigma(T)\gamma(t_l + T\varepsilon)| + |\sigma(T)\gamma(t_l - T\varepsilon)| \le 2T + CT\varepsilon^2 + o(T).$$

Therefore

$$\mathrm{dist}_x \circ \gamma(t_l + T\varepsilon) + \mathrm{dist}_x \circ \gamma(t_l - T\varepsilon) \le 2\mathrm{dist}_x \circ \gamma(t_l) + CT\varepsilon^2 + o(T).$$

Therefore for $T \to 0$

$$(\mathrm{dist}_x \circ \gamma)^+(t_l) \le (\mathrm{dist}_p \circ \gamma)^-(t_l) + C\varepsilon.$$

Hence for $\varepsilon \to 0$ we obtain

$$(\mathrm{dist}_x \circ \gamma)^+(t_l) \le (\mathrm{dist}_x \circ \gamma)^-(t_l).$$

Therefore by [I 1.9(B)] we obtain the needed inequality for every t.

Let γ_m be an m-shortest path between $p, q \in X$. Then $\gamma = \lim_{m \to \infty} \gamma_m$ is a shortest path between p and q. It is easy to see that γ is convex (as a limit of convex curves) and parameterized by the arclength (because $length(\gamma_m) \to length(\gamma)$); hence γ is a quasigeodesic. Therefore by [I 1.17] we obtain that X is an Alexandrov space of curvature $\ge k$. This completes the proof of the Gluing Theorem ♠.

§3 RADIUS SPHERE THEOREM

Никодиму

The following Theorem was proved by Karsten Grove and Peter Petersen [GP]. Another proof follows immediately from [PP1 1.2, 1.4.1]. The following proof is only a good demonstration of how beautiful quasigeodesics are.

Proposition. Let Σ be an Alexandrov space of curvature ≥ 1, with radius $> \pi/2$. Then for every $p \in \Sigma$ the space of directions Σ_p has a radius $> \pi/2$.

Proof. Assume that Σ_p has radius $\le \pi/2$, and let $\xi \in \Sigma_p$ be a direction such that $\mathrm{clos} B_\xi(\pi/2) = \Sigma_p$. Take a quasigeodesic of length $\pi/2$ starting at p in the direction ξ. Then the other endpoint q of this quasigeodesic satisfies $\mathrm{clos} B_q(\pi/2) = \Sigma$. (Indeed, for any point $r \in \Sigma$ we have $\angle rpq \leq \pi/2$, therefore $|rq| \leq \pi/2$ by the comparison inequality; see [I 1.7(B′)]). This contradicts our assumption that Σ has radius $> \pi/2$ ♠.

Theorem. Let Σ be an Alexandrov space of curvature ≥ 1, with radius $> \pi/2$. Then Σ is homeomorphic to the sphere S^n.

Proof. Assume we have proved the Theorem for $\dim\Sigma < n$. Let us prove it for $\dim\Sigma = n$.

Let xy be a diameter of Σ. Let z be a critical point of dist_x. Then we have

$$\tilde{\angle} xzy \leq \angle xzy \leq \pi/2.$$

By assumption $|xz|, |zy|, \pi/2 \leq |xy|$. Therefore the last inequality can hold only for $z = y$. Therefore dist_x has no critical points but x and y. By [P1] Σ is homeomorphic to $S(\Sigma_x)$. By the Proposition we have $\mathrm{Rad}(\Sigma_x) > \pi/2$. Hence by the induction assumption Σ_x is homeomorphic to S^{n-1}. Therefore Σ is homeomorphic to S^n ♠.

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