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DIFFEOMORPHISM FINITENESS, POSITIVE PINCHING, AND SECOND HOMOTOPY

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Abstract

Our main results can be stated as follows:

1. For any given numbers m, C and D, the class of m-dimensional simply connected closed smooth manifolds with finite second homotopy groups which admit a Riemannian metric with sectional curvature bounded in absolute value by $|K| \leq C$ and diameter uniformly bounded from above by D contains only finitely many diffeomorphism types.
2. Given any m and any $\delta > 0$, there exists a positive constant $i_0 = i_0(m, \delta) > 0$ such that the injectivity radius of any simply connected compact m-dimensional Riemannian manifold with finite second homotopy group and $\mathrm{Ric} \geq \delta$, $K \leq 1$, is bounded from below by $i_0(m, \delta)$.

In an appendix we discuss Riemannian megafolds, a generalized notion of Riemannian manifolds, and their use (and usefulness) in collapsing with bounded curvature.

0 Introduction

This note is a continuation of the work begun in [PetrRTu] (this issue) and centers around the problems of establishing finiteness theorems and injectivity radius estimates for certain classes of closed Riemannian manifolds. Our first result can be stated as follows:

Theorem 0.1 (π_2-Finiteness theorem). *For given m, C and D, there is only a finite number of diffeomorphism types of simply connected closed m-dimensional manifolds M with finite second homotopy groups which admit*

The main part of this research was done while both authors were visiting the Max Planck Institute at Bonn in 1996-1997, and it was finished in April 1998, when W.T. was visiting A.P. at the Institute of Mathematical Sciences at Stony Brook, and during our stay at the Max Planck Institute for Mathematics in the Sciences at Leipzig in 1998-1999. We would like to thank these institutions for their support and hospitality.

Riemannian metrics with sectional curvature $|K(M)| \leq C$ *and diameter* $\operatorname{diam}(M) \leq D$.

Note that in this theorem, no assumption can be removed. Indeed, the assumption of an upper curvature bound is necessary, since, for example, there exists an infinite sequence of nonnegatively curved, but topologically distinct S^3 bundles over S^4 ([GrovZi]). On the other hand, uniformly pinched sequences of Aloff-Wallach, Eschenburg and Basaikin spaces (see [AlW], [E], [Ba]) show the necessity of requiring that the second homotopy group always be finite. These examples show moreover that in this sense also the following result is optimal:

Corollary 0.2 (A "classification" of simply connected closed manifolds). *For given* m, C *and* D, *there exists a finite number of closed smooth manifolds* E_i *such that any simply connected closed* m*-dimensional manifold* M *admitting a Riemannian metric with sectional curvature* $|K(M)| \leq C$ *and diameter* $\operatorname{diam}(M) \leq D$ *is diffeomorphic to a factor space* $M = E_i/T^{k_i}$, *where* $0 \leq k_i = \dim E_i - m$ *and* T^{k_i} *acts freely on* E_i.

(For a slight refinement of the corollary see Remark 2.3.)

Here is a short account of the principal finiteness results (we know of) which only require volume, curvature, and diameter bounds: For manifolds M of a given fixed dimension m, the conditions

- $\operatorname{vol}(M) \geq v > 0$, $|K(M)| \leq C$ and $\operatorname{diam}(M) \leq D$ imply finiteness of diffeomorphism types ([C] and [Pet]); this conclusion continues to hold for $\operatorname{vol}(M) \geq v > 0$, $\int_M |R|^{m/2} \leq C$, $|\operatorname{Ric}_M| \leq C'$, $\operatorname{diam}(M) \leq D$ ([AnC1]);
- $\operatorname{vol}(M) \geq v > 0$, $K(M) \geq C$ $\operatorname{diam}(M) \leq D$ imply finiteness of homotopy types ([GrovPete]), homeomorphism types ([Pe]) and Lipschitz homeomorphism types (Perelman, unpublished). If $m > 4$, these conditions imply finiteness diffeomorphism types (compare [GrovPW]);
- $K(M) \geq C$ and $\operatorname{diam}(M) \leq D$ imply a uniform bound for the total Betti number ([Gro1]).

The π_2-Finiteness theorem and Corollary 0.2 require two-sided bounds on curvature, but no lower uniform volume bound. Thus, in spirit these results are somewhere between Cheeger's Finiteness and Gromov's Betti number theorem. Let us now point out some of their consequences.

The π_2-Finiteness theorem implies by Myers' theorem in particular the following finiteness result:

Given any m and $\delta > 0$, there is only a finite number of diffeomorphism types of simply connected closed m-dimensional manifolds M with finite second homotopy groups which admit Riemannian metrics with Ricci curvature $\mathrm{Ric} \geq \delta > 0$ and sectional curvature $K \leq 1$.

Using Corollary 0.2 one can show (see [Tu]) that in low dimensions the assumption on the finiteness of the second homotopy groups in Theorem 0.1 can actually be dropped:

For given C and D, there is only a finite number of diffeomorphism types of simply connected closed m-manifolds, $m < 7$, which admit Riemannian metrics with sectional curvature $|K| \leq C$ and diameter $\leq D$.

This last result (for an independent proof see [FR2]) explains in particular why 7 is the first dimension where infinite sequences of closed simply connected manifolds of mutually distinct diffeomorphism type and uniformly positively pinched sectional curvature (see [AlW], [E]) can appear.

For positively curved manifolds, by building on the results from [PetrRTu] (on which, by the way, the π_2-Finiteness theorem does not depend at all), one can in fact obtain stronger versions of Theorem 0.1.

Theorem 0.3 (π_2-theorem). *For each natural number m and any $\delta > 0$ there exists a positive constant $i_0(m, \delta) > 0$ such that the injectivity radius i_g of any δ-pinched Riemannian metric g on a simply connected compact m-dimensional manifold M with finite second homotopy group is uniformly bounded from below by $i_g \geq i_0(m, \delta)$.*

Note that in Gromov-Hausdorff convergence terms the π_2-theorem can be formulated as follows:

Theorem 0.3′. *There is no collapsing sequence of simply connected manifolds with finite second homotopy groups and fixed dimension and positively pinched curvature $1 \geq K \geq \delta > 0$.*

Theorem 0.3 proves in particular a principal case of the following conjecture of Klingenberg and Sakai:

CONJECTURE ([KlS2]). *Let M be a closed manifold and $\delta > 0$. Then there exists $i_0 = i_0(M, \delta) > 0$ such that the injectivity radius i_g of any δ-pinched metric g on M, i.e., any metric with sectional curvature $\delta \leq K_g \leq 1$, is bounded from below by $i_g \geq i_0$.*

A different proof of Theorem 0.3 (but also based on [PetrRTu]) has been given by Fang and Rong ([FR1], in this issue). Theorem 0.3 actually also holds under the more relaxed curvature conditions of positive Ricci pinching:

Using a synthetic version of positive Ricci curvature for (a very special class of) Alexandrov spaces, we show why for the existence of lower injectivity radius bounds the finiteness of the second homotopy groups is a sufficient condition also under the conditions $K \leq 1$ and $\mathrm{Ric} \geq \delta > 0$ (for an alternative proof see also the end of the present paper's Appendix):

Theorem 0.4. *There is no collapsing sequence of simply connected manifolds with finite second homotopy groups and fixed dimension with curvature conditions $K \leq 1$ and $\mathrm{Ric} \geq \delta > 0$. In other words:*

Given any m and any $\delta > 0$, there exists a positive constant $i_0 = i_0(m, \delta) > 0$ such that the injectivity radius of any simply connected compact m-dimensional Riemannian manifold with finite second homotopy group and $\mathrm{Ric} \geq \delta$, $K \leq 1$, is bounded from below by $i_0(m, \delta)$.

Note that since a positive Ricci pinching condition does not allow to make use of Synge's lemma (as in [Kl1]), this last result is new even in even dimensions.

We show that this result is optimal even if (as in the Klingenberg-Sakai Conjecture) we fix the topological type. Namely, there is a sequence of metrics g_n on $S^2 \times S^3$ which satisfy the bounds $K_{g_n} \leq 1$ and $\mathrm{Ric} \geq \delta > 0$, but for which the spaces $(S^2 \times S^3, g_n)$ collapse to $S^2 \times S^2$. This and more interesting examples will be discussed in section 4.

In fact, all results from [PetrRTu] (among them the Continuous Collapse theorem and Stable Collapse theorem, see Theorems 1.5–1.7 below and section 3) are also valid for positive Ricci pinching conditions. Here is one illustration (the bounded version of the Klingenberg-Sakai conjecture for Ricci pinching conditions):

Theorem 0.5. *Let M be a closed manifold, d_0 be a metric on M and $\delta > 0$. Then there exists $i_0 = i_0(M, d_0, \delta) > 0$ such that the injectivity radius i_g of any Ricci-δ-pinched d_0-bounded metric g on M (i.e., any Riemannian metric g with $\mathrm{Ric}_g \geq \delta$, $K_g \leq 1$ and $\mathrm{dist}_g(x, y) \leq d_0(x, y)$) is bounded from below by $i_g \geq i_0$.*

For a brief history of the problem of finding lower positive bounds for the injectivity radii of simply connected δ- and Ricci-δ-pinched Riemannian m-manifolds M^m which depend only on the pinching constant δ and/or (the dimension of) the manifold, we refer the reader to [PetrRTu]. As original sources we mention the papers [Po], [Kl1,2], [BuT], [CGr], [KlS1,2], and [AMe].

For a Riemannian manifold M let $\mathfrak{R}_2 : \Lambda^2(T(M)) \to \Lambda^2(T(M))$ be the curvature operator which occurs in the Bochner formula for two-forms, i.e., the one so that for any 2-form ϕ on M one has that

$$D^2\phi = \nabla^*\nabla\phi + \mathfrak{R}_2(\phi)\,.$$

Let us mention one more result:

Theorem 0.6. *There is no collapsing sequence of simply connected compact Riemannian m-manifolds with uniformly bounded diameters which satisfies the curvature conditions $K \le 1$ and $\mathfrak{R}_2 \ge \delta > 0$. In other words:*

Given any m, D and any $\delta > 0$, there exists a positive constant $i_0 = i_0(m,\delta,D) > 0$ such that the injectivity radius of any simply connected compact m-dimensional Riemannian manifold with diameter $\le D$ and $\mathfrak{R}_2 \ge \delta$, $K \le 1$, is bounded from below by $i_0(m,\delta,D)$.

Note that since in dimension 3 the conditions $\mathrm{Ric} > 0$ and $\mathfrak{R}_2 > 0$ are equivalent, Theorems 0.4 and 0.6 can be considered as direct generalizations of the Burago-Toponogov result ([BuT]) to general dimensions. Also note that for dimension $m \ge 4$ these two conditions are independent (i.e., neither one of them implies the other).

The remaining parts of this paper are organized as follows:

In section 1, the relevant preliminaries are presented. The proofs of the π_2-theorem and the π_2-Finiteness theorem and its corollary are given in section 2. The extensions of the π_2-theorem as well as the above-mentioned results from [PetrRTu] to positive Ricci pinching conditions are given in section 3. In section 4 we construct some examples to show that under these conditions, the π_2-theorem is indeed optimal. There we also disprove a conjecture of Fukaya on the dimension of limit spaces of collapsing sequences of manifolds with uniformly positively pinched curvature. In section 5 we discuss several open problems. In an appendix we describe a generalized notion of Riemannian manifold and its use in the context of our results and collapsing with bounded curvature.

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1 Preliminaries and Rough Structure of the Proofs

In this section, we introduce some notation and review relevant results. A little more detailed discussion is given in [PetrRTu]. For basic notions and results about collapsed manifolds, (equivariant) Hausdorff convergence, and Alexandrov spaces the reader is referred to [BuGroPe], [CFuGro], [Fu3] and [GroLP].

Let $M = (M^m, g)$ be a Riemannian manifold of dimension m and let $FM = F(M^m)$ denote its bundle of orthonormal frames. When fixing a bi-invariant metric on $O(m)$, the Levi-Civita connection of g gives rise to a canonical metric on FM, so that the projection $FM \to M$ becomes a Riemannian submersion and so that $O(m)$ acts on FM by isometries. Another fibration structure on FM is called $O(m)$ *invariant*, if the $O(m)$ action on FM preserves both its fibres and its structure group.

A *pure N-structure* on M^m is defined by an $O(m)$ invariant fibration, $\tilde{\eta} : FM \to B$, with fibre a nilmanifold isomorphic to $(N/\Gamma, \nabla^{\mathrm{can}})$ and structural group contained in the group of affine automorphisms of the fibre, where N is a simply connected nilpotent group and ∇^{can} the canonical connection on N for which all left invariant vector fields are parallel. A pure N-structure on M induces, by $O(m)$-invariance, a partition of M into "orbits" of this structure (see [CFuGro]), and is then said to have *positive rank* if all these orbits have positive dimension. A pure N-structure $\tilde{\eta} : FM \to B$ over a Riemannian manifold (M, g) gives rise to a sheaf on FM whose local sections restrict to local right invariant vector fields on the fibres of $\tilde{\eta}$; see [CFuGro]. If the local sections of this sheaf are local Killing fields for the metric g, then g is said to be *invariant* for the N-structure (and $\tilde{\eta}$ is then also sometimes referred to as *pure nilpotent Killing structure for g*).

Theorem 1.1 ([CFuGro], [R1]). *Let for $m \geq 2$ and $D > 0$ $\mathfrak{M}(m, D)$ denote the class of all m-dimensional compact connected Riemannian manifolds (M, g) with sectional curvature $|K_g| \leq 1$ and diameter* $\mathrm{diam}(g) \leq D$.

Then, given any $\varepsilon > 0$, there exists a positive number $v = v(m, D, \varepsilon) > 0$ such that if $(M, g) \in \mathfrak{M}(m, D)$ satisfies $\mathrm{vol}(g) < v$, *then M^m admits a pure N-structure $\tilde{\eta} : FM \to B$ of positive rank so that*

(a) *There is a smooth metric g_ε on M which is invariant for the N-structure $\tilde{\eta}$ and for which all fibres of $\tilde{\eta}$ have diameter less than ε, satisfying*
$$e^{-\varepsilon} g < g_\varepsilon < e^{\varepsilon} g\,, \quad |\nabla_g - \nabla_{g_\varepsilon}| < \varepsilon\,, \quad |\nabla^l_{g_\varepsilon} R_{g_\varepsilon}| < C(m, l, \varepsilon)\,;$$
(b) *There exists $c(m) < \infty$ such that the invariant metric g_ε in* (a) *also satisfies the curvature bounds*
$$\min K_g - c(m)\varepsilon \leq K_{g_\varepsilon} \leq \max K_g + c(m)\varepsilon\,;$$
(c) *There exist constants $i = i(m, \varepsilon) > 0$ and $C = C(m, \varepsilon)$ such that, when equipped with the metric induced by g_ε, the injectivity radius of B is $\geq i$ and such that the second fundamental form of all fibres of $\tilde{\eta}$ is bounded by C.*

Remark 1.2. Parts (a) and (c) of Theorem 1.1 follow from [CFuGro, Theorem 1.3 and Theorem 1.7]. Part (b) of Theorem 1.1 is proved in [R1]. The fact that the N-structure $\tilde{\eta}$ in Theorem 1.1 is indeed a *pure* structure follows from the presence of a diameter bound (compare [Fu3]). The assertion about the injectivity radius of B in part (c) of Theorem 1.1 can be extracted from [CFuGro, section 5].

The $O(m)$ invariance of a pure N-structure $\tilde{\eta} : FM \to B$ implies that the $O(m)$ action on FM descends to an $O(m)$ action on B and that the fibration on FM descends to a possibly singular fibration on M, $\eta : M^m \to B/O(m)$, such that the following diagram commutes.

$$\begin{array}{ccc} F(M^m) & \xrightarrow{\tilde{\eta}} & B \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \tilde{\pi}} \\ M^m & \xrightarrow{\eta} & B/O(m) \end{array}$$

If $\pi_1(M^m)$ is finite, then the homotopy exact sequence shows that the fibre of a pure N-structure on M is a torus. If, in particular, M^m is simply connected, then since in this case the structure group of the torus fibration is trivial, a pure N-structure on a simply connected M is defined, up to an isomorphism, by a global torus action.

Theorem 1.1 and the above remark imply the following result, which we are going to use throughout this paper. (The first part of Theorem 1.3(c) below is a consequence of Perelman's Stability theorem ([Pe]).)

Theorem 1.3. *Assume that (M_n, g_n) is a sequence of simply connected compact Riemannian m-manifolds with sectional curvature bounds $\lambda \le K(g_n) \le \Lambda$ and diameters $\operatorname{diam}(M_n) \le D$ which collapses to an Alexandrov space X of dimension $m-k$. Then, given any $\varepsilon > 0$, for $n = n(\varepsilon)$ sufficiently large the following holds:*

(a) *There exists on the frame bundle FM_n of M_n an $O(m)$ invariant T^k fibration structure $T^k \to FM_n \to B_n$ for which the induced fibration on M_n is given by a smooth global effective T^k action with empty fixed point set all of whose orbits have diameter less than ε;*

(b) *There exists on M_n a T^k invariant metric $\tilde{g}_n$ which satisfies*

$$e^{-\varepsilon} g_n < \tilde{g}_n < e^{\varepsilon} g_n \,, \qquad \lambda - \varepsilon \le K(\tilde{g}_n) \le \Lambda + \varepsilon \,;$$

(c) *The orbit space M_n/T^k is homeomorphic to X and, when equipped with the metric induced by $\tilde{g}_n$, the Gromov-Hausdorff distance between X and M_n/T^k is less than ε;*

(d) *There exist constants $i = i(\varepsilon) > 0$ and $C = C(\epsilon)$, such that when the frame bundle FM_n is equipped with the metric defined by the Levi-Civita connection of $\tilde{g}_n$, then FM_n has sectional curvature bounded by $|K(FM_n)| \leq C$ and the injectivity radius of $B_n = FM_n/T^k$ is $\geq i$.*

(e) *If in addition $\mathrm{Ric}(g_n) \geq \lambda'$, then the invariant metrics $\tilde{g}_n$ can also be chosen to satisfy $\mathrm{Ric}(\tilde{g}_n) \geq \lambda' - \varepsilon$.*

We note that 1.3(e) does not follow directly from the above. Its proof is an exact copy of the proof of Proposition 2.5 in [R1], if one replaces the expression $A(t)$ used there by $A(t) = \sup_{e,x} \mathrm{Ric}_(g_t)$, where e is a one-dimensional direction in T_xM. Let us also note that in 1.3(d), one can in addition also obtain that the sectional curvatures of all manifolds B_n are uniformly bounded in absolute value. This follows for instance from [Fu2] (but we will not use this fact).

We will refer to the torus actions arising from Theorem 1.3(a) as *collapse-related torus actions* or *collapsing torus actions associated to the (sufficiently collapsed) metrics g_n.*

For the proof of the π_2-theorem we will use the Stable Collapse theorem from [PetrRTu]. To this means recall the following notion of *stability* of a collapsing sequence of metric spaces:

Definition 1.4. *A sequence of metric spaces M_n is called stable if there is a topological space M and a sequence of metrics d_n on M such that (M, d_n) is isometric to M_n and such that the metrics d_n converge uniformly as functions on $M \times M$ to a continuous pseudometric.*

In [PetrRTu], the following results were obtained:

Theorem 1.5 (Stable Collapse theorem) ([PetrRTu]). *Suppose that a compact manifold M admits a sequence of metrics $(g_n)_{n\in\mathbb{N}}$ with sectional curvatures $\lambda \leq K_{g_n} \leq \Lambda$, such that, as $n \to \infty$, the metric spaces (M, g_n) Hausdorff converge to a compact metric space X of lower dimension. Then, provided that $\{(M, g_n)\}$ contains a stable subsequence, the metrics g_n cannot be uniformly positively pinched, i.e., λ cannot be positive.*

Theorem 1.6 (Continuous Collapse theorem) ([PetrRTu]). *Suppose that a compact manifold M admits a continuous one parameter family $(g_t)_{0<t\leq 1}$ of Riemannian metrics with sectional curvature $\lambda \leq K_{g_t} \leq \Lambda$, such that, as $t \to 0$, the family of metric spaces (M, g_t) Hausdorff converges to a compact metric space X of lower dimension. Then these metrics cannot be uniformly positively pinched, i.e., λ cannot be positive.*

Theorem 1.7 (Bounded version of the Klingenberg-Sakai conjecture) ([PetrRTu]). *Let M be a closed manifold and d_0 be a metric on M and $\delta > 0$. Then there exists $i_0 = i_0(M, d_0, \delta) > 0$ such that the injectivity radius i_g of any δ-pinched d_0-bounded metric g on M, i.e., any Riemannian metric g with sectional curvature $\delta \leq K_g \leq 1$ and $\mathrm{dist}_g(x, y) \leq d_0(x, y)$, is bounded from below by $i_g \geq i_0$.*

To prove the π_2-theorem (Theorem 0.3 in the Introduction), we therefore simply have to show that any collapsing sequence of m-dimensional compact simply connected Riemannian manifolds with finite second homotopy groups and uniformly bounded curvatures contains a stable subsequence and then apply the Stable Collapse theorem.

The Stable Collapse theorem relies on the following result:

Theorem 1.8 (Gluing theorem) ([PetrRTu]). *Let $\{M_n\}$ be a stable sequence of simply connected Riemannian manifolds with uniformly bounded sectional curvatures $\lambda \leq K_{g_n} \leq \Lambda$ such that the sequence of metric spaces M_n Hausdorff converges to a compact metric space X of lower dimension. Then there exists a* noncompact *complete Alexandrov space $Y = Y(X, (g_n))$ with the same lower curvature bound λ.*

The Stable Collapse theorem then follows by contradiction from the Gluing theorem and the following extension of Myers' theorem:

Theorem 1.9 ([BuGroPe]). *A complete Alexandrov space with lower positive curvature bound has finite diameter and hence is compact.*

The Continuous Collapse theorem and the bounded version of the Klingenberg-Sakai Conjecture theorem are proven by showing that inside a continuous and collapsing one parameter family of metrics, or, respectively, inside a collapsing sequence of d_0-bounded metrics one can always find an infinite sequence of collapsing metrics which is stable in the sense of Definition 1.4, so that the Stable Collapse theorem applies.

Note that the procedure of finding such stable subsequences in [PetrRTu] works for sectional curvature just bounded in absolute value. Thus, the π_2-theorem, the Continuous Collapse theorem and the bounded version of the Klingenberg-Sakai Conjecture theorem will immediately extend to positive Ricci pinching conditions once we have shown that the proof of the Stable Collapse theorem extends to positive Ricci pinching conditions – and for this we only have to establish an analogue of the Bonnet-Myers theorem for the Alexandrov spaces Y that arise from the Gluing theorem.

2 The π_2-theorem and Diffeomorphism Finiteness

In this section, we will prove the π_2-Diffeomorphism Finiteness theorem, the classification theorem 0.2 and the π_2-theorem for sectional curvature pinching:

Theorem 2.1 (π_2-Finiteness theorem). *For given m, C and D, there is only a finite number of diffeomorphism types of simply connected closed m-dimensional manifolds M with finite second homotopy groups which admit Riemannian metrics with sectional curvature $|K(M)| \leq C$ and diameter $\operatorname{diam}(M) \leq D$.*

Theorem 2.2 (A "classification" of simply connected closed manifolds). *For given m, C and D, there exists a finite number of closed smooth manifolds E_i such that any simply connected closed m-dimensional manifold M admitting a Riemannian metric with sectional curvature $|K(M)| \leq C$ and diameter $\operatorname{diam}(M) \leq D$ is diffeomorphic to a factor space $M = E_i/T^{k_i}$, where $0 \leq k_i = \dim E_i - m$ and T^{k_i} acts freely on E_i.*

Remark 2.3. Using the same technique, one can strengthen the conclusions as follows:

For given m, C and D, there exist finitely many closed manifolds E_i with a torus action, (E_i, T^{s_i}), such that any simply connected closed m-dimensional manifold M admitting a Riemannian metric with sectional curvature $|K(M)| \leq C$ and diameter $\operatorname{diam}(M) \leq D$ is diffeomorphic to a factor space $M = E_i/T^{k_i}$, where $0 \leq k_i = \dim E_i - m$, T^{k_i} is a subgroup of T^{s_i}, and T^{k_i} acts freely on E_i.

Theorem 2.4. *There is no collapsing sequence of simply connected manifolds with finite second homotopy groups and fixed dimension and positively pinched curvature $1 \geq K \geq \delta > 0$.*

For the proof of the above theorems, let us first introduce some notation and prove two lemmas.

Definition 2.5.A. *Let (Z_n, G, ρ_n) and (Z, G, ρ) be compact spaces with metrics ρ_n and ρ, and isometric actions of a compact group G. We say that (Z_n, G, ρ_n) converges to (Z, G, ρ) in the G-equivariant Hausdorff topology, if there is a metric d on the disjoint union of $\{Z_n\}$ and Z such that:*

(a) *$d|_{Z_n} \equiv \rho_n$ for all n and $d|_Z \equiv \rho$;*
(b) *the spaces Z_n (as subsets) converge to Z in the standard Hausdorff sense, i.e., for any $\epsilon > 0$ we have (with respect to the d-metric) that $Z \subset B_\epsilon(Z_n)$ and $Z_n \subset B_\epsilon(Z)$ when n is sufficiently large;*

(c) *there exist automorphisms $A_n : G \to G$ such that, with respect to d, the G action on the disjoint union of all $\{(Z_n, A_n(G))\}$ and (Z, G) is an isometric action.*

DEFINITION 2.5.B. *Let (M, G) and (M', G) be G manifolds, where G is a Lie group. Then (M, G) and (M', G) are said to be (G) diffeomorphic, if there exists a diffeomorphism h from M to M' and an automorphism A of G which conjugate the G actions, i.e., for which for all $x \in M$ and $g \in G$ it holds that $h(gx) = A(g)h(x)$.*

When we will construct or speak about G-diffeomorphisms (or G-convergence), the automorphisms A (A_n) will not be explicitly mentioned.

KEY LEMMA 2.6. *Let $(F, G \times T^k)$ and $(F', G \times T^k)$ be simply connected compact $G \times T^k$ manifolds with finite second homotopy groups, where G is a connected compact Lie group whose fundamental group is finite. Assume that the T^k subactions on F and F' are free and that there is a G diffeomorphism $h : (F/T^k, G) \to (F'/T^k, G)$. Then there is a $G \times T^k$ diffeomorphism $\tilde{h} : (F, G \times T^k) \to (F', G \times T^k)$ such that the following diagram commutes:*

$$\begin{array}{ccc} (F, G \times T^k) & \xrightarrow{\tilde{h}} & (F', G \times T^k) \\ \big\downarrow \pi & & \big\downarrow \pi' \\ (F/T^k, G) & \xrightarrow{h} & (F'/T^k, G) \end{array}$$

Proof of Key Lemma 2.6. Set $(Y, G) := (F/T^k, G) =_h (F'/T^k, G)$. First note that from the homotopy exact sequence for fibre bundles we have that

$$\pi_2(F) \to \pi_2(Y) \xrightarrow{e} \pi_1(T^k) \to \pi_1(F) = 0\,,$$

where the mapping $e \in H^2(Y, \mathbb{Z}^k)$ is the T^k bundle version of the Euler class for circle bundles.

So if Tor is the torsion part of $\pi_2(Y)$, then $\pi_2(Y)/\operatorname{Tor} = \mathbb{Z}^k$ and there is an isomorphism $\pi_2(Y)/\operatorname{Tor} \xrightarrow{e} \pi_1(T^k)$. This isomorphism induces a unique isomorphism $A : T^k \to T^k$, where the first torus T^k acts on F and the second on F'. Thus, from now on we can think that the same torus T^k is acting on both F and F', and that the T^k-Euler class of these actions is the same.

Therefore it is possible to construct a diffeomorphism h^* from F to F' which conjugates the T^k actions, so that (F, T^k) and (F', T^k) are T^k

diffeomorphic and which makes the following diagram commute:

$$\begin{array}{ccc} (F,T^k) & \xrightarrow{h^*} & (F',T^k) \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi'} \\ (F/T^k,G) & \xrightarrow{h} & (F'/T^k,G) \end{array}$$

To show that $(F,G\times T^k)$ is diffeomorphic to $(F',G\times T^k)$, we now only need to correct the constructed diffeomorphism h^* in such a way that the $G\times T^k$ actions are conjugate.

The idea of the following construction is essentially the same as one used in [GrovK].

We will proceed in the following way: After applying $h^*: F\to F'$, we simply rotate each T^k orbit $T^k\cdot h^*(x)$ of F' by multiplying by an appropriately chosen element $w^{-1}(x)\in T^k$. In other words, we obtain a smooth map $w^{-1}: F\to T^k$ which is T^k invariant (i.e., which satisfies $w^{-1}(\tau x)=w^{-1}(x)$ for all $\tau\in T^k$ and $x\in F$), and the map $\tilde h := w^{-1}\cdot h^*$ will then be the desired one.

Since π (and π') commutes with the G action, from the last commutative diagram we have that $\pi'\circ h^*(gx)=\pi'(gh^*(x))$. Therefore $h^*(gx)$ and $gh^*(x)$ live in the same orbit (of the free T^k-action), and thus there exists a mapping $\eta: G\times F\to T^k$ such that

$$\eta(g,x)h^*(gx)=gh^*(x).$$

The mapping $\eta(g,x)$ is T^k invariant, i.e., η satisfies $\eta(g,\tau x)=\eta(g,x)$ for any $\tau\in T^k$, $g\in G$ and $x\in F$. Moreover, one has that

$$\eta(g'g,x)=\eta(g',gx)\eta(g,x)\,. \tag{$*$}$$

Now set $w(x)=\text{mean value}_G(\eta(g,x))$, and note that this average of $\eta(\cdot,x)$ over G is well defined. In fact, since G has finite fundamental group, there is a continuous lift of η to $\tilde\eta: G\times F\to\mathbb{R}^k$, where $\mathbb{R}^k$ is the universal covering Lie group of $T^k=\mathbb{R}^k/\mathbb{Z}^k$. Now all such lifts differ from each other by an element of $\mathbb{Z}^k$, so that for different lifts of η the mean values of those lifts also only differ by elements of $\mathbb{Z}^k$. Therefore the projections of these mean values back to the torus T^k give us for fixed x always the same element, so that $w(x)$ is well defined.

Integrating $(*)$ by g' we get that for all $x\in F$, the mean value $w(x)$ satisfies

$$w(x)=\eta(g,x)w(gx)\,.$$

Now consider the map $\tilde h: F\to F'$, defined by $\tilde h(x):=w^{-1}(x)h^*(x)$ (where $w^{-1}(x)w(x)=id\in T^k$).

We claim that $\tilde{h}$ is a diffeomorphism which conjugates the $(G \times T^k)$ actions on F and F'.

Let us first see why this map is a diffeomorphism: Note that $\tilde{h}$ is a bijection, because since w^{-1} is T^k invariant, $\tilde{h}$ is one-to-one on each T^k orbit. Moreover, $\tilde{h}$ is a smooth map because w, being the average of smooth functions $G \to T^k$, is smooth. To see that also the inverse of $\tilde{h}$ is smooth, one could check by using local coordinates that the differential of $\tilde{h}$ is invertible in every point. However, a shorter argument for this last assertion is the following one: Choose any T^k invariant volume form on F'. Then, since w^{-1} is simply rotating the T^k orbits, the smooth map sending $h^*(x)$ to $w^{-1}(x)h^*(x)$ is volume preserving and thus a diffeomorphism of F'. Therefore $\tilde{h}$ is a diffeomorphism.

It remains to show that $\tilde{h}$ preserves the $G \times T^k$ actions: Indeed, in noting that $w^{-1}(x)g = gw^{-1}(x)$ (since the G and T^k subactions commute and $g \in G$ and $w^{-1}(x) \in T^k$) we have that

$$\begin{aligned} &g\tilde{h}(x) \\ &= w^{-1}(x)gh^*(x) = w^{-1}(x)\eta(g,x)h^*(gx) = w^{-1}(x)\eta(g,x)w(gx)\tilde{h}(gx) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \tilde{h}(gx)\,. \end{aligned}$$

The following lemma is just slightly more general than Theorem 6.9 of [Fu3] (which in [Fu3] is stated without proof).

LEMMA 2.7. *Let $(Y_i, G) \to (Y, G)$ be a sequence of smooth compact m-dimensional Riemannian G manifolds Y_i with diameter* $\operatorname{diam}(Y_n) \le D$, *injectivity radius $\ge i_0 > 0$ and curvature bounded from below by $K \ge \lambda$, which G-Hausdorff converges to an Alexandrov space Y of the same dimension, where G is a compact connected Lie group.*

Then Y is a C^1-smooth manifold with a Riemannian metric of class C^0, and for i sufficiently large there exist C^1 diffeomorphisms $\pi_i : Y_i \to Y$, which are also bi-Lipschitz almost isometries, such that the corresponding sequence of G actions on Y is in fact converging in C^1. In particular, by [GrovK], for i sufficiently large all G actions are conjugate by a C^1 diffeomorphism which is also a bi-Lipschitz almost isometry.

Proof of Lemma 2.7. The idea of the proof of Lemma 2.7 lies in the following simple fact: Assume that a sequence of concave C^1-smooth functions f_i converges in C^0 to a C^1 smooth function. Then in the interior of any interval, the convergence takes in fact place in C^1.

That the limit space Y is a manifold with C^1 smooth structure and Riemannian metric of class C^0 follows directly from [AnC2] or [OShi] by our assumption on the injectivity radii of the Riemannian manifolds Y_i. (This can actually also be checked directly by showing that distance functions on Y give us C^1-charts.)

We now construct the mappings π_i by using a construction from [BuGroPe]. From our assumptions we have that Y has injectivity radius bounded from below by i_0 and curvature $\geq \lambda$. For each point $p \in Y$, we find a collection of points $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m$ in Y so that for the comparison angles in the model space of constant curvature λ, the following is true: The angles $\tilde{\angle} a_i p b_i$ satisfy $\tilde{\angle} a_i p b_i > \pi - \epsilon$, and for $i \neq j$ and $|pa_i|, |pa_j| < i_0$, one has that $\tilde{\angle} a_i p b_j, \tilde{\angle} a_i p a_j, \tilde{\angle} b_i p b_j > \pi/2 - \epsilon$.

Then the collection of distance functions to each a_k,

$$f_k(x) := |a_k x|\,,$$

gives us in a neighborhood U of p a C^1 smooth chart $f = (f_1, \dots, f_m) : U \to R^m$. Now we can consider points in Y_i which are close to the above points in Y and construct another smooth chart in a neighborhood U_i of a point in Y_i which is close to p. After eventually shrinking U and U_i, these two charts then give us local C^1 diffeomorphisms $\phi_{i,U} : Y_i \supset U_i \to U \subset Y$.

Now consider a finite covering $\{U_l\}$ of Y by C^1 smooth charts of the above kind. For each corresponding local map $\phi_{i,U_l} : U_{i,l} \to U_l$ one can then construct a C^1-almost identity diffeomorphism h_l of U_l such that all local C^1 diffeomorphisms $h_l \circ \phi_i$ can be glued together to yield a global C^1 diffeomorphism $\pi_i : Y_i \to Y$. (For a concrete construction of these global diffeomorphisms using special imbeddings of Y_i and Y into a Euclidean space $\mathbb{R}^S$ of high dimension, the reader may compare with [Y, p. 325]. Since in our case Y does not necessarily have an upper curvature bound, these imbeddings are only of class C^1. To obtain the mapping π_i from Yamaguchi's construction, one then has only to replace the orthogonal projection from a normal bundle by a projection along some fixed transversal distribution of $(S-m)$-dimensional planes.)

Since then, as $i \to \infty$, on each $U_{i,l}$ the C^1 distance between ϕ_{i,U_l} and $\pi_i|_{U_{i,l}}$ converges to 0, we only have to show that the lemma is true for the C^1 mappings $\phi_i = \phi_{i,U_l}$. Since G is connected and compact, it is furthermore enough to prove our lemma under the assumption $g \in B_\epsilon(e) \subset G$, where ϵ is suitably small.

Thus we must only show that for any $x \in Y$ and any unit vector $\xi \in T_x(Y)$ and $g \in B_\epsilon(e) \subset G$ one has that

$$\lim_{i \to \infty} \phi_i\big(g \phi_i^{-1}(\xi)\big) = g\xi\,.$$

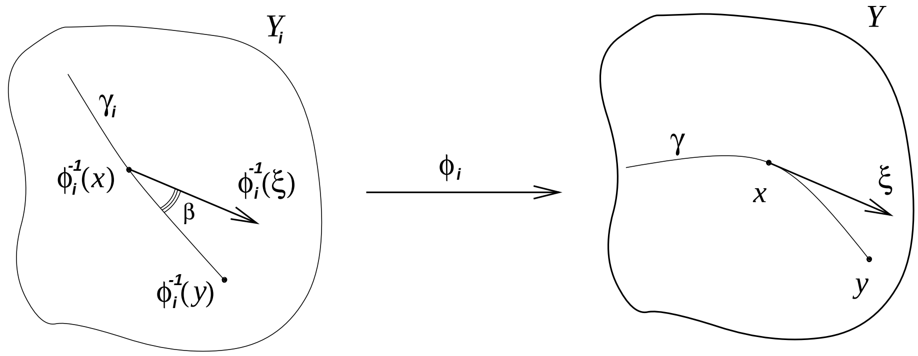
(Here we use the same notation for both a mapping and for its differential.)

Since each ϕ_i is close to a standard Hausdorff approximation, and since the actions of G on Y_i converge to the G action on Y, we have that $\alpha := \lim_{i\to\infty} \phi_i(g\phi_i^{-1}(\xi))$ and $g\xi$ are elements in $T_{gx}(Y)$.

Therefore it is enough to show that for any coordinate function f_k of a local chart (U, f) around x one has that $\partial f_k/\partial\alpha = \partial f_k/\partial(g\xi)$.

There is a unique minimal unit speed geodesic $\gamma = \gamma(t) : [-l, l] \to Y$, $\gamma(0) = x$, starting at x in the direction of ξ. Choose l in such a way that γ ends in some point $y \in U$, $y \neq x$, and such that for $g \in B_\epsilon(e) \subset G$ the geodesic $g\gamma(t)$ is still contained in U. The geodesic $g\gamma(t)$ then connects the points gx and gy and has initial direction $g\xi$.

Now consider (see the diagram below) in Y_i the geodesic $\gamma_i(t)$ from $\phi_i^{-1}(x)$ to $\phi_i^{-1}(y)$. Extend it in the opposite direction beyond $\phi_i^{-1}(x)$. We then claim that, as $i \to \infty$, the angle β, between $\phi_i^{-1}(\xi)$ and the initial direction of $\gamma_i(t)$ at $t = 0$ goes to zero.



To show this, note that γ is the limit of γ_i. Therefore for each f_k it follows that $f_k \circ \phi_i \circ \gamma_i(t)$ converges to $f_k \circ \gamma(t)$.

Our curvature conditions imply that for some suitably chosen c, the functions $f_k \circ \phi_i \circ \gamma_i(t) - ct^2$ are concave. The function $f_k \circ \gamma(t) - ct^2$ is even C^1 smooth, because f_k is C^1 and γ is a geodesic. Therefore, by our observation at the beginning of the proof we have that for all $k = 1, \dots, m$, the derivatives $\frac{d}{dt}(f_k \circ \phi_i \circ \gamma_i(t))_{t=0}$ converge to $\frac{d}{dt}(f_k \circ \gamma(t))_{t=0}$. This proves our intermediate claim that for $i \to \infty$ the angle between $\phi_i^{-1}(\xi)$ and $\gamma_i'(0)$ converges to zero, so that the vectors $\phi_i(\gamma_i'(0))$ converge to $\xi = \gamma'(0)$.

Finally, consider the geodesics $g\gamma_i$. They must converge to $g\gamma$, and from the same reasoning as above we obtain that $\phi_i(g\gamma_i'(0))$ converges to $g\gamma'(0)$. Hence the lemma is proved.

Proof of the π_2-Finiteness theorem. Step 1. Assume that the theorem is false. Then, for some m, C and D, there exists an infinite sequence $(M_i)_{i\in N}$ of *mutually non-diffeomorphic* compact simply connected m-dimensional Riemannian manifolds with finite second homotopy groups and with sectional curvatures uniformly bounded by $|K(M_i)| \leq C$ and uniformly bounded diameters $\operatorname{diam}(M_i) \leq D$. By Cheeger's Finiteness theorem, the injectivity radii of the manifolds M_i must converge to zero as $i \to \infty$. By Gromov's compactness theorem one can find a converging subsequence $M_i \overset{GH}{\to} X$, where X is an Alexandrov space with $m - k = \dim X < \dim M_i = m$.

Step 2. By Theorem 1.3 one can slightly perturb the metrics g_i on M_i such that for sufficiently big i, the metrics on M_i are smooth and invariant with respect to the collapsible T^k actions associated to them, such that $|K(M_i)| \leq C'$, $\operatorname{diam}(M_i) \leq D'$ and such that the diameter of all T^k orbits will uniformly converge to 0. Moreover (also by Theorem 1.3), the oriented frame bundles $F(M_i)$, equipped with the metrics induced by the Levi-Civita connections of the invariant metrics on the M_i, will have uniformly bounded curvatures $|K(F(M_i))| \leq C''$ and uniformly bounded diameters as well. From now on we assume that all M_i carry such invariant metrics.

Since the T^k actions on the M_i are effective and isometric, the $SO(m)$ principal bundles $F(M_i)$ admit natural $SO(m) \times T^k$ actions, so that the $SO(m)$ subaction as well as the T^k subaction on each $F(M_i)$ are free and isometric actions.

Step 3. Now we construct a sequence of spaces $(F_i)_{i\in N}$ to which Key Lemma 2.6 is applicable.

Let $F_i := \widetilde{F(M_i)}$ denote the universal Riemannian coverings of $F(M_i)$. Each F_i is either isometric to $F(M_i)$, or else it is a two-sheeted covering of $F(M_i)$. (Here we must assume that $\dim M_i \geq 3$, but since there are no closed simply connected two- and one-manifolds with finite π_2, no one can complain.)

Therefore all F_i will satisfy similar curvature and diameter bounds as the frame bundles $F(M_i)$.

It follows that we can lift the $SO(m)$ action on $F(M_i)$ to a covering and isometric action of $G := Spin(m)$ on F_i . Moreover, by lifting the T^k action on M_i to F_i, we obtain on each F_i an isometric $G \times T^k$ action such that the T^k subaction is free on each F_i.

Step 4. After eventually passing to a subsequence, we may assume that the sequence of compact simply connected $G \times T^k$ manifolds F_i converges in the equivariant $(G \times T^k)$ Hausdorff distance (see [Fu3]). Let Y be its

Hausdorff limit. Note that the T^k action on Y is trivial, and that Y is also a G limit of the Riemannian manifolds F_i/T^k. In particular, by Theorem 1.3, Y is a C^1-smooth compact manifold of dimension dim $F_i - k$ with lower curvature bound and a lower bound for the injectivity radius.

Step 5. We apply Lemma 2.7 to the sequence of manifolds $(F_i/T^k, G)$, and after that Key Lemma 2.6 to the manifolds $(F_i, G \times T^k)$. It follows that for sufficiently big i all F_i are $G \times T^k$ diffeomorphic. In particular, all manifolds $(M_i, T^k) = (F_i/G, T^k)$ are (T^k equivariantly) diffeomorphic to each other.

Therefore our collapsing sequence $(M_i)_{i\in N}$ contains an infinite subsequence of diffeomorphic manifolds. This is a contradiction, so that the π_2-Finiteness theorem is proved.

For the proof of Theorem 2.2, we will employ the following definition:

Definition 2.8. *Let M be a simply connected compact manifold. Then a simply connected manifold E is called* universal T^k bundle *of M if one has that*

(a) *the second homotopy group $\pi_2(E)$ of E is finite;*
(b) *for some natural number k, the manifold E admits the structure of a T^k bundle $T^k \to E \to M$. (In particular, if $\pi_2(M)$ is finite, then $k = 0$ and the universal T^k bundle of M is M itself.)*

Using the exact homotopy sequence, one sees that universal T^k bundles exist and that they are unique.

Proof of Theorem 2.2. Fix given numbers m, C and D. If M is any simply connected closed m-dimensional manifold admitting a Riemannian metric with sectional curvature bounded by $|K(M)| \le C$ and diameter bounded by $\operatorname{diam}(M) \le D$, let E be its universal T^k-bundle. The dimension of E can be estimated from Gromov's Betti number theorem (see [Gro1]): $\dim(E) = \dim(M) + b_2(M) \le m + L_m^{1+\sqrt{C}D}$. Therefore we have an upper bound for the dimension of all such universal bundles.

Now note that on each such a bundle one can construct a metric with just slightly worse curvature and diameter bounds, say, $|K| \le C'$ and diameter $\le D'$, where C' and D' only depend on C and D. Since all universal T^k bundles have by definition finite second homotopy groups, by the π_2-Finiteness theorem there are only finitely many of them, so that the theorem is proved.

Proof of the π_2-theorem. This proof actually almost coincides with the proof of the π_2-Finiteness theorem. The only new ingredient here consists of applying the Stable Collapse theorem from [PetrRTu].

Suppose that for some m and $\delta > 0$ there exists a sequence $(M_i)_{i\in N}$ of m-dimensional compact simply connected Riemannian manifolds with finite second homotopy groups and uniformly positively pinched sectional curvatures $0 < \delta \le K(M_i) \le 1$ whose injectivity radii converge to zero as $i \to \infty$. We may assume that the sequence (M_i) Hausdorff converges to an Alexandrov space of lower dimension $m-k$.

After performing the smoothing procedure described in Theorem 1.3, we obtain on all manifolds M_i metrics that will remain to have uniformly positively pinched curvature, say, $0 < \delta' \le K(M_i) \le 1$, and which now are also invariant under their associated collapsible T^k actions.

Now let F_i be as in the proof of the π_2-Finiteness theorem (Step 3), and recall that $G = Spin(m)$.

Fix almost isometric diffeomorphisms $\tilde{\chi}_i : (F_i/T^k, G) \to (Y, G)$, which exist by Lemma 2.7. Then Key Lemma 2.6 gives us diffeomorphisms $\tilde{h}_{i,l}$: $(F_i, G \times T^k) \to (F_l, G \times T^k)$ which make the following diagram commute:

$$\begin{array}{ccc} (F_i, G\times T^k) & \xrightarrow{\tilde{h}_{i,l}} & (F_l, G\times T^k) \\ \big\downarrow{\scriptstyle \pi_i} & & \big\downarrow{\scriptstyle \pi_l} \\ (F_i/T^k, G) & \xrightarrow{\tilde{\chi}_l^{-1}\circ\tilde{\chi}_i} & (F_l/T^k, G) \end{array}$$

Therefore the diffeomorphisms $\tilde{\chi}_i : (F_i/T^k, G) \to (Y, G)$ induce almost isometric homeomorphisms $\chi_i : M_i/T^k = F_i/(G \times T^k) \to Y/G = X$ on the factors, where X is the Hausdorff limit of the sequence (M_i). In the same way, the diffeomorphisms $\tilde{h}_{i,l}$ induce diffeomorphisms $\bar{h}_{i,l} : (M_i, T^k) \to (M_l, T^k)$ for which the following diagram commutes:

$$\begin{array}{ccc} (M_i, T^k) & \xrightarrow{\bar{h}_{i,l}} & (M_l, T^k) \\ \big\downarrow{\scriptstyle \pi_i} & & \big\downarrow{\scriptstyle \pi_l} \\ M_i/T^k & \xrightarrow{\chi_l^{-1}\circ\chi_i} & M_l/T^k \end{array}$$

Now fix some l and set in Definition 1.4 $M := M_l$ and $d_i(\bar{h}_{i,l}(x), \bar{h}_{i,l}(y)) := \mathrm{dist}_{M_i}(xy)$. Then the sequence (M_i, g_i) is stable. Thus applying the Stable Collapse Theorem 1.5 finishes the proof.

Remark 2.9. It is possible to simplify the proofs of the π_2 and the Stable Collapse theorem. This is done by using the following arguments:

Note that the above proof in fact shows that the sequence (M_i) satisfies a more special condition: Namely, in this case not only the sequence (M_i), but also the sequence of bundles (F_i) is stable, and the homeomorphisms $\tilde{h}_{i,l}$ (which one can use in the definition of stable collapse, see 1.4) already conjugate the collapse related T^k actions on F_i (which are free actions!). By considering a subtorus $T^{k-1} \subset T^k$, one only needs to apply the simplest case of the Stable Collapse theorem, namely, the case of a stable collapse where the collapse related T^k actions are given by a free S^1 action (see [PetrRTu]), to obtain a noncompact complete Alexandrov space, say, a space Z (in [PetrRTu] this space is always called "Y", but here this letter is already occupied). So far this is nothing exciting; the space Z simply is some noncompact space with curvature bounded from below by some negative constant. However, the excitement now comes from the following fact: One can reconstruct the G action from the limit of the local actions of G on the limit "tubes" $\widetilde{C}_\alpha^\infty$ (in the notation of [PetrRTu]) to obtain an isometric action of G on Z. (Again, this is only true if $\dim M \geq 3$, see the remark in Step 3 above.) Then, because G is compact, the factor space Z/G will be noncompact. Now Z/G locally looks like the limit of the local factors $\widetilde{C}_\alpha^i/G$, which are locally isometric to M_i. Therefore we have our lovely contradiction to Myers' theorem: Z/G is a complete, noncompact Alexandrov space with a strictly positive lower curvature bound.

In the same way one can also simplify the proof of the Stable Collapse theorem, since in fact the following theorem holds: If (M_n) is a stable sequence of compact simply connected Riemannian manifolds with uniformly bounded curvatures and diameters collapsing to an Alexandrov space X, then after smoothing (see Theorem 1.3) the sequence of orthonormal frame bundles $F_n = F(M_n)$, equipped with the induced metrics, contains an infinite subsequence which is also stable.

3 The π_2-theorem and Collapsing under Ricci Pinching Conditions

In the π_2-theorem, until now we were dealing with positively pinched *sectional* curvature, namely, the case $1 \geq K(M) \geq \delta > 0$. In this section, we would like to explain why the π_2-theorem as well as the Stable and Continuous Collapse and bounded version of the Klingenberg-Sakai Conjecture theorems from [PetrRTu] (see section 1) are in fact true for the case of *positive Ricci pinching*, i.e., under the conditions $K \leq 1$ and $\mathrm{Ric} \geq \delta > 0$, so that in fact the following results hold:

π_2-theorem for Ricci pinching. *There is no collapsing sequence of* *simply connected manifolds with finite second homotopy groups and fixed dimension with curvature conditions* $K \leq 1$ *and* $\mathrm{Ric} \geq \delta > 0$.

Continuous Collapse theorem for Ricci pinching. *Suppose that a compact manifold* M *admits a continuous one parameter family* $(g_t)_{0<t\leq 1}$ *of Riemannian metrics satisfying the curvature conditions* $K_{g_t} \leq 1$ *and* $\mathrm{Ric}_{g_t} > 0$*, such that, as* $t \to 0$*, the family of metric spaces* (M, g_t) *Hausdorff converges to a compact metric space* X *of lower dimension. Then these metrics cannot be uniformly positively Ricci pinched, i.e., there is no* $\delta > 0$ *such that* $\mathrm{Ric}_{g_t} \geq \delta$ *for all* t.

Stable Collapse theorem for Ricci pinching. *Suppose that a compact manifold* M *admits a sequence of metrics* $(g_n)_{n\in\mathbb{N}}$ *satisfying the curvature conditions* $K_{g_n} \leq 1$ *and* $\mathrm{Ric}_{g_n} > 0$*, such that, as* $n \to \infty$*, the metric spaces* (M, g_n) *Hausdorff converge to a compact metric space* X *of lower dimension. Then, provided that* $\{(M, g_n)\}$ *contains a stable subsequence, the metrics* g_n *cannot be uniformly positively Ricci pinched, i.e., there is no* $\delta > 0$ *such that* $\mathrm{Ric}_{g_n} \geq \delta$ *for all* n.

Bounded version of the Klingenberg-Sakai Conjecture theorem for Ricci pinching. *Let* M *be a closed manifold,* d_0 *be a metric on* M *and* $\delta > 0$*. Then there exists* $i_0 = i_0(M, d_0, \delta) > 0$ *such that the injectivity radius* i_g *of any Ricci-*δ*-pinched* d_0*-bounded metric* g *on* M*, i.e., any Riemannian metric* g *with* $\mathrm{Ric}_g \geq \delta$*,* $K_g \leq 1$ *and* $\mathrm{dist}_g(x, y) \leq d_0(x, y)$*, is bounded from below by* $i_g \geq i_0$.

The general reason for why all these theorems extend to positive Ricci pinching conditions is that our proofs do not really depend on the very condition of sectional curvature pinching. In order to deal with the Ricci-condition, and generalize our results to positive Ricci pinching, we simply must know a little about limit spaces with such more relaxed curvature bounds.

The following remarks will fill these gaps. They do not pretend to be of complete generality in any sense, like, say, trying to give a general synthetic definition of Ricci curvature for Alexandrov spaces; they are just designed for dealing with our particular problem.

First of all, let us make some trivial comments on the the condition of positive Ricci pinching:

Obviously, this condition is more general than the condition of positively pinched sectional curvature. But the difference is not that big. In particular, $K(M) \leq 1$ and $\mathrm{Ric}(M) \geq \delta > 0$ imply $|K(M)| \leq \dim(M)$, i.e., under positive Ricci pinching one automatically has that also sectional curvature is uniformly bounded. Therefore, if we do not have a collapse, under the Ricci pinching condition any Hausdorff limit is an N-manifold.

By an N-manifold we understand a Riemannian manifold with two-sided bounded curvature in the sense of Alexandrov, i.e., every triangle is neither too fat nor too thin (see [BeN]). Now N-manifolds behave in many respects just as ordinary Riemannian manifolds. In particular, in such spaces one can define a Ricci tensor, and it has the same geometric sense as in the Riemannian setting (see [N], [BeN]), so that the notion of N-manifold with $\mathrm{Ric} \geq c$ is well defined. Phrased differently, an N-manifold of dimension m has $\mathrm{Ric} \geq c$ if it can be obtained as a Lipschitz limit of a sequence of m-dimensional Riemannian manifolds $(M_i)_{i\in\mathbb{N}}$ with a two sided uniform curvature bound and $\mathrm{Ric}_{M_i} \geq c - \varepsilon_i$, where $\varepsilon_i \to 0$ as $i \to \infty$.) In the very same way one defines N-orbifolds and N-orbifolds with $\mathrm{Ric} \geq c$.

Definition 3.1. *Let (X,μ) be a complete metric space with a locally finite measure. We say that (X,μ) has n-*Ricci curvature* $\geq \delta$, if for any point $x \in X$ there is a neighbourhood U_x of x with the following properties: There exists an open n-dimensional N-orbifold N_x with* $\mathrm{Ric} \geq \delta$ *and an isometric action of an Abelian group A_x on N_x, so that $U_x = N_x/A_x$, and so that μ is proportional to the factor measure induced from N_x. In this case we call N_x a* Ricci chart *of X.*

Definition 3.2. *A function $f : X \to R$ is λ-subharmonic if its pullback to any Ricci chart is λ-subharmonic.*

Note that the subharmonicity of a function on X depends only on (X,μ), but not on a Ricci chart.

These two definitions make it possible to adapt the Laplacian comparison proof of Myers' theorem for Riemannian manifolds (or orbifolds) (see, for example, [Z]) to spaces with n-Ricci curvature $\geq \delta$:

Proposition 3.3. *A space (X,μ) with n-Ricci curvature $\geq \delta > 0$ has finite diameter* $\mathrm{diam}(X) \leq c(n,\delta)$. *In particular, there is no noncompact complete metric space with n-Ricci curvature $\geq \delta > 0$.*

On the other hand, from the above definitions and properties one easily sees: If we have a stable collapse of M^n with positive Ricci pinching, then the noncompact complete space Y constructed in the Gluing theorem of [PetrRTu] (see Theorem 1.5) carries a measure μ so that (Y,μ) has n-Ricci curvature $\geq \delta$. (Here is the place where we need Theorem 1.3(e)).

Therefore, by what has been said above, the Stable Collapse theorem will extend to positive Ricci pinching conditions.

The measure μ on Y can be constructed as follows: We first consider the limit measure μ_X on X of the normalized measures $\mu_i/\mu_i(M_i)$ on $M_i = (M, g_i)$, where μ_i is the n-dimensional Hausdorff measure on M_i. (See [Fu1] for more on this measure on X.) Then, since there is a free and isometric action of $\mathbb{R}^{k'}$ on Y such that $X = Y/\mathbb{R}^{k'}$, one can construct an $\mathbb{R}^{k'}$ invariant measure μ on Y whose "factor measure" is μ_X.

So the Stable and Continuous Collapse and thus the π_2 and bounded version of the Klingenberg-Sakai Conjecture theorems also hold for positive Ricci pinching conditions.

4 Examples of Collapses with Positive Pinching

A. Let us first show that Theorem 0.4 is indeed optimal, even if we restrict ourselves to sequences of metrics on a fixed manifold and to collapses with nonnegative sectional curvature:

Example 4.1 (The $S^2 \times S^3$-example). There exists on $M = S^2 \times S^3$ a sequence of metrics g_n with sectional curvatures $0 \le K_{g_n} \le 1$ and $\mathrm{Ric}_{g_n} \ge \delta > 0$, but for which, as $n \to \infty$, the spaces (M, g_n) collapse to $S^2 \times S^2$.

To see this, let $E := (S^3 \times S^3, g_{can})$ be the Riemannian product of two standard 3-spheres. View E as a subset of $\mathbb{C}^2 \times \mathbb{C}^2$, so that the standard torus $T^2 = S^1_1 \times S^1_2$ acts on E by isometries and so that each S^1 factor acts on the corresponding S^3 by complex multiplication. For integers $p, q \in \mathbb{Z}$, let $S^1_{p,q} < T^2$ denote the S^1-subgroup of T^2 which corresponds to $p[S^1_1] + q[S^1_2] \in \pi_1(T^2)$. Note that if p and q are relatively prime, then $S^1_{p,q}$ acts freely and isometrically on E.

Consider now the sequence of factor spaces $M_n := E/S^1_{1,n}$. Using the Gysin sequence, one observes that for all $n = 0, 1, 2, \dots \in \mathbb{N}$, the homology groups and second Stiefel Whitney classes of M_n and $S^2 \times S^3$ coincide. By Barden's diffeomorphism classification of simply connected 5-manifolds ([B]), we can therefore identify all M_n with $M = S^2 \times S^3$.

It is easy to see that, as $n \to \infty$, the sequence of manifolds M_n Hausdorff converges to $S^2 \times S^2 = S^3 \times S^3/T^2$. Moreover, from for instance ([E, Proposition 22]) it follows that the sectional curvatures of M_n converge to the sectional curvatures of $S^3 \times S^2 = S^3 \times S^3/S^1_{0,1}$. In particular, we have that for sufficiently big n (here in fact for any n) all M_n satisfy a positive Ricci pinching condition with nonnegative sectional curvature.

REMARK 4.2. We would like to note that in a different context, Example 3.4 was earlier discussed by M. Wang and W. Ziller in [WZi]; compare also [Gro2]. The results in [WZi] in particular imply the following: For any natural number $q \neq 0$, the manifolds $M = \mathbb{CP}^1 \times S^{2q+1}$ carry a sequence of metrics with $\mathrm{Ric} \equiv 1$ so that M, when equipped with these metrics, collapses with sectional curvature uniformly bounded in absolute value to $\mathbb{CP}^1 \times \mathbb{CP}^q$.

B. Note that in the last example the collapsing S^1 actions on $(S^3 \times S^2, g_n)$ are not conjugate. Therefore one could ask whether under positive Ricci pinching conditions such a collapse is still possible if we require in addition that all collapse-related torus actions belong to one fixed conjugacy class.

However, we will now give an example of a collapse with positive Ricci pinching of a simply connected six-manifold where all collapsing torus actions are given by a free S^1 action with *fixed* conjugacy class. This example builds on $S^3 \times S^2 \# S^3 \times S^2$ and shows in particular that the definition of stability and the assumptions of the Stable Collapse theorem cannot be weakened to mere conjugacy of the torus actions associated to collapsed metrics.

The following lemma follows directly from ([B, Theorem 2.2]):

LEMMA 4.3. *Let X be the connected sum $X := S^3 \times S^2 \# S^3 \times S^2$. Then any automorphism $\theta : H^2(X) \to H^2(X)$ is induced by a diffeomorphism $f : X \to X$.*

COROLLARY 4.4. *Assume that (M, S^1) and (M', S^1) are two simply connected manifolds with free S^1 action and factor spaces M/S^1 and M'/S^1 both diffeomorphic to $X = S^3 \times S^2 \# S^3 \times S^2$. Then there is a diffeomorphism $\tilde{f} : M \to M'$ which conjugates the S^1 actions.*

Proof of Corollary 4.4. Let $e \in H^2(M/S^1)$ and $e' \in H^2(M'/S^1)$ be the Euler classes that classify M and M' as principal circle bundles over X. Since M and M' are simply connected, both e and e' are indivisible. Therefore there exists an isomorphism $\theta : H^2(M/S^1) \to H^2(M'/S^1)$ such that $\theta(e) = e'$. By Lemma 4.3 we can find a diffeomorphism $f : M/S^1 \to M'/S^1$ which makes both S^1 actions on M and M' to have the same Euler class. This implies that we can construct a conjugation diffeomorphism $\tilde{f} : M \to M'$ which makes the diagram below commutative, so that the corollary is proved.

$$\begin{array}{ccc} M & \xrightarrow{\tilde{f}} & M' \\ \downarrow & & \downarrow \\ M/S^1 & \xrightarrow{f} & M'/S^1 \end{array}$$

EXAMPLE 4.5 (A collapsing sequence with positive Ricci pinching and fixed conjugacy class). Let as above X be the connected sum $X := S^3 \times S^2 \# S^3 \times S^2$.

By [ShYa], X admits a Riemannian metric g with positive Ricci curvature. By [GPaTu] now the following holds: For each nontrivial cohomology class $e \in H^2(X)$ one can find a corresponding closed 2-form ω on X and a smooth function $h : X \to R$ such that the S^1 principal bundle over X determined by the cohomology class $e = [\omega]$ carries a S^1 invariant metric with positive Ricci curvature so that ω is the curvature form of this metric and so that $h(x)$ equals the length of the S^1 fibre over $x \in X$.

Choose a basis $e_1, e_2 \in H^2(X)$. By what we said above, for e_1 there is a closed 2-form ω_1 on X and a smooth function $h : X \to R$ such that the S^1 principal bundle over X determined by $[\omega_1] = e_1$ carries an S^1 invariant metric with positive Ricci curvature with ω_1 as curvature form and h as fibre-length function.

Now take any smooth closed 2-form ω_2 whose cohomology class corresponds to e_2, and consider the S^1 principal bundle M_n over X with indivisible Euler class $ne_1 + e_2$. On M_n we can construct an S^1 invariant warped product metric g_n with curvature form $n\omega_1 + \omega_2$ and fibre-length function $h_n = (1/n)h$.

Direct calculation then shows the following: If $p_n \in M_n$ and $p \in M$ are points which all project to the same point in X, then for $n \to \infty$ the curvature tensors R_{p_n} converge to R_p. Therefore all curvature bounds on (M_n, g_n) will converge to the curvature bounds of (M, g), so that for sufficiently large n we have that $\mathrm{Ric}(g_n) \geq \delta > 0$ and $K(g_n) \leq C < \infty$ for some fixed C and $\delta > 0$. On the other hand, for $n \to \infty$ the manifolds (M_n, g_n) collapse to X. By Corollary 4.4 all M_n are diffeomorphic and all collapsing S^1 actions are conjugate.

QUESTION 4.6. Let X be as above. Consider the product space $X \times [0, 1]$ and glue $X \times 0$ to $X \times 1$ by some diffeomorphism f of X for which the corresponding automorphism of $H^2(X)$ has infinite order. Let Y be the resulting space, which we can view as an X bundle over S^1. Is it then possible to find a Riemannian metric on Y so that each X fibre has positive

Ricci curvature?

A positive answer to this question would allow us to construct on a six-manifold M^6 a continuous family of metrics (g_t) that satisfied the positive Ricci pinching curvature bounds $K \leq C < \infty$, $\mathrm{Ric} \geq \delta > 0$, but which would be collapsing. In particular, this would imply that in the Continuous Collapse theorem, the assumption of the convergence of the family (M, g_t) to a fixed limit space X cannot be removed.

C. To close this section, we will construct counterexamples to the following conjecture:

CONJECTURE 4.7 ([Fu3, 15.7]). *Let X be a Gromov-Hausdorff limit of a sequence of uniformly positively pinched m-dimensional simply connected Riemannian manifolds. Then* $\dim(X) \geq m - 1$.

However, we will now show that collapsing with uniform pinching to spaces with codimension higher than one indeed takes place:

EXAMPLE 4.8. There exist sequences $(M_i^7)_{i\in\mathbb{N}}$ of uniformly positively pinched Eschenburg spaces which collapse to a 4-dimensional Alexandrov space $X^4 = T^2\backslash SU(3)/T^2$.

Example 4.8 in particular shows that the following dimension estimate is optimal:

PROPOSITION 4.9 ([R2]). *Let X be as in Conjecture 4.7. Then* $\dim(X) \geq (m+1)/2$.

REMARK 4.10. Using Basaikin spaces, we can also construct counterexamples to Fukaya's conjecture in dimension 13. Here we obtain sequences that collapse with uniform positive pinching to 9-dimensional limits of the form $X^9 = T^5\backslash U(5)/(Sp(2) \times S^1)$.

Before we start to show how to obtain Example 4.8, let us first recall from [E] the definition of the Eschenburg spaces: Let p, q, k, l be integers which are relatively prime. Then the quadruple $(p, q, k, l) \in \mathbb{Z}^4$ is called *admissible* if and only if each of the following pairs of integers is relatively prime:

$$(k-p, l-q)\,,\ (k-p, l+p+q)\,,\ (k-q, l+p+q)\,,$$
$$(k-q, l-p)\,,\ (k+p+q, l-p)\,,\ (k+p+q, l-q)\,.$$

Let $G = SU(3)$ and U be a closed subgroup of $G^2 = G \times G$. Then U acts on G by $(u, g) \mapsto u_1 g u_2^{-1}$, where $g \in G$ and $u = (u_1, u_2) \in U$.

Every admissible quadruple (k, l, p, q) determines a closed one parameter subgroup $U = U_{klpq}$ of G^2 by requiring that $U \subset T^4$, where $T^4 = T^2 \times T^2$ is the maximal torus of diagonal matrices in $SU(3)\times SU(3)$, and by requiring that $v_{klpq} := 2\pi i(\mathrm{diag}(k,l,-k-l), \mathrm{diag}(p,q,-p-q))$ be a generator of the kernel of the exponential map of $\mathfrak{u}$. Moreover, for any admissible quadruple (k,l,p,q) the group U_{klpq} acts freely on G, and the quotient space $M_{klpq} := U_{klpq}\backslash G$ is a compact simply connected seven-dimensional manifold.

It is proved in [E] that there are many admissible quadruples (k_0,l_0,p_0,q_0) for which the corresponding spaces $M_{k_0l_0p_0q_0}$ admit a metric of positive sectional curvature.

Note that when equipped with the metrics from [E], the curvature bounds of the spaces M_{klpq} depend continuously on the direction of the vector (k,l,p,q). In fact, from ([E, Proposition 22]) one easily deduces the following:

LEMMA 4.11. *Let (k,l,p,q) and, for $i = 1,2,\dots \in \mathbb{N}$, $(k_i,l_i,p_i,q_i)\in\mathbb{Z}^4$ be admissible quadruples. Assume that, as $i\to\infty$, the directions $\frac{(k_i,l_i,p_i,q_i)}{\|(k_i,l_i,p_i,q_i)\|}$ converge to the direction $\frac{(k,l,p,q)}{\|(k,l,p,q)\|}$. Then the sectional curvature bounds of the corresponding spaces $M_{k_il_ip_iq_i}$ converge to the curvature bounds of M_{klpq}.*

Therefore, for every positively curved Eschenburg space $M_{k_0l_0p_0q_0}$ there exists an open cone $C\subset\mathbb{R}^4$, containing the line through (k_0,l_0,p_0,q_0), such that for any admissible quadruple $(p,q,k,l)\in C$ the corresponding space M_{pqkl} has curvature $0<\delta_0<K(M_{pqkl})\le 1$, where δ_0 depends only on the pinching of $M_{k_0l_0p_0q_0}$.

The construction of Example 4.8. Take a totally irrational direction in C, i.e., a line which is not contained in any rational hyperplane. Choose a sequence of admissible points $\{(p_i,q_i,k_i,l_i)\}_{i\in\mathbb{N}}\subset C$ which approach this direction, and consider the sequence of corresponding Eschenburg spaces $M_i = M_{p_iq_ik_il_i}$. Then it is easy to see that the M_i Hausdorff converge to $X = T^2\backslash \mathrm{SU}(3)/T^2$.

Therefore, to complete our construction, it remains only to prove the existence of a sequence of admissible quadruples with the above properties. This follows from the following claim. Its proof is an exercise in elementary geometry of numbers, but we are giving it here because we had a lot of fun proving it:

CLAIM 4.12. *Let $h_i : \mathbb{Z}^n\to\mathbb{Z}^2$ be a finite collection of homomorphisms, and assume that there exists an element $\alpha\in\mathbb{Z}^n$ such that every $h_i(\alpha)$ is a relatively prime pair of integers. Then every open cone $C\subset\mathbb{R}^n\supset\mathbb{Z}^n$ contains infinitely many elements $\alpha'\in\mathbb{Z}^n$ with the same property.*

Proof of the claim. Choose any rational direction $x \in C$, $x \in \mathbb{Z}^n$ such that for every i, $h_i(\alpha)$ and $h_i(x)$ are not collinear, and consider the sequence $\alpha_m = mx + \alpha$. Then for any i there is a number n_i such that $h_i(\alpha_{mn_i})$ is relatively prime. Therefore, for any i one also has that $h_i(\alpha_{mN})$ is relatively prime, where $N = \prod_i n_i$. For sufficiently large m it follows that $\alpha_{mN} \in C$. Thus C contains infinitely many elements α_j for which all $h_i(\alpha_j)$ are relatively prime, and our claim is proved.

5 Remarks and Questions

Related to the above conjecture of Fukaya we have a conjecture for manifold Hausdorff limits of sequences of positively curved manifolds which are subject only to a lower positive curvature bound.

Conjecture 5.1. *Let X be a Hausdorff limit of a sequence of simply connected m-dimensional Riemannian manifolds with sectional curvatures bounded from below by $K \geq 1$. Assume that X is a Riemannian manifold (of positive dimension). Then* $\dim X \geq m - 1$.

The following conjecture, we believe, is due to Yamaguchi, and would follow from a positive answer to Conjecture 5.1.

Conjecture 5.2. *Let X be a Hausdorff limit of a continuous one parameter family of Riemannian metrics on an m-dimensional manifold with sectional curvatures bounded from below by $K \geq 1$. Assume that X is a Riemannian manifold. Then* $\dim X = m$. *In other words: there is no continuous collapse with lower positive curvature bound to a Riemannian manifold.*

We would also like to mention again our question from section 4 (see also the discussion given there):

Question 5.3. Let $X = S^3 \times S^2 \# S^3 \times S^2$. Consider the product space $X \times [0, 1]$ and glue $X \times 0$ to $X \times 1$ by some diffeomorphism f of X for which the corresponding automorphism of $H^2(X)$ has infinite order. Let Y be the resulting space, which we can view as an X bundle over S^1. Is it then possible to find a Riemannian metric on Y so that each X fibre has positive Ricci curvature?

Recall once more the notion of universal T^k bundle (Definition 2.8):

Definition. *Let M be a simply connected compact manifold. Then a simply connected manifold E is called* universal T^k bundle *of M if one has that*

(a) *The second homotopy group* $\pi_2(E)$ *of* E *is finite;*
(b) *For some natural number* k*, the manifold* E *admits the structure of a* T^k *bundle* $T^k \to E \to M$*.*

Now let us take a look at the Klingenberg-Sakai conjecture in the case where the second homotopy group of the manifold has infinite order.

Conjecture 5.4. *Let* B *be a compact simply connected manifold. Suppose that over* B *there are infinitely many different principal* T^k *bundles with total space diffeomorphic to a fixed simply connected manifold* F*.*

Then the diffeomorphism group of the universal T^k *bundle* E *of* F *contains a subgroup which generates an infinite group of automorphisms of the cohomology ring* $H^*(E)$*.*

It seems that a positive solution to this purely topological conjecture could make it possible to prove the Klingenberg-Sakai Conjecture 0.6 in the introduction for all Aloff-Wallach, Eschenburg, and Basaikin spaces. In particular, this would establish the Klingenberg-Sakai conjecture for all positively curved manifolds that are known today.

Let us also note that the usefulness of the notion of stable collapse in this paper as well as [PetrRTu] suggests to understand it better and in a more general setting.

Question 5.5. Let (M, d_n) be a sequence of metric spaces which converges to a compact metric space X. What kind of topological and/or curvature conditions for M and d_n will insure that any such sequence is stable?

For example, is it true that any converging sequence of metrics on S^m with uniform lower curvature bound and upper bound on the diameter has a stable subsequence?

Let us also recall that to prove Theorem 0.3, we showed that any converging sequence of 2-connected manifold contains a stable subsequence. But is it true that in this case the whole sequence we started with is actually already stable?

Question 5.6. Is it possible to also “classify” all *non*-simply connected manifolds in a way that is close to Theorem 0.2 ?

Question 5.7. For which kinds of curvature pinching does there exist an analogue of the π_2-theorem (see Theorems 0.3, 0.4, and 0.6 in the Introduction)?

Since we already have two cases where this works (positive Ricci pinching, see Theorem 0.4, and positive pinching of the “second Bochner curvature” $\mathfrak{R}_2$, see Theorem 0.6), there might be more general (or at least other) pinching conditions which imply the existence of uniform estimates for the injectivity radius.

In this regard it is also interesting to ask if the diameter bound in Theorem 0.6 could actually be removed.

Question 5.7 is closely related to the following one (the notation we use here is explained in the Appendix):

QUESTION 5.8. Assume that a sequence of manifolds M_n, with uniformly bounded curvatures and diameters, Grothendieck-Lipschitz converges to a megafold $\mathfrak{M}$. Then what can be said about the relations between the topology of the M_n and the topology of $\mathfrak{M}$?

In particular we do not know a counterexample to the following:

Assume that a sequence of odd-dimensional spheres $M_n := (S^{2m+1}, g_n)$, with uniformly bounded curvatures and diameters, Grothendieck-Lipschitz converges to a megafold $\mathfrak{M}$. Then either $\mathfrak{M}$ is homeomorphic to S^{2m+1} and the M_n Lipschitz converge to $\mathfrak{M}$, or $H^*_{dR}(\mathfrak{M}) = H^*_{dR}(\mathbb{C}\mathrm{P}^m \times S^1)$.

Appendix: Collapsing and Grothendieck-Lipschitz Convergence

In this appendix we would like to provide the right notions for the above business (including [PetrRTu]). Though our proofs above do not use these concepts at all, in some sense they were always present, but the reason why we did not use them explicitly is that (at least for geometers) at first contact it is not easy to like them.

However, as the reader will see below, this approach, when compared to the standard one, not only simplifies our proofs a lot; it also gives a clear reason why they work. We also hope that the compactness theorem below (see Theorem A.5) will give rise to further results and applications.

It is our pleasure to note that about two years ago Professor Gromov was trying to explain to the first author some ideas similar to the following (but we understood them only now). We would also like to thank Vladimir Voevodsky for friendly help with topoi.

First, using Grothendieck topologies* let us give a (very formal) definition of an appropriate generalization of the notion of Riemannian manifold (for an informal one see below). For readers who are not familiar with topos theory we suggest skipping directly to A.2, at least for a first reading. (We

*As general references for Grothendieck topologies and topos theory, we mention the books [J] and [MMo].

will, as far as notation is concerned, usually not distinguish between a topological space and its corresponding Grothendieck topos.)

DEFINITION A.1. *A Riemannian m-megafold $\mathfrak{M} = (\mathfrak{M}, g)$ is a Grothendieck topos such that there is an epimorphism $Y \to X$ onto the final object X of $\mathfrak{M}$ so that the slice topos $\mathfrak{M}/Y$ is isomorphic to a topos corresponding to an m-manifold and such that the following is true:*

(i) *$\mathfrak{M}$ is linearly separable, i.e. any two continuous maps $f_i : [0,1] \to \mathfrak{M}$, $i = 1, 2$, which coincide on $[0,1)$ coincide on the whole interval $[0,1]$.*

(ii) *On $\mathfrak{M}/Y$ a specific choice of a Riemannian metric g is made such that for any two morphisms $Z \overset{p}{\to} Y$ and $Z \overset{q}{\to} Y$, the induced pullback metrics on $\mathfrak{M}/Z$ coincide.*

(Note that since the topos which corresponds to a topological space is nothing but the category of all local homeomorphisms to this space, we can talk about pull-back metrics without ambiguity. Also note that the Riemannian metric g on $\mathfrak{M}$ does in the following sense NOT depend on the choice of the epimorphism $Y \to X$: The choice of a Riemannian metric on $\mathfrak{M}/Y$, as above, uniquely determines a metric on each $\mathfrak{M}/Z$ (if it is a manifold), and this metric makes any morphism $Z \overset{p}{\to} Z'$ correspond to a local isometry $\mathfrak{M}/Z \overset{p'}{\to} \mathfrak{M}/Z'$.)

In particular, if $\mathcal{F}$ is a foliation on some Riemannian manifold M with (locally) equidistant fibres, then the factor $M/\mathcal{F}$ admits a natural structure of a Riemannian megafold of dimension $\dim M - \dim \mathcal{F}$.

Another class of examples can be constructed in the following way: Let M be a Riemannian manifold, and G be *any* group of isometries of M. If now as open sets one takes all local homeomorphisms $U \to M$ and as morphisms all commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ M & \xrightarrow{\gamma} & M \end{array}$$

where $\gamma : M \to M$ denotes multiplication by some (and every) element of G, then this category generates a Riemannian megafold which we will denote by $(M : G)$. Note that $(M : G)$ has the same dimension as M.

This example can be generalized to the case of natural pseudogroup actions on M, whose definition is as follows:

DEFINITION A.2. *A* pseudogroup action *(or pseudogroup of transformations) on a manifold M is given by a set G of pairs of the form $p = (D_p, \bar{p})$,* *where D_p is an open subset of M and $\bar{p}$ is a homeomorphism $D_p \to M$, so that the following properties hold:*

(1) $p, q \in G$ *implies* $p \circ q = (\bar{q}^{-1}(D_p \cap \bar{q}(D_p)), \bar{p} \circ \bar{q}) \in G$;
(2) $p \in G$ *implies* $p^{-1} = (\bar{p}(D_p), \bar{p}^{-1}) \in G$;
(3) $(M, id) \in G$;
(4) *if $\bar{p}$ is a homeomorphism from an open set $D \subset M$ into M and $D = \bigcup_\alpha D_\alpha$, where D_α are open sets in M, then the property $(D, \bar{p}) \in G$ is equivalent to $(D_\alpha, \bar{p}|_{D_\alpha}) \in G$ for any α.*

We call the pseudo-group action natural *if in addition the following is true:*

$(i)'$ *If $(D, \bar{p}) \in G$ and $\bar{p}$ can be extended as a continuous map to a boundary point $x \in \partial D$, then there is an element $(D', \bar{p}') \in G$ such that $x \in D'$, $D \subset D'$ and $\bar{p}'|_D = \bar{p}$.*

By taking $U = \mathfrak{M}/Y$ (as in Definition A.1) one sees that any Riemannian megafold $(\mathfrak{M}, g)$ can be represented as $((U, g) : G)$, where $(D, \bar{p}) \in G$ if and only if there are two morphisms $q_1, q_2 : V \to U$ such that $D = q_1(V)$ and $\bar{p} = q_2 \circ q_1$. Note that A.1(i) implies A.2(i)$'$, i.e., the G (pseudogroup) action is natural. (It is moreover easy to see that the pair (U, G) contains all information about $\mathfrak{M}$.)

Therefore any megafold admits a representation $((U, g) : G)$, where G is a natural action by local isometries on a (possibly open) Riemannian manifold (U, g). This representation is similar to representing a manifold by using a collection of charts and gluing maps.

Recall that for manifolds everything like tensors, forms, maps between manifolds, etc., can be defined either purely intrinsically or by using local coordinate representations, and note that the very same is true for megafolds. Here one can either use Definition A.1, or define everything via megafold representations. For example: two Riemannian m-megafolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are said to be isometric iff there exists an isomorphism of topoi $I : \mathfrak{M}_1 \to \mathfrak{M}_2$ such that for any object Y of $\mathfrak{M}_1$ for which the slice topos $\mathfrak{M}_1/Y$ corresponds to a manifold, the induced isomorphism $\mathfrak{M}_1/Y \to \mathfrak{M}_2/I(Y)$ corresponds to an isometry of Riemannian manifolds. Equivalently, $\mathfrak{M}_1 = ((U_1, g_1) : G_1)$ and $\mathfrak{M}_2 = ((U_2, g_2) : G_2)$ are isometric iff there exists a megafold representation $((U, g) : G)$ and a locally isometric covering $p_i : (U, g) \to (U_i, g_i)$ such that the pulled back actions of the pseudogroups G_i on U coincide with the G action on U.

Note that if the pseudogroup action G is properly discontinuous and free, then the corresponding megafold is simply a Riemannian manifold, and if the pseudogroup action is only assumed to be properly discontinuous, then one obtains a Riemannian orbifold (V-manifold).†

In this regard note also that the infinitesimal motions of the pseudogroup G give rise to a Lie algebra of Killing fields on (U, g), and from this one can recover an isometric local action of a connected Lie group on (U, g). Let us call this group G_o.

It is obvious that $G_o = G_o(\mathfrak{M})$, i.e., G_o does not depend on the special representation $(U : G)$ (of a connected Riemannian megafold $(\mathfrak{M}, g)$). If the $G_o(\mathfrak{M}, g)$ action is trivial, then $(\mathfrak{M}, g)$ is a Riemannian orbifold.

Definition A.3. *A Riemannian megafold $(\mathfrak{M}, g)$ is called* complete *if any finite length curve $f : [0, 1) \to \mathfrak{M}$ can be extended to the end (i.e., there is a continuous map $\bar{f} : [0, 1] \to \mathfrak{M}$ such that $f = \bar{f}|_{[0,1)}$).*

A Riemannian megafold $(\mathfrak{M}, g) = ((U, g) : G)$ is called H-complete if G together with any converging sequence of local isometries also contains their limit.

Now we come to the main point of this note:

Definition A.4. *A sequence of Riemannian megafolds $(\mathfrak{M}_n, g_n)$ is said to* Grothendieck-Lipschitz converge (GL-converge) *to a Riemannian megafold $(\mathfrak{M}, g)$ if there are representations $(\mathfrak{M}_n, g_n) = ((U_n, g_n) : G_n)$ and $(\mathfrak{M}, g) = ((U, g) : G)$ such that*

(a) *The (U_n, g_n) Lipschitz converge to (U, g), and*
(b) *For some sequence $\epsilon_n \to 0$ there is a sequence of $e^{\pm\epsilon_n}$-bi-Lipschitz homeomorphisms $h_n : (U_n, g_n) \to (U, g)$, such that the pseudogroup actions on (U_n, g_n) converge (with respect to the homeomorphisms h_n) to a pseudogroup action on (U, g).*

I.e., for any converging sequence of elements $p_{n_k} \in G_{n_k}(U_{n_k}, \mathfrak{M}_{n_k})$ there exists a sequence $p_n \in G_n$ which converges to the same local isometry on U, and the pseudogroup of all such limits, acting on U, coincides with the pseudogroup action $G(U, \mathfrak{M})$.

It is obvious that the Grothendieck-Lipschitz limit of Riemannian megafolds is always H-complete.

Here are two simple examples of GL-convergence:

Consider the sequence of Riemannian manifolds $S^1_\epsilon \times \mathbb{R}$, which for $\epsilon \to 0$ Gromov-Hausdorff converge to $\mathbb{R}$. Then this sequence converges in the GL-

†This is actually also the motivation for the name "megafold": A manifold is an object which is glued from *many* pieces by UNIQUE gluings; a megafold is obtained by gluing many pieces along MANY (in general actually infinitely many) gluings.

topology to a Riemannian megafold $\mathfrak{M}$, which can be described as follows: It is covered by one single chart $U = \mathbb{R}^2$, and the pseudogroup G simply consists of all vertical shifts of $\mathbb{R}^2$. I.e., $\mathfrak{M}$ is nothing but $(\mathbb{R}^2 : \mathbb{R})$ where $\mathbb{R}$ acts by parallel translations. (Note that $(\mathbb{R}^2 : \mathbb{R}) \neq \mathbb{R}^2/\mathbb{R}$, these megafolds even have different dimensions!)

The Berger spheres, as they Gromov-Hausdorff collapse to S^2, converge in Grothendieck-Lipschitz topology to the Riemannian megafold $(S^2\times\mathbb{R}{:}\mathbb{R})$. Here $\mathbb{R}$ acts by parallel shifts of $S^2 \times \mathbb{R}$.

A Riemannian metric on a megafold $((U, g) : G)$ defines a pseudometric on the set of G orbits. In particular one has that the diameter of a Riemannian megafold is well defined. Now here is the basic result:

Theorem A.5. (i) *The set of Riemannian m-manifolds with bounded sectional curvature $|K| \leq 1$ and diameter* diam $\leq D$ *is precompact in the Grothendieck-Lipschitz topology.*

(ii) *The set of complete Riemannian m-megafolds with bounded sectional curvature $|K| \leq 1$ (in the sense of Alexandrov) and diameter* diam $\leq D$ *is compact in the Grothendieck-Lipschitz topology. The same holds for the corresponding set of complete and H-complete Riemannian megafolds.*

Proof. Since the proofs of (i) and (ii) are almost identical, we will only prove the first statement.

Assume we are given a sequence of Riemannian manifolds (M_n, g_n) of dimension m with sectional curvature $|K| \leq 1$ and diam $\leq D$.

For each n we can find a finite collection of points $p_{i,n} \in M_n$, $i \in \mathbb{N}$, where $i \leq N(m, D)$, such that the images of $\pi/2$-balls under the exponential mappings $\exp_{p_{i,n}} : B_{i,n} = B_{\pi/2} \subset T_{p_{i,n}} \to M_n$ will cover all M_n. Let (U_n, g_n) be the disjoint union of these balls, equipped with the pullback metric. Note that even if we assume that M_n is a manifold, the covering $B_{i,n}$ describes M_n as a megafold.

Now let us consider the Grothendieck-Lipschitz limit of the M_n.

Passing to a subsequence if necessary we can assume that the coverings U_n Lipschitz converge to some U. Moreover the same arguments as in [Fu3] show that (after maybe again passing to a subsequence) the pseudogroup actions $G_n = G(U_n, M_n)$ converge to a pseudogroup action G on U.

Now U and G define a Riemannian megafold $\mathfrak{M}$, which is obviously the Grothendieck-Lipschitz limit of the manifolds M_n. □

Let us state some natural questions which arise from this theorem:

1. Which Riemannian megafolds can be approximated by manifolds with bounded curvature and diameter?

It follows from [CFuGro] that if $(\mathfrak{M}, g)$ is a limit of Riemannian manifolds with bounded curvature, then $G_o(\mathfrak{M})$ is nilpotent. A direct construction moreover shows that this condition is also sufficient.

(Note that since a pure N-structure on a simply connected manifold is given by a torus action (see section 1), one also has the following: If a megafold can be approximated by simply connected manifolds with bounded curvature, then $G_o(\mathfrak{M}) = \mathbb{R}^k$.)

2. How can one recover the Gromov-Hausdorff limit space from the Grothendieck-Lipschitz limit?

Let $\mathfrak{M} = ((U, g) : G)$ be a GL-limit of Riemannian manifolds. Then $\mathfrak{M}$ is H-complete, and the GH-limit is the space of G orbits with the induced metric, in other words: The Gromov-Hausdorff is nothing but $(U, g)/G$.

Now note that for Riemannian megafolds one can define the de Rham complex just as well as for manifolds. A megafold version of the Gluing theorem (see section 1) can then be stated as follows:

Theorem A.6. *Let (M_n, g_n) be a sequence of compact simply connected Riemannian m-manifolds with bounded curvatures and diameters which Grothendieck-Lipschitz converges to a Riemannian megafold $(\mathfrak{M}, g)$.*

If $H^2_{dR}(M_n) = 0$ then $\mathfrak{M}$ is either a Riemannian manifold and the manifolds M_n converge to $\mathfrak{M}$ in the Lipschitz sense, or $H^1_{dR}(\mathfrak{M}) \neq 0$.

Theorem A.6 actually also holds if instead of the H^2 condition one only assumes that the sequence (M_n, g_n) is stable.

The proof of this result follows from the one of the Gluing theorem in [PetrRTu] and the above. However, anyone who knows the proof of the Gluing theorem will see that, using the megafold notion now, many technical problems in [PetrRTu] completely disappear (and only nice parts of it remain). In fact, the proof of Theorem A.6 (given the H^2-condition) almost coincides with the proof of Theorem A.7 (see the sketch below).

Note that Riemannian megafolds share almost all the properties of Riemannian manifolds. In fact (one might think of this as a meta-theorem) we are not aware of a single theorem in Riemannian geometry which would not admit a straightforward generalization to the megafold case – and the above Compactness theorem makes megafolds even nicer.

It is in particular straightforward to show that if $\mathrm{Ric} > 0$, then $H^1_{dR}(\mathfrak{M}) = 0$. Moreover, a Grothendieck-Lipschitz limit of manifolds with uniformly bounded sectional curvatures and $\mathrm{Ric} \geq \delta > 0$ is a Riemannian megafold with $\mathrm{Ric} \geq \delta > 0$.

Now, to obtain for example the π_2-theorem for Ricci pinching conditions (Theorem 0.4) we only have to apply the Bochner formula for 1-forms to the GL-limit of a sequence of positively Ricci-pinched manifolds (and this, moreover, allows one to get rid of the whole section 3 of this paper). (For more details see also the proof of Theorem A.8).

Riemannian megafolds are actually not that general objects as they might seem at first sight. Indeed, given a Riemannian megafold $(\mathfrak{M}, g)$ we can consider its orthonormal frame bundle $(F\mathfrak{M}, \tilde{g})$, equipped with the induced metric. Now consider some representation of it, say, $(F\mathfrak{M}, \tilde{g}) = ((U, \tilde{g}) : G)$. Then the G pseudogroup action is free on U, so that its closure $\bar{G}$ also acts freely. Therefore the corresponding factor, equipped with the induced metric, is a Riemannian manifold $Y = (U/\bar{G}, \bar{g})$, and there is a Riemannian submersion $(F\mathfrak{M}, \tilde{g}) \to Y$ whose fibre is G_o/Γ_o, where Γ_o is a dense subgroup of G_o. (Roughly speaking, Γ_o is generated by the intersections of G_o and G.) If we assume that $\mathfrak{M}$ is simply connected, then $G_o = \mathbb{R}^k$ and Γ_o is the homotopy sequence image of $\pi_2(Y)$. (In our case simply-connectedness is equivalent to the fact that there is no other complete megafold $(W : G')$ such that $\mathfrak{M} = (W : G)$ and $G' \subset G$, but this notion can also be defined for general topoi, see [J].)

From this last characterization of Riemannian megafolds it is not hard to obtain the following:

Theorem A.7. *Assume that (M_n, g_n) is a sequence of simply connected compact Riemannian m-manifolds with uniformly bounded curvatures and diameters which GL-converges to a Riemannian megafold $\mathfrak{M}$.*

Then $\mathfrak{M}$ is either a Riemannian manifold and the M_n converge to $\mathfrak{M}$ in the Lipschitz sense, or $H^2_{dR}\mathfrak{M} \neq 0$.

Let us briefly recall some facts about Bochner formulas:

There is a linear operator $\mathfrak{R}_n : \Lambda^n(T(M)) \to \Lambda^n(T(M))$ on a Riemannian manifold M, such that

$$D^2\phi = \nabla^*\nabla\phi + \mathfrak{R}_n(\phi)\,,$$

where ϕ is any n-form on M (see [LaMic, p. 155]). In particular, if on a Riemannian manifold or megafold it holds that $\mathfrak{R}_n > 0$, then every harmonic n-form on it must vanish, so that $H^n_{dR} = 0$.

It is well known that $\mathfrak{R}_1$ is nothing but the Ricci curvature. The operator $\mathfrak{R}_2$ is also still understandable: For dim ≤ 2 it is better to think that it is undefined, for dimension $= 3$ the condition $\mathfrak{R}_2 \geq 0$ is equivalent to Ric ≥ 0, in dimension $= 4$ the property $\mathfrak{R}_2 \geq 0$ is the same as $K^{isotr}_{\mathbb{C}} \geq 0$ (for $K^{isotr}_{\mathbb{C}}$ and its algebra see [MiMoo] and [H]).

Furthermore, at least in even dimensions, the condition $\mathfrak{R}_2 \geq 0$ follows from the condition $K_{\mathbb{C}}^{isotr} \geq 0$. The relations between $\mathfrak{R}_2$ and $K_{\mathbb{C}}^{isotr}$ are in many respects the same as the ones between Ricci and sectional curvature.

For dimension ≥ 4 the conditions $\mathfrak{R}_2 \geq 0$ and Ric ≥ 0 are independent (i.e., neither one is a consequence of the other). Therefore, and also in view of Theorem 0.4, the following generalization of the Burago-Toponogov theorem ([BuT]) is interesting in several respects.

Theorem A.8. *There is a positive constant $i_0(m, \delta, D) > 0$ such that the injectivity radius of any simply connected compact m-dimensional Riemannian manifold M with $K(M) \leq 1$, $\mathfrak{R}_2(M) \geq \delta > 0$ and diameter $\leq D$ is bounded from below by $i(M) \geq i_0(m, \delta, D)$.*

Proof of the theorem. Assume that this is wrong. Then there is a sequence of compact simply connected Riemannian m-manifolds M_n such that $K(M_n) \leq 1$, $\mathfrak{R}_2(M_n) \geq \delta > 0$, and $\mathrm{diam}(M_n) \leq D$, but whose injectivity radii satisfy $i(M_n) \to 0$ as $n \to \infty$ (i.e., this sequence is collapsing). Passing to a subsequence if necessary we can assume that this sequence converges in the Grothendieck-Lipschitz sense, $M_n \stackrel{GL}{\to} \mathfrak{M}$. Now $\mathfrak{R}_2(\mathfrak{M}) \geq \delta$, therefore $H^2_{dR}(\mathfrak{M}) = 0$. Thus Theorem A.7 implies that $\mathfrak{M}$ is a manifold and that the M_n converge to $\mathfrak{M}$ in the Lipschitz sense. Therefore this sequence does not collapse, which is a contradiction. □

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