QUASIGEODESICS AND GRADIENT CURVES IN ALEXANDROV SPACES

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Introduction

1. A comparison theorem for complete Riemannian manifolds with sectional curvatures $\geq k$ says that distance functions in such manifolds are more concave than in the model space S_k of constant curvature k. In other words, the restriction of any distance function dist_p to any geodesic γ (always parametrised by the arclength) satisfies a certain concavity condition $(*)_k$. For example, the condition $(*)_o$ reads

$$(*) \qquad \mathrm{dist}_p^2 \circ \gamma(t) - t^2 \qquad \text{is concave in } \ t$$

In fact, the condition $(*)_k$ for a Riemannian manifold is equivalent to the corresponding lower curvature bound. On the other hand, it characterises geodesics among all curves, parametrised by the arclength.

Geodesics in (complete, finite dimensional) Alexandrov spaces with curvature $\geq k$ also satisfy $(*)_k$. However, in this case $(*)_k$ usually holds for some other curves as well. The class of all arclength parametrised curves satisfying $(*)_k$, or quasigeodesics, is our main object in this paper. We prove that, unlike geodesics in Alexandrov spaces, quasigeodesics can be constructed with arbitrary initial data and are extendable (§§4,5), and have a natural compactness property (§2). We also give a local description of quasigeodesics (1.7).

Several applications of quasigeodesics are discussed in [Pet]. Here we present a typical one.

Proposition. *Let Σ^n be a compact Alexandrov space of curvature ≥ 1, with radius $> \pi/2$. Then for any $p \in \Sigma$ the space of directions Σ_p has radius $> \pi/2$.*

Remark. This proposition implies, by an inductive argument, that Σ^n is homeomorphic to the sphere S^n. Another proof of this "radius sphere theorem" is due to Grove and Petersen [GP]. Yet another proof follows immediately from [PP, 1.2, 1.4.1].

Proof of the proposition. Assume that Σ_p has radius $\leq \pi/2$, and let $\xi \in \Sigma_p$ be a direction, such that $\mathrm{clos}\, B_\xi(\pi/2) = \Sigma_p$. Suppose there is a geodesic of length $\pi/2$ starting at p in direction ξ. Then the other endpoint q of this geodesic satisfies $\mathrm{clos}\, B_q(\pi/2) = \Sigma$. (Indeed, for any point $r \in \Sigma$ we have $\angle rpq \leq \pi/2$, therefore $|rq| \leq \pi/2$ by the comparison inequality). This contradicts our assumption that Σ has radius $> \pi/2$. Now in general there may be no such geodesic, but we can always construct a quasigeodesic with the prescribed data, and the argument goes through. □

2. Another important feature of manifolds with sectional curvature $\geq k$ is the contracting property of the exponential map in a neighborhood of the origin in the tangent space. Here the tangent space is the space of constant curvature k. In an Alexandrov space, for the purpose of defining a contracting exponential map, the tangent space at a point p can be defined as the k-cone on the space of directions Σ_p (that is the ordinary cone if $k = 0$ and the spherical suspension if $k = 1$). However, the exponential map may not be defined in any open neighborhood of the origin. In §3 we define a gradient-exponential map, which is defined and contracting on the whole tangent space if $k \leq 0$ and on the half of it if $k > 0$. It maps the rays of the tangent space to the corresponding gradient curves for the distance function from the base point; the gradient curves have a special parametrisation, which ceases to coincide with the arclength as soon as the curve ceases to be a shortest line. In addition, it turns out that the restrictions of arbitrary distance functions to the gradient curves with this special parametrisation satisfy a certain monotonicity condition, which is often as useful as the concavity condition $(*)_k$. This monotonicity condition also allows us to use gradient curves as the first step in the construction of quasigeodesics.

Gradient curves and gradient-exponential maps can be constructed in infinite-dimensional Alexandrov spaces as well (see the Appendix). This construction complements the work of Plaut to settle the problem of equivalence of Hausdorff and topological dimensions for Alexandrov spaces.

3. Several natural questions about quasigeodesics remain open. Probably the most important are

(1) Is it true that quasigeodesics with almost all initial data are unique? (There are simple examples of geodesics without common arcs having the same initial data.) Is there an analog of the Liouville theorem for “quasigeodesic flow”?

(2) Is it true that any quasigeodesic can be approximated by quasigeodesics made up of shortest lines? or by broken geodesics with small negative turn?

4. **Historic remarks**. Quasigeodesics on convex surfaces in $\mathbb{R}^3$ were introduced by Alexandrov [A1]. He defined them by a local condition (nonnegative left and right turns), which seems to be hard to generalize for ambient spaces of higher dimensions. A number of results were proved by Alexandrov [A2] and Pogorelov [Pog]. Later Alexandrov extended his definition to the curves in two-dimensional manifolds of bounded (integral) curvature; quasigeodesics in such spaces were discussed in [AB]. Many arguments were based on the fundamental fact that the spaces under consideration can be approximated by polyhedral metrics with some geometric control; for multidimensional Alexandrov spaces such results are not known. Quasigeodesics in multidimensional polyhedra were considered by Milka [M]. Our approach to gradient curves was influenced by the work of Sharafutdinov [Sh], who considered gradient curves for concave non-smooth functions on Riemannian manifolds of nonnegative sectional curvature.

5. **Prerequisites**. We will freely use the basic results about Alexandrov spaces, which can be found in [BGP]. The discussion in §6 relies on certain arguments from [PP].

Notation

We denote by M complete n-dimensional Alexandrov spaces of curvature $\geq k$.

S_k is the simply connected complete surface of constant curvature k; we fix an origin $o \in S_k$; the constant $\pi/\sqrt{k}$ equals ∞ for $k \leq 0$.

Σ denotes complete finite dimensional Alexandrov spaces of curvature ≥ 1.

For a point p in an Alexandrov space, Σ_p and C_p denote the space of directions and the tangent cone at p respectively; $o \in C_p$ is the origin; Σ_p is embedded in C_p as the unit sphere. $\log_p : M \to C_p$ is a multivalued map; for each $q \in M$ the set $\log_p(q)$ consists of all elements $v \in C_p$ such that $|ov| = |pq|$ and the direction of v is the direction of some shortest line $\overline{pq}$.

The directional derivative of a function f is denoted by f'; it is defined on a space of direction Σ_p and can be naturally extended to C_p; this extension is called a differential and denoted by df; thus $df(\lambda v) = \lambda df(v)$ for any $v \in C_p$, $\lambda \geq 0$.

The comparison angle $\tilde{\angle} pqr$ is the angle at $\tilde{q}$ in the triangle $\tilde{p}\tilde{q}\tilde{r}$ on S_k whose side have lengths $|\tilde{p}\tilde{q}| = |pq|$, $|\tilde{p}\tilde{r}| = |pr|$, $|\tilde{q}\tilde{r}| = |qr|$. For a Lipshitz curve γ, with Lipschitz constant 1, we denote by $\tilde{\angle} p\gamma(t_1)^{\smile}\gamma(t_2)$ the angle in the triangle on S_k whose sides have lengths $|p\gamma(t_1)|$, $|p\gamma(t_2)|$ and $|t_2 - t_1|$; similarly, if γ_1, γ_2 are two such curves, $\gamma_1(0) = \gamma_2(0) = p$, then $\tilde{\angle}\gamma_1(t_1)^{\smile}p^{\smile}\gamma_2(t_2)$ denotes the angle in the triangle on S_k whose sides have lengths $|t_1|, |t_2|$, $|\gamma_1(t_1)\gamma_2(t_2)|$; if such a triangle does not exist, then the comparison angle is equal to 0.

$B_p(R) \subset M$ denotes the open metric ball of radius R centered at p, $S_p(R)$ the corresponding metric sphere.

For a function ϕ on $\mathbb{R}$, ϕ^+ and ϕ^- denote its right and left derivatives respectively.

§1. Preliminaries and Definitions

1.1 Lipshitz curves. (cf. [B]) In this paper a curve will always mean a parametrised Lipschitz curve, very often with Lipschitz constant 1. A Lipschitz curve is always rectifiable; moreover, the length of its arc, say $\gamma|_{[t_1,t_2]}$, can be computed as $\int_{t_1}^{t_2} |\dot{\gamma}|(t)dt$, where $|\dot{\gamma}|(t) := \lim_{\tau\to 0} |\gamma(t)\gamma(t+\tau)|/|\tau|$ exists for almost all t. In particular, if γ is parametrised by the arc length, then $|\dot{\gamma}|(t) = 1$ a.e.

1.2 Development. (cf. [A3]) Fix a real k. Let $\gamma : [a, b] \to X$ be a 1-Lipschitz curve in a metric space X, $p \in X$, $0 < |p\gamma(t)| < \pi/\sqrt{k}$ for all $t \in [a, b]$. Then there exists a unique (up to rotation) curve $\tilde{\gamma} : [a, b] \to S_k$, parametrised by the arclength, and such that $|o\tilde{\gamma}(t)| = |p\gamma(t)|$ for all t and the segment $o\gamma(t)$ turns clockwise as t increases. (This is easy to prove.) Such a curve $\tilde{\gamma}$ is called the development of γ with respect to p on the k-plane S_k. The development $\tilde{\gamma}$ is called convex if for every $t \in (a, b)$ and for sufficiently small $\tau > 0$ the curvilinear triangle, bounded by the segments $o\tilde{\gamma}(t \pm \tau)$ and the arc $\tilde{\gamma}|_{t-\tau,t+\tau}$, is convex.

1.3 Differential inequalities. Let ϕ be a continuous function on (a,b), $t \in (a,b)$. We write $\phi''(t) \le B$ if $\phi(t+\tau) \le \phi(t) + A\tau + B\tau^2/2 + o(\tau^2)$ for some $A \in \mathbb{R}$; $\phi''(t) < \infty$ means that $\phi''(t) \le B$ for some $B \in \mathbb{R}$. If f is another continuous function on (a,b), then $\phi'' \le f$ means $\phi''(t) \le f(t)$ for all t. We will use several equivalent conditions as well:

(1) $\phi - F$ is concave, where F is the solution of $F'' = f$.
(2) $\phi''(t) \le f(t)$ for all $t \ne t_0$, and $\phi^+(t_0) \le \phi^-(t_0)$.
(3) $\phi''(t) < \infty$ for all t and $\phi''(t) \le f(t) + \delta$ for almost all t and all $\delta > 0$.
(4) There exist sequences $\{\phi_i\}, \{f_i\}$ uniformly converging to ϕ and f respectively, and such that $\phi_i'' \le f_i$ for each i.

The equivalence is easy to check, except maybe the sufficiency of (3), which requires a lemma from [T,11.82].

1.4 Comparison theorems. Let M be a complete Riemannian manifold with sectional curvatures $\ge k$, $\gamma : [a,b] \to M$ be a geodesic, parametrised by the arclength, $p \in M$, $0 < |p\gamma(t)| < \pi/\sqrt{k}$ for all t. Then, according to a comparison theorem for the shape operator, the function $\mathrm{dist}_p \circ \gamma$ satisfies a certain concavity condition. One way to express this condition is to say that

$(L1)$ The development of γ w.r.t. p on the k-plane is convex.

Another way is to write it as a differential inequality

$$f'' \le 1 - kf \ , \text{ where } \quad f = \rho_k \circ \mathrm{dist}_\rho \circ \gamma \ , \quad \text{and}$$

$(L2)$
$$\rho_k(x) = \begin{cases} 1/k(1-\cos(x\sqrt{k})) \ , & \text{if } k > 0 \\ x^2/2 \ , & \text{if } k = 0 \\ 1/k(1-\cosh(x\sqrt{-k})) \ , & \text{if } k < 0 \end{cases}$$

((L1) and (L2) are equivalent because the inequality (L2) becomes an identity for a geodesic on the k-plane.)

It is well known that the local (in t) conditions (L1), (L2) imply global comparison statements (versions of Toponogov comparison theorem).

(G1). *Let $q_1 = \gamma(t_1)$, $q_2 = \gamma(t_2)$, $q_3 = \gamma(t_3)$, $t_1 < t_2 < t_3$, $t_3 - t_1 \le |pq_1| + |pq_3| < 2\pi/\sqrt{k} - (t_3 - t_1)$. Let $\tilde{p}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3 \in S_k$ form a comparison triangle, so that $|\tilde{p}\tilde{q}_1| = |pq_1|$, $|\tilde{p}\tilde{q}_3| = |pq_3|$, $|\tilde{q}_1\tilde{q}_2| = t_2 - t_1$, $|\tilde{q}_2\tilde{q}_3| = t_3 - t_2$, $|\tilde{q}_1\tilde{q}_3| = t_3 - t_1$. Then $|pq_2| \ge |\tilde{p}\tilde{q}_2|$.*

(G2). *Let $q_1 = \gamma(t_1)$, $q_2 = \gamma(t_2)$, $t_2 > t_1$, $t_2 - t_1 < \pi/\sqrt{k}$, and let $\tilde{p}, \tilde{q}_1, \tilde{q}_2 \in S_k$ form a comparison triangle, so that $|\tilde{p}\tilde{q}_1| = |pq_1|$, $\angle\tilde{p}\tilde{q}_1\tilde{q}_2 = \angle pq_1q_2$. Then $|pq_2| \le |\tilde{p}\tilde{q}_2|$.*

(G3). *For any t, the comparison angle $\tilde{\angle} p\gamma(t)\smile\gamma(t+\tau)$ is non-increasing in τ for $0 \le \tau < \pi/\sqrt{k}$.*

(A geometric proof of (G1)–(G3) from (L1) is based on Alexandrov's lemma, see [BGP,2.5].)

1.5 Convex curves and quasigeodesics.

Definition. A 1-Lipschitz curve γ in a metric space X is called k-convex if it satisfies (L1) for each $p \in X$ such that $0 < |p\gamma(t)| < \pi/\sqrt{k}$ for all t. A k-convex curve is a k-quasigeodesic (QG) if it is parametrised by the arclength.

Clearly, a convex curve satisfies (L2) and (G1),(G3). Conversely, each of these conditions can be taken as a definition of convex curves. The condition (G2) can also be adapted to hold for convex curves; namely, one should replace the angle $\angle pq_1q_2$ by $\arccos(-(\mathrm{dist}_p \circ \gamma)^+(t_1))$.

1.6 Alternative definition of Alexandrov spaces. Although we introduced quasigeodesics in arbitrary metric spaces, all our further results are about QG in Alexandrov spaces. The only exception is the following alternative description of Alexandrov spaces themselves.

Proposition. *A length space X is an Alexandrov space of curvature $\geq k$ if every two points in X can be connected by a shortest line, which is a k-quasigeodesic.*

Proof. Repeat the arguments from [BGP, 2.7–8] substituting “shortest line which is a k-quasigeodesic” for “shortest line”. □

1.7 A local, invariant description of quasigeodesics.

Proposition. *Let $\gamma : [a, b] \to M$ be a curve, parametrised by the arclength, in an Alexandrov space of curvature $\geq k$. Then γ is a $k-QG$ iff for every $t \in (a, b)$*

$$\tfrac{1}{2}(\mathrm{dist}_q^2 \circ \gamma)''(t) \leq 1 + o(|q\gamma(t)|).$$

Proof. The “only if” implication follows easily from the definitions. The “if” implication will be proved here in the case $k = 1$; in the other cases the proof is similar.

We have to prove that $(-\cos \circ \mathrm{dist}_p \circ \gamma)'' \leq \cos \circ \mathrm{dist}_p \circ \gamma$. In fact we will check the conditions 1.3(3).

Fix $t \in (a, b)$ and $p \in M$ such that $0 < |p\gamma(t)| < \pi$. Pick a point q on a shortest line $\overline{p\gamma}(t)$ close to $\gamma(t)$. The comparison inequalities imply that

$$(\cos|p\gamma(t')|-\cos|q\gamma(t')|\cos|pq|)\sin|q\gamma(t)|+(\cos|\gamma(t)\gamma(t')|-\cos|q\gamma(t')|\cos|q\gamma(t)|)\sin|pq| \geq 0$$

or

$$-\cos|p\gamma(t')| \leq (-\cos|q\gamma(t')|\sin|p\gamma(t)| + \cos|\gamma(t)\gamma(t')|\sin|pq|)/\sin|q\gamma(t)| \ .$$

Our assumption implies that

$$(-\cos \circ \mathrm{dist}_q \circ \gamma)''(t) \leq 1 + o(|q\gamma(t)|) \ .$$

On the other hand, obviously

$$(\cos \circ \mathrm{dist}_{\gamma(t)} \circ \gamma)''(t) \leq 0 \ .$$

Therefore,

$$(-\cos \circ \mathrm{dist}_p \circ \gamma)''(t) < \infty \ .$$

Moreover, since γ is parametrised by the arclength, for almost all t we have

$$|\gamma(t)\gamma(t')| = |t - t'| + o(|t - t'|)$$

hence

$$(\cos \circ \mathrm{dist}_{\gamma(t)} \circ \gamma)''(t) \le -1 \ .$$

It follows that for all such t,

$$(-\cos \circ \mathrm{dist}_p \circ \gamma)''(t) \le ((1 + o(|q\gamma(t)|)) \sin |p\gamma(t)| - \sin |pq|)/ \sin |q\gamma(t)| \le \cos |p\gamma(t)| + \delta$$

if q was chosen sufficiently close to $\gamma(t)$. $\square$

Corollary. *If $k \ge k'$ then the classes of $k-QG$ and $k'-QG$ in an Alexandrov space of curvature $\ge k$ coincide.*

§2. Tangent Vectors and Compactness

It is clear that a uniform limit of k-convex curves is a k-convex curve, cf. 1.3(4). Our main goal in this section is to show that the length of the limit curve is equal to the limit of lengths of converging curves; in particular, a limit of QG is a QG.

2.1 Tangent cones and tangent vectors. An Alexandrov space has a tangent cone at each point; at most points it is isometric to euclidean space, but there may be singular points as well. Still it is convenient to treat elements of any tangent cone C as vectors. We define the norm of a vector $v \in C$ by $|v| = |ov|$, and the scalar product of two vectors $u, v \in C$ by $\langle u, v\rangle = (|u|^2 + |v|^2 - |uv|^2)/2$. Clearly, $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for $\lambda \ge 0$.

Two vectors $u, v \in C$ are called polar, if $\langle u, w\rangle + \langle v, w\rangle \ge 0$ for all $w \in C$, and opposite, if $\langle u, w\rangle + \langle v, w\rangle = 0$ for all $w \in C$. Clearly, u and v are opposite iff $2|u| = 2|v| = |uv|$. The existence of a pair of nonzero opposite vectors in C gives rise to an isometric splitting $C = \mathbb{R} \times C'$; in contrast, any vector in any cone C has a polar one, see 3.1.1.

Let $\gamma : [a, b] \to M$ be a Lipschitz curve, $t \in [a, b]$. A vector $v \in C_{\gamma(t)}$ is called a left tangent vector of γ at t if $v = \lim_{j\to\infty} \tau_j^{-1} \log_{\gamma(t)} \gamma(t - \tau_j)$ for some sequence $\tau_j \to 0^+$. A right tangent vector is defined in a similar way. There always exist right and left tangent vectors at each point (except the endpoints, where only right or only left tangent vectors exist), but they are not necessarily unique. The right and left tangent vectors of γ at t will be denoted by $\gamma^+(t)$ and $\gamma^-(t)$ respectively.

Proposition. (a) *The left and right tangent vectors are unique and opposite for almost all t.*

(b) *If γ is k-convex, then the tangent vectors are unique and polar for all t.*

(c) *If $\gamma_1 : [a, b] \to M$, $\gamma_2 : [b, c] \to M$ are k-convex, $\gamma_1(b) = \gamma_2(b)$, $\gamma_1^-(b)$ is polar to $\gamma_2^+(b)$, then the curve $\gamma : [a, c] \to M$ obtained by gluing γ_1 and γ_2 together, is also k-convex.*

Proof. Let f be a distance function, such that $(f\circ\gamma)^+(t)$ exists. Then for any right tangent vector $\gamma^+(t)$ we have $(f\circ\gamma)^+(t) = df(\gamma^+(t))$. Similarly, if $(f\circ\gamma)^-(t)$ exists, then for any $\gamma^-(t)$ we have $(f\circ\gamma)^-(t) = -df(\gamma^-(t))$. Choose a family of distance functions $f_i = \mathrm{dist}_{p_i}$, such that $\{p_i\}$ is a countable dense subset of M. Now the statement (a) follows from the fact that each Lipschitz function $f_i\circ\gamma$ has derivative a.e., while (b) holds because each function $f_i\circ\gamma$ has right and left derivatives satisfying $(f_i\circ\gamma)^+(t)\le(f_i\circ\gamma)^-(t)$ for all t, if γ is k-convex. The assertion (c) follows from 1.3(2). □

Convention. We will sometimes use the right and left tangent vectors in the situations where they are not necessarily unique. In these cases an appearance of, say, γ^+ in an assumption means "there exists a right tangent vector γ^+ such that …", whereas its appearance in a conclusion means "for all right tangent vectors γ^+ …". We will also use signs "±" and "∓" to write one formula instead of two similar ones.

2.2 Theorem. *Let $\gamma_i : [a,b]\to M$ be a sequence of k-convex curves converging uniformly to a curve γ. Then* $\mathrm{length}(\gamma) = \lim\ \mathrm{length}(\gamma_i)$*. Moreover, the conclusion holds if γ_i are curves in different Alexandrov spaces M_i, with the same lower curvature bound and the same dimension, converging to M in Gromov-Hausdorff sense.*

Proof. The statement is an immediate consequence of the formula for the length of a Lipschitz curve (see 1.1), Proposition 2.1(1), and the following

2.2.1 Key Lemma. *If t is a point where $\gamma^+(t)$ and $\gamma^-(t)$ are unique and opposite, then $|\gamma^\pm(t)| = \lim|\gamma_i^\pm(t)|$.*

Proof of the lemma. For simplicity consider only the case when $k=0$ and M has curvature ≥ 0. Let $p=\gamma(t)$, $p_i=\gamma_i(t)$. Passing to a subsequence, we may assume that C_{p_i} converge in the Gromov-Hausdorff sense to a cone C of the same dimension, and $\gamma_i^\pm(t)$ tend to $\omega^\pm\in C$. (In general, C is different from C_p.) We can also construct a non-contracting map $\log : M\to C$, such that for every $x\in M$ there exists a sequence $x_i\in M_i$ converging to x and such that $\log_{p_i}(x_i)\subset C_{p_i}$ converge to $\log(x)$. (First define $\log$ on a countable dense subset of M using diagonal argument, then extend it to the whole M.) Obviously, $|\log(x)| = |px|$ for all $x\in M$.

Fix a point $q\in M$ and a sequence $q_i\in M_i$ as in the definition of log; let $f_i=\frac12\mathrm{dist}^2_{q_i}\circ\gamma_i$, $f=\frac12\mathrm{dist}^2_q\circ\gamma$. Since γ_i are k-convex, $k=0$, we have

$$f_i(t+\tau)\le f_i(t)\pm f_i^\pm(t)\tau+\tau^2\ .$$

Using the first variation formula and passing to the limit we get

$$f(t+\tau)\le f(t)\mp\langle\omega^\pm,\log(q)\rangle\tau+\tau^2\ . \tag{1}$$

Assume that there is only one shortest line between p and q. Then f is differentiable at t because $\gamma^+(t)$ and $\gamma^-(t)$ are opposite. In this case (1) implies

$$\begin{aligned}&\mp f'(t)=\langle\omega^\pm,\log(q)\rangle\ ,\quad\text{and, in particular,}\\&\langle\omega^+,\log(q)\rangle+\langle\omega^-,\log(q)\rangle=0\ .\end{aligned}\tag{2}$$

Let $\nu^{\pm} = \lim \tau_j^{-1} \log \gamma(t \pm \tau_j)$ for some sequence $\tau_j \to 0^+$. Then $\mp f'(t) = -\lim \tau_j^{-1}(f(t \pm \tau_j) - f(t)) = -\lim \frac{1}{2}\tau_j^{-1}(|q\gamma(t \pm \tau_j)|^2 - |qp|^2) \geq -\lim \frac{1}{2}\tau_j^{-1}(|\log(q)\log(\gamma(t \pm \tau_j))|^2 - |\log q|^2) = \lim \tau_j^{-1} < \log q, \log(\gamma(t \pm \tau_j))\rangle = \langle \log, \nu^{\pm}\rangle$, whence

$$\langle \omega^{\pm}, \log(q)\rangle \geq \langle \nu^{\pm}, \log(q)\rangle \ . \tag{3}$$

On the other hand, clearly $|\nu^{\pm}| = |\gamma^{\pm}|$, and $|\nu^+\nu^-| = \lim \tau_j^{-1}|\log(\gamma(t+\tau_j))\log(\gamma(t-\tau_j))| \geq \lim \tau_j^{-1}|\gamma(t+\tau_j)\gamma(t-\tau_j)| = |\gamma^+\gamma^-| = 2|\gamma^{\pm}|$, whence ν^+ and ν^- are opposite. It follows that

$$\langle \nu^+, \log(q)\rangle + \langle \nu^-, \log(q)\rangle = 0 \ . \tag{4}$$

Comparing (2),(3),(4) we conclude that

$$\langle \nu^{\pm}, \log(q)\rangle = \langle \omega^{\pm}, \log(q)\rangle \ . \tag{5}$$

This conclusion holds for all q which can be connected to p by only one shortest line. The set of such q has full measure in M (see [OS,3.1]), hence its log-image has positive measure in C. It follows easily that $\omega^{\pm} = \nu^{\pm}$. Thus,

$$|\gamma^{\pm}| = |\nu^{\pm}| = |\omega^{\pm}| = \lim |\gamma_i^{\pm}| \ .$$

$\square$

2.3 Corollaries. (1) *If γ is k-convex then $|\gamma^+(t)| = \lim_{\tau\to 0^+} |\gamma^+(t+\tau)|$ and $|\gamma^-(t)| = \lim_{\tau\to 0^+} |\gamma^-(t-\tau)|$ for all t.*

Proof. Let $p = \gamma(t)$, $\tau_i \to 0^+$. Consider a sequence of $\tau_i k$-convex curves γ_i in the spaces $M_i = \tau_i^{-1} \cdot M$, defined by $\gamma_i(s) = \tau_i^{-1} \cdot \gamma(t + s\tau_i)$, $s \geq 0$. Clearly M_i converge to C_p, and γ_i converge to a curve γ_∞ in C_p, defined by $\gamma_\infty(s) = s\gamma^+(t)$, $s \geq 0$. Applying the Key Lemma for $s = 1$ we get $|\gamma^+(t)| = |\gamma_\infty^+(1)| = \lim |\gamma_i^+(1)| = \lim |\gamma^+(t+\tau_i)|$. The proof for left tangent vectors is similar. $\square$

(2) *If γ is a quasigeodesic then $|\gamma^{\pm}(t)| = 1$ for all t. In particular, QG satisfies* (G2) *in its original form.*

(3) *Under the assumptions of Theorem 2.2, a limit of QG is QG.*

(4) *If $\overline{pq}$ is a shortest line, and γ is a QG starting at p in the same direction then γ and $\overline{pq}$ have a common arc. In particular, any QG in a Riemannian manifold is a geodesic.*

Proof. Consider the developments of γ with respect to points q_i close to q, and use (2) for the right tangent vector at p.

§3. Gradient Curves

A Lipschitz function f in a domain $U \subset M$ is called λ-concave if for every shortest line γ in U, $(f \circ \gamma)'' \leq -2\lambda$; f is called semiconcave if for every $p \in U$ there exists a number $\lambda(p)$ such that f is $\lambda(p)$-concave in a neighborhood of p. For example, if M is an Alexandrov space, $p \in M$, then dist_p is semiconcave in $M \backslash \{p\}$, whereas dist_p^2 is semiconcave in M. Clearly a semiconcave function is differentiable at every point and its differentials are concave homogeneous functions on the tangent cones.

Let f be any function differentiable at p. A vector $v \in C_p$ is called the gradient of f at p, and is denoted by $\nabla f(p)$, if $df(v) = |v|^2$ and the function $df(u)/|u|$ attains its positive maximum value at v. We let $\nabla f(p) = 0$ if $df(u) \leq 0$ for all $u \in C_p$; in this case p is called a critical point for f. If df is concave on C_p then it is not hard to show that $\nabla f(p)$ is unique and can be characterised by the property $df(u) \leq \langle \nabla f, u \rangle$ for all $u \in C_p$. (Indeed, assume that there exist $u \in C_p$ violating this condition and use concavity of df to show that $df(w)/|w| > |\nabla f|$ for vectors w near ∇f on the shortest line between u and ∇f.)

3.1.1 Example. Let $M = C_p$, $f(u) = -\langle u, v \rangle$ for some $v \in C_p$. Then $\nabla f(p)$ is polar to v, and $|\nabla f(p)| \leq |v|$.

3.2 Gradient curves. Let f be a semiconcave function without critical points in $U \subset M$. A locally Lipschitz curve $\gamma : (a, b) \to U$ is called a gradient curve for f (or f-gradient curve), if $f \circ \gamma(t) = t$ and $\gamma^+(t) = \nabla f(\gamma(t))/|\nabla f(\gamma(t))|^2$ for all $t \in (a, b)$.

Proposition. *For each $p \in U$ there is a unique complete f-gradient curve starting at p.*

(Complete means having no limit points in U.)

The proof is based on the following

3.2.1 Lemma. *Let f be λ-concave in U. Then*

(a) *For any shortest line $\gamma : [a, b] \to U$ we have*

$$\langle \gamma^+(a), \nabla f(\gamma(a)) \rangle + \langle \gamma^-(b), \nabla f(\gamma(b)) \rangle \geq 2\lambda(b-a)$$

(b) *$|\nabla f|$ is semicontinuous in U, i.e.*

$$\liminf_{p_i \to p} |\nabla f(p_i)| \geq |\nabla f(p)| \qquad \textit{for all} \ \ p \in U$$

Proof of the lemma. (a) Since $f \circ \gamma(t) + \lambda t^2$ is a concave function of t, we have

$$(f \circ \gamma)^+(a) - (f \circ \gamma)^-(b) \geq 2\lambda(b-a) \ .$$

This implies (a) because

$$\begin{aligned} (f \circ \gamma)^+(a) = df(\gamma^+(a)) &\leq \langle \gamma^+(a), \nabla f(\gamma(a)) \rangle \qquad \text{and} \\ -(f \circ \gamma)^-(b) = df(\gamma^-(b)) &\leq \langle \gamma^-(b), \nabla f(\gamma(b)) \rangle \end{aligned}$$

(b) Fix a sequence $p_i \to p$ and choose a sequence $q_i \to p$ such that $(f(q_i) - f(p))/|pq_i| \to |\nabla f(p)|$ and $|pq_i|/|pp_i| \to \infty$. The λ-concavity of f implies that $\liminf(df(v_i)/|v_i|) \geq |\nabla f(p)|$ for $v_i \in \log_{p_i}(q_i)$, hence $\liminf |\nabla f(p_i)| \geq |\nabla f(p)|$. $\square$

3.2.2 Proof of the proposition (sketch; see Appendix for more details). Since the statement is essentially local, we may assume that f is λ-concave in U and that $\inf_{q\in U}|\nabla f(q)|>0$. The gradient curve can be constructed as limit of broken geodesics, made up of short segments with directions close to the gradient directions. The convergence, as well as uniqueness, is ensured by the assertion 3.2.1(a), while 3.2.1(b) guarantees that the limit is indeed a gradient curve, having a unique right tangent vector at each point. $\square$

3.3 Monotonicity estimates for gradient curves. ¿From here forth we consider gradient curves for distance functions. For technical reasons we restrict ourselves to the case of nonnegative curvature, cf. 3.6

Let γ be a complete gradient curve for $f=\mathrm{dist}_p$, defined for $t\in(0,a)$. Consider a reparametrised curve $\gamma\circ\rho^{-1}$, where $\rho(t)$ is determined by the conditions

$$d\rho/\rho = dt/t\cdot|\nabla f(\gamma(t))|^{-2}\ , \qquad \rho/t\to 1 \quad \text{as} \quad t\to 0\ . \tag{1}$$

Proposition. (a) *The curve $\gamma\circ\rho^{-1}$ can be correctly defined for $\rho\in(0,\infty)$.*

(b) *For arbitrary $q\in M$ the comparison angle $\tilde\angle qp^{\smile}\gamma\circ\rho^{-1}(\rho)$ is non-increasing in ρ.* (c) *If γ_1 is another* dist_p*-gradient curve, and ρ_1 is defined in a similar way, then the comparison angle $\tilde\angle\gamma\circ\rho^{-1}(\rho)^{\smile}p^{\smile}\gamma_1\circ\rho_1^{-1}(\rho_1)$ is non-increasing as ρ and ρ_1 increase in such a way that $d\rho/\rho=d\rho_1/\rho_1$.*

Proof. We will use differential inequalities which may make sense only almost everywhere in the parameter domain; the conclusions can be easily justified using semicontinuity of the gradient.

(a) Integrating (1) formally we get $\rho(t)=t\exp(I_\gamma(t))$, where

$$I_\gamma(t)=\int_0^t \tau^{-1}(|\nabla f(\gamma(\tau))|^{-2}-1)d\tau\ .$$

At this point we make an assumption (justified in the end of the proof) that $I_\gamma(t)$ converges at $t=0$. In this case it is clear that $\rho\ge t$ because $|\nabla f|\le 1$. Moreover, $|(\gamma\circ\rho^{-1})^+(\rho)|=\frac{t}{\rho}|\nabla f(\gamma(t))|^{-1}\le 1$, hence $\gamma\circ\rho^{-1}$ is 1-Lipshitz. Therefore, if ρ is bounded on a complete f-gradient curve γ, then γ converges to a single critical point, and we simply let this point be the image of the curve $\gamma\circ\rho^{-1}$ for $\rho\ge\rho(a)$.

(b) and (c). The angle monotonicity condition in (c) can be written as

$$dh\le\frac{\rho^2+h^2-\rho_1^2}{2\rho h}\,d\rho+\frac{\rho_1^2+h^2-\rho^2}{2\rho_1 h}\,d\rho_1\ ,$$

where $h=h(\rho,\rho_1)=|\gamma(t)\gamma(t_1)|$.

On the other hand, the actual dh can be estimated as

$$dh\le-\langle\nabla f(\gamma(t)),\xi\rangle|\nabla f(\gamma(t))|^{-2}dt-\langle\nabla f(\gamma_1(t_1)),\xi_1\rangle|\nabla f(\gamma_1(t_1))|^{-2}dt_1\ ,$$

where $\xi \in \sum_{\gamma(t)}$ and $\xi_1 \in \sum_{\gamma_1(t_1)}$ are directions of a shortest line $\overline{\gamma(t)\gamma(t_1)}$. The scalar products can be estimated as

$$-\langle \nabla f(\gamma(t)), \xi \rangle \leq -df(\xi) \leq \cos \tilde{\angle} p\gamma(t)\gamma_1(t_1) = \frac{t^2 + h^2 - t_1^2}{2th} \ ,$$

and similarly

$$-\langle \nabla f(\gamma_1(t_1)), \xi_1 \rangle \leq \frac{t_1^2 + h^2 - t^2}{2t_1 h} \ .$$

Thus our assertion (c) reduces to

$$\begin{aligned} (2) \qquad & \frac{\rho^2 + h^2 - \rho_1^2}{2\rho h}\ d\rho + \frac{\rho_1^2 + h^2 - \rho^2}{2\rho_1 h}\ d\rho_1 \\ & \geq \frac{t^2 + h^2 - t_1^2}{2th} |\nabla f(\gamma(t))|^{-2} dt + \frac{t_1^2 + h^2 - t^2}{2t_1 h} |\nabla f(\gamma_1(t_1))|^{-2} dt_1 \ . \end{aligned}$$

This inequality becomes an identity for ρ and ρ_1 defined by (1) and satisfying $\frac{d\rho}{\rho} = \frac{d\rho_1}{\rho_1}$. (Strictly speaking, this is true only until γ or γ_1 hits a critical point; after that we get a strict inequlaity, as can be easily checked.) To check (b) it suffices to prove (2) when $\rho_1 = t_1$ and $d\rho_1 = dt_1 = 0$. In this case (2) reduces to a correct inequality $\rho^2 \geq t^2$.

It remains to verify our assumption about convergence of $I_\gamma(t)$. It obviously converges if γ is a shortest line on some subinterval $(0, a')$. In the general case, we can approximate our gradient curve γ by gradient curves γ_i which coincide with γ on (a_i, a) and are shortest lines on $(0, a_i)$, $a_i \to 0$. Choose a finite collection of points q_j near p, such that the directions of shortest lines $\overline{pq}_j$ form a $\pi/4$-net in Σ_p. Applying the conclusion of (b) to curves γ_i and points q_j we see that the parameters ρ_i are uniformly bounded in some neighborhood of p. Therefore the integrals $I_{\gamma_i}(t)$ are uniformly bounded for small t, and $I_\gamma(t)$ converges. □

3.3.3 Corollary. *If $\gamma|_{(0,a')}$ and $\gamma_1|_{(0,a_1')}$ are shortest lines, then $\tilde{\angle}\gamma \circ \rho^{-1}(\rho)^\smile p^\smile \gamma_1 \circ \rho_1^{-1}(\rho_1) \leq \tilde{\angle}\gamma(a')p\gamma_1(a_1')$ whenever $\rho \geq a'$, $\rho_1 \geq a_1'$.*

PROOF. Assume that $\rho/a' \geq \rho_1/a_1'$ and let $\overline{\rho} = \rho a_1'/\rho_1$. Then $\tilde{\angle}\gamma \circ \rho^{-1}(\rho)^\smile p^\smile \gamma_1 \circ \rho_1^{-1}(\rho_1) \leq \tilde{\angle}\gamma \circ \rho^{-1}(\overline{\rho})^\smile p\gamma_1(a_1')$ according to (c) and $\tilde{\angle}\gamma \circ \rho^{-1}(\overline{\rho})^\smile p\gamma_1(a_1') \leq \tilde{\angle}\gamma(a')p\gamma_1(a_1')$ according to (b), whence the result. □

3.4 Unique gradient curves in all directions. Let $p \in M$, $\xi \in \Sigma_p$. We are going to construct a unique complete dist_p-gradient curve $\gamma : [0, a) \to M$ such that $\gamma^+(0) = \xi$. Let $\xi_i \to \xi$ be directions of shortest lines $\overline{pq}_i$, $q_i \to p$. Extend each of $\overline{pq}_i$ to a complete dist_p-gradient curve γ_i. Since the gradient of dist_p is bounded away from zero near p, the curves γ_i are uniformly Lipshitz there, and we can consider a limit curve γ near p. The semicontinuity of the gradient implies easily that γ is also a dist_p-gradient curve; of course γ can be extended to a complete one. To check $\gamma^+(0) = \xi$ apply 3.3(b) to curves γ_i and points q_j, with $i >> j$. Uniqueness follows from 3.3(c).

3.5 Gradient-exponential map. Now we are in a position to define the gradient-exponential map, $g\exp_p : C_p \to M$. Namely, given $v \in C_p$, construct a complete dist_p-gradient curve γ starting at p in direction $v/|v|$, and let $g\exp_p(v) = \gamma \circ \rho^{-1}(|v|)$. Of course, $g\exp_p \circ \log_p = \mathrm{Id}$. The gradient-exponential map is non-expanding on the whole C_p — this follows from 3.3(c).

3.6 There are versions of 3.3–3.5 for arbitrary lower curvature bound. They tend to be more complicated when it is positive. For example, if it is 1, then the definition (1) of ρ becomes $\frac{d\rho}{\tan\rho} = \frac{dt}{\tan t}|\nabla f(\gamma(t))|^{-2}$, 3.3(b) holds true for $\rho < \pi/2$, 3.3(c) is valid for $\rho < \pi/2$, $\rho_1 < \pi/2$ when ρ and ρ_1 vary in such a way that $\frac{d\rho}{\sin\rho\cos t} = \frac{d\rho_1}{\sin\rho_1\cos t_1}$, Corollary 3.3.3 holds true for $\rho, \rho_1 < \pi/2$, and the gradient-exponential map is defined and is non-expanding on the half of the spherical suspension $S(\Sigma_p)$.

§4. Construction of pre-quasigeodesics

Our goal in this and the next section is to construct infinite QG with arbitrary initial data. This will be done in two steps. At first we construct so-called pre-quasigeodesics. (This class of curves does not seem to have any independent significance; it just comes out in an attempt to construct QG in a natural way.) Then, in the next section, we obtain QG as limits of appropriately chosen pre-quasigeodesics.

We assume that lower curvature bound $k = 0$. The case $k < 0$ is very similar whereas in the case $k > 0$ there are certain additional problems. Therefore in this case we simply consider our space as having nonnegative curvature, and use in the end the invariant description 1.7.

4.1 Monotone curves.

Definition. A 1-Lipschitz curve $\gamma : [a, b) \to M$ is called monotone if $\tilde{\angle} q\gamma(a)^\smile\gamma(t)$ is non-increasing in t for every $q \in M$. According to the discussion in 1.5, any arc of a convex curve is monotone and conversely if any arc of a curve is monotone then this curve is convex.

Technical definition. A Lipschitz curve $\gamma : [0, \infty) \to M$ is called normal if its Lipschitz constant is equal to $|\gamma^+(0)|$.

According to 3.4, 3.3(b), given a point $p \in M$ and a unit vector $v \in C_p$ one can always construct a normal monotone curve γ with $\gamma(0) = p$, $\gamma^+(0) = v$. In fact, the condition $|v| = 1$ can be relaxed to $|v| \le 1$ due to the following observation.

4.1.1 If γ is monotone (convex) then for arbitrary $\lambda \le 1$ the curve $\gamma\gamma(t) = \gamma(\lambda t)$ is also monotone (convex). (Note that the corresponding statement for $k > 0$ is generally false.) The proof is straightforward.

4.2 Compactness.

Proposition. *A uniform limit of monotone curves is monotone. Moreover, if γ_i are converging normal monotone curves, $\gamma_i(0) = p$, then the limit curve γ is also normal, and $\gamma_i^+(0) \to \gamma^+(0)$. In particular, the right tangent vector of a normal monotone curve at the origin is unique.*

PROOF. The first statement is obvious. To prove the second one, consider an arbitrary function f of the type $\frac{1}{2}\mathrm{dist}_q^2$. Since γ_i are monotone, we have

$$f(\gamma_i(t)) \leq f(p) + df(\gamma_i^+(0))t + t^2$$

whence for any limit v of a subsequence of $\gamma_i^+(0)$,

$$f(\gamma(t)) \leq f(p) + df(v)t + t^2 \ .$$

On the other hand,

$$f(\gamma(t_j)) = f(p) + df(\gamma^+(0))t_j + o(t_j)$$

for some sequence $t_j \to 0$. It follows that $df(v) \geq df(\gamma^+(0))$, for any choice of v, $\gamma^+(0)$ and f. Since each of γ_i is normal, the limit curve γ is $|v|$-Lipschitz, and therefore $|v| \geq |\gamma^+(0)|$. Now if $f_j = \frac{1}{2}\mathrm{dist}_{q_j}^2$, where q_j tends to p in such a way that the direction of $\overline{pq}_j$ tends to that of v, then $df_j(v) < df_j(\gamma^+(0))$ for large j unless $\gamma^+(0) = v$. □

4.3 Gluing.

Proposition. *Let γ_1, γ_2 be normal monotone curves, such that $\gamma_1(a) = \gamma_2(0)$, $\gamma_1^+(a) = \gamma_2^+(0)$ for some $a \geq 0$. Then a curve γ, defined by*

$$\gamma(t) = \begin{cases} \gamma_1(t) \ , & \text{if } \ 0 \leq t \leq a \\ \gamma_2(t-a) \ , & \text{if } \ t \geq a \end{cases}$$

is normal and monotone.

PROOF. The normality of γ is obvious, so we only need to check monotonicity of $\tilde{\angle} q\gamma(0)^\smile\gamma(t)$ for $t \geq a$. It is equivalent to the condition $(\mathrm{dist}_q \circ \gamma)^+(t) \leq \cos\tilde{\angle} q\gamma(t)^\smile\gamma(0)$, while from monotonicity of γ_2 we know that $(\mathrm{dist}_q \circ \gamma)^+(t) \leq \cos\tilde{\angle} q\gamma(t)^\smile\gamma(a)$. According to Alexandrov's lemma, the inequality $\tilde{\angle} q\gamma(t)^\smile\gamma(0) \leq \tilde{\angle} q\gamma(t)^\smile\gamma(a)$ is equivalent to $\tilde{\angle} q\gamma(a)^\smile\gamma(t) + \tilde{\angle} q\gamma(a)^\smile\gamma(0) \leq \pi$, or $\cos\tilde{\angle} q\gamma(a)^\smile\gamma(t) + \cos\tilde{\angle} q\gamma(a)^\smile\gamma(0) \geq 0$. This inequality is true because $\cos\tilde{\angle} q\gamma(a)^\smile\gamma(0) \geq (\mathrm{dist}_q \circ \gamma_1)^+(a)$ (since γ_1 is monotone), $\cos\tilde{\angle} q\gamma(a)^\smile\gamma(t) \geq -(\mathrm{dist}_q \circ \gamma_2)^+(0)$ (since γ_2 is monotone), and the right-hand sides of these two inequalities match (since $\gamma_1^+(a) = \gamma_2^+(0)$). □

4.3.1 Corollary. *If γ_1, γ_2 are convex then γ is convex as well.*

4.4 Construction of convex curves.

Proposition. *Given $p \in M$, $v \in C_p$, $|v| \leq 1$, there exists a normal convex curve γ with $\gamma(0) = p$, $\gamma^+(0) = v$.*

PROOF. Fix $\delta > 0$. Construct inductively normal monotone curves α_i, such that $\alpha_0 = p$, $\alpha_0^+(0) = v$, $\alpha_{i+1}(0) = \alpha_i(\delta)$, $\alpha_{i+1}^+(0) = \alpha_i^+(\delta)$. Define a curve γ_δ by $\gamma_\delta(t) = \alpha_i(t - \delta i)$ for $i\delta \leq t \leq (i+1)\delta$. According to 4.3, γ_δ is a normal monotone curve. Any limit γ of a subsequence of γ_δ as $\delta \to 0$ is obviously a normal convex curve with the required initial data. $\square$

4.5 Pre-quasigeodesics.

Definition. A convex curve $\gamma : [a, b) \to M$ is called a pre-quasigeodesic (pre-QG) if for every $s \in (a, b)$ the curve $\gamma^s(t) = \gamma(s + t/|\gamma^+(s)|)$ is also convex. (If $|\gamma^+(s)| = 0$ then we require $\gamma(t) = \gamma(s)$ for all $t \in (s, b)$.)

Note that if γ is a pre-QG then each γ^s is pre-QG as well. A pre-QG is called complete if it is defined on $[0, \infty)$. Note that a complete pre-QG need not have infinite length.

Proposition. *Given $p \in M$, $v \in C_p$, $|v| = 1$, there exists a complete pre-QG γ with $\gamma(0) = p$, $\gamma^+(0) = v$.*

PROOF. Fix $\delta > 0$. Use 4.4 to construct inductively normal convex curves α_i such that $\alpha_0(0) = p$, $\alpha_0^+(0) = v$, $\alpha_{i+1}(0) = \alpha_i(\delta)$, $\alpha_{i+1}^+(0) = \alpha_i^+(\delta)$, and such that the curves $\alpha\alpha_i$ defined by $\alpha\alpha_i(t) = \alpha_i(t/|\alpha_i^+(0)|)$ are also convex. (At each step we construct a normal convex $\alpha\alpha_{i+1}$ with $\alpha\alpha_{i+1}(0) = \alpha_i(\delta)$, $\alpha\alpha_{i+1}^+(0) = \alpha_i^+(\delta)/|\alpha_i^+(\delta)|$, and then obtain α_{i+1} by reparametrisation; if $|\alpha_i^+(\delta)| = 0$ then all further α_j map $[0, +\infty)$ to the point $\alpha_i(\delta)$.) Define curves $\gamma_{j,\delta}$ and $\gamma\gamma_{j,\delta}$ by $\gamma_{j,\delta}(t) = \alpha_{i+j}(t - \delta i)$ for $i\delta \leq t \leq (i+1)\delta$ and $\gamma\gamma_{j,\delta}(t) = \gamma_{j,\delta}(t/|\alpha_j^+(0)|)$.

According to 4.3, each $\gamma_{j,\delta}$ is normal and convex. Morever, each $\gamma\gamma_{j,\delta}$ is normal and convex as well. (Indeed, $\gamma\gamma_{j,\delta}$ can be obtained by gluing the curves $\beta_{i,j}$ defined by $\beta_{i,j}(t) = \alpha\alpha_i(t \cdot |\alpha_i^+(0)|/|\alpha_j^+(0)|)$, $i \geq j$, each of these $\beta_{i,j}$ is normal and convex, because $|\alpha_j^+(0)| \leq |\alpha_j^+(\delta)| = |\alpha_{j+1}^+(0)| \leq \cdots \leq |\alpha_i^+(0)|$, cf. 4.1.1.)

Let γ be any limit of a subsequence of $\gamma\gamma_{0,\delta}$ as $\delta \to 0$. Then γ is a normal convex curve with the required initial data, cf. 4.2. Moreover, it follows from the Key Lemma 2.2.1 that for almost all $s \geq 0$ the curve γ^s is a limit of an appropriate sequence of curves $\gamma\gamma_{j,\delta}$. Therefore γ^s is convex for almost all $s \geq 0$, and an application of the corollary 2.3(1) shows that the same is true for all s. $\square$

4.6 Entropy.

Definition. Let $\gamma : [a, b) \to M$ be a pre-QG. A measure μ on $[a, b)$ defined by $\mu(t, t') = \log|\gamma^+(t)| - \log|\gamma^-(t)|$ is called the entropy of γ. (Note that $|\gamma^+(t)| \geq |\gamma^-(t')|$ if $t' > t$ because γ^t is 1-Lipschitz.) It follows from 2.1(a) and 2.3(1) that $\mu\{t\} = \log|\gamma^-(t)| - \log|\gamma^+(t)|$ for each $t \in (a, b)$. A pre-QG γ is a QG iff $|\gamma^+(0)| = 1$ and the entropy $\mu \equiv 0$. Thus our goal will be to construct a pre-QG with given initial data and zero entropy.

§5. Construction of quasigeodesics

5.1 Reduction to a local problem. Let $\gamma_1 : [a,b) \to M$ and $\gamma_2 : [b,c) \to M$ be pre-QG, and suppose that $\gamma_1(b) = \gamma_2(b)$, $|\gamma_1^-(b)| \le |\gamma_2^+(b)|$ and $\gamma_1^+(b)$ is polar to $\gamma_1^-(b)$. Then a curve γ, defined by

$$\gamma(t) = \begin{cases} \gamma_1(t) , & \text{if } t \le b , \\ \gamma_2(t) , & \text{if } t \ge b \end{cases}$$

is a pre-QG with entropy

$$\mu(t,t') = \begin{cases} \mu_1(t,t') , & \text{if } t' \le b , \\ \mu_2(t,t') , & \text{if } t \ge b , \\ \mu_1(t,b) + \mu_2(b,t') + \log|\gamma^-(b)| - \log|\gamma_2^+(b)| , & \text{if } t < b < t' . \end{cases}$$

This follows from 2.1(c), 4.1.1 and 4.6.

In particular, every pre-QG is extendable, because for any vector one can find a polar one which is not longer, see 3.1.1.

Note the problem of existence of infinite QG with given initial data, as well as the problem of extendability of QG, has been reduced to the verification of two statements

(1) There exists local QG with arbitrary initial data
(2) For every unit vector there exists a polar unit vector

In fact, the second statement follows from extendability of QG in lower dimensional spaces. Indeed, given a unit vector $\eta \in \Sigma_q$ construct an arbitrary QG (in Σ_q) of length π, starting at η; the comparison inequality (G2) implies that the second endpoint ζ of this QG satisfies $|\eta\xi| + |\xi\zeta| \le \pi$ for all $\xi \in \Sigma_q$, which is equivalent to the statement that η, ζ are polar in C_q.

The proof of statement (1) occupies the rest of this section.

Since the problem is local, we will assume that M is compact; the general case requires only minor modifications.

5.2. We are going to construct local QG with prescribed initial data as limits of pre-QG with the same initial data. Note that a uniform limit of pre-QG is a pre-QG, and the entropies weakly converge to the entropy of the limit curve — this follows from 2.2.1, 2.3(1). Therefore we have to find a way to estimate the entropy.

At this point we fix the initial data $p_0 \in M$, $\xi_0 \in \Sigma_p$ and choose a small number $\overline{a}$ such that the directions of shortest lines $\overline{p_0 p}$ of length $10\overline{a}$ form a $1/100$-net in Σ_{p_0}, and for any two shortest lines $\overline{p_0p_1}, \overline{p_0p_2}$ of length $\le 10\overline{a}$ we have $\angle p_1p_0p_2 - \tilde{\angle} p_1p_0p_2 \le 1/100$. From now forth we will consider only pre-QG γ defined on $[0,\infty)$ with $\gamma(0) = p_0$, $\gamma^+(0) = \xi_0$, and their reparametrised arcs γ^s, $s \in [0,\overline{a})$.

5.3 Lemma. *Let $s \in [0,\overline{a})$, $p = \gamma(s)$, $\xi = \gamma^+(s)/|\gamma^+(s)|$, $q \in M$, $\eta \in \Sigma_p$ — the direction of $\overline{pq}$, $\alpha = \angle(\xi,\eta)$. Suppose that $\alpha < 1/10$. Then*

(a) $|(\gamma^s)^+(|pq|/4)| \ge 1 - 2\alpha^2$.
(b) $\mu(0,|pq|/4) \le 4\alpha^2$.

In particular

(c) $|\gamma^+(\overline{a})| > 1 - 1/100, \;\; \mu[0,\overline{a}) < 1/100$

(d) *For any* $\nu > 0$ *there exists* $s' \in (s,\overline{a})$ *such that if* $q = \gamma(s')$ *then* $\mu(s,s') < \nu(s' - s + \alpha)$. *Moreover,* s' *can be chosen arbitrarily close to* s.

PROOF. Since γ^s is convex, $|q\gamma^s(|pq|/2)| \leq (1+\alpha^2)|pq|/2$. On the other hand, $|pq| \leq |q\gamma^s(|pq|/2) + \frac{1}{4}|pq||(\gamma^s)^+(0)| + \frac{1}{4}|pq||(\gamma^s)^+(|pq|/4)$, since $(\gamma^s)^+(t)$ is non-increasing. Subtracting we get (a); (b) follows immediately. The assertion (c) follows from (a),(b) applied for $s = 0$ and appropriately chosen q. To prove (d) consider a sequence $s_i = s + 4^{-i}(\overline{a} - s)$. If for some large i we have $\mu(s,s_i) < 10\mu(s,s_{i+1})$ then using (b) we easily get $\mu(s,s_i) < \nu\alpha$; on the other hand, if $\mu(s,s_i) \geq 10\mu(s,s_{i+1})$ for all large i then obviously $\mu(s,s_i) = o(s_i - s)$ as $i \to \infty$, and the assertion (d) follows. $\square$

5.4 The preceding lemma gives us no means to estimate the point charges of μ. In fact we should not worry about such charges, because we can always modify our pre-QG to remove all of them above a certain level. Indeed, if $\mu\{s\}$ is "large" then we can replace the arc γ^s by another one, choosing as $(\gamma^s)^+(0)$ a unit vector, polar to $\gamma^-(s)/|\gamma^-(s)|$.

5.5 Consider a model example. Let $\gamma : (0,\overline{a}) \to s_0$ be a Lipschitz curve, (locally) convex in the usual sense of euclidean geometry, and let μ be a (locally) finite measure on $(0,\overline{a})$, satisfying 5.3(d) and having no point charges. Then $\mu \equiv 0$. This conclusion essentially follows from the fact that for any approximation of γ by an inscribed broken geodesic, the sum of arc-chord angles α is (locally) uniformly bounded.

The following estimate is essentially an attempt to relate the arc-chord angles of a convex curve in M to the corresponding angles for its development. For technical reasons, we express it in somewhat different terms.

Lemma. *Let* f *be a function of the type* $\frac{1}{2}\mathrm{dist}_x^2$, $\; p = \gamma^s(0), \;\; q = \gamma^s(t), \;\; \xi = (\gamma^s)^+(0)$, $\eta \in \Sigma_p$ *— the direction of* $\overline{pq}$. *Then* $(f \circ \gamma^s)^+(0) - (f \circ \gamma^s)^-(t) \geq df(\xi) - df(\eta) - 2t$, *provided that* $df(\eta) \geq 0$.

PROOF. Clearly, $f(q) \leq f(p) + df(\eta)|pq| + |pq|^{\leq} f(p) + df(\eta)t + t^2$. On the other hand, $f(p) \leq f(q) - (f \circ \gamma^s)^-(t)t + t^2$, since γ^s is convex. Therefore, $(f \circ \gamma^s)^-(t) \leq df(\eta) + 2t$, whence the result. $\square$

5.6 The preceding lemma allows us to estimate from above the expressions of the form $df(\xi) - df(\eta)$ rather than $\angle(\xi,\eta)$. If Σ_p is the standard unit sphere then, given ξ and η, it is easy to find many functions f for which the difference $df(\xi) - df(\eta)$ is positive and of order $\angle(\xi,\eta)$. However, for a general Σ_p there may be no such functions at all. In the next lemma we describe a specific situation where such functions can be found.

Let $A_0 = \frac{1}{2}\inf_{x\in M}\mathrm{Vol}(\Sigma_x)$. Clearly $A_0 > 0$ since M is compact.

Lemma. *There exists a small constant* $c_0 = c_0(n, A_0)$ *such that if* $R > 0$, $\; A \geq A_0$, $q \in M$, $\; p \in B_q(c_0R)$, $\; p_i$ *tends to* p *and directions of shortest lines* $\overline{pp}_i$ *tend to* $\xi \in \Sigma_p$, $\mathrm{Vol}(\Sigma_p) \geq A\mathrm{Vol}(S^{n-1})$, $\;\; \mathrm{Vol}(\Sigma_{p_i}) \leq A(1+c_0)\mathrm{Vol}(S^{n-1})$, $\;\; R^{1-n}\mathrm{Vol}(S_q(R)) \geq A(1 - c_0)\mathrm{Vol}(S^{n-1})$, *then for any* $\eta \in B_\xi(\pi/4) \subset \Sigma_p$ *there exists a subset* $V_\eta \subset S_q(R)$ *with*

$\mathrm{Vol}(V_\eta) \geq c_0 \mathrm{Vol}(S_q(R))$ *such that for each* $x \in V_\eta$ *the function* $f = \mathrm{dist}_x$ *satisfies* $df(\xi) - c_0|\xi\eta| \geq df(\eta) \geq 0$.

Proof. It is easy to check that if $M_i^n \to M^n$ in Gromov-Hausdorff sense, and $x_i \in M_i$ tend to $x \in M$, then $\mathrm{Vol}(\Sigma_x) \leq \liminf \mathrm{Vol}(\Sigma_{x_i})$. In particular, considering the convergence of rescaled M to its tangent cone C_p, we see that $\mathrm{Vol}(S(\Sigma_\xi)) \leq A(1+c_0)\mathrm{Vol}(S^{n-1})$, where $S(\Sigma_\xi)$ denotes the spherical suspension of Σ_ξ. Now imagine that $c_0 = 0$. In this case, $\exp_\xi$ is an isometry of $S(\Sigma_\xi)$ onto Σ_p, whereas $\exp_p \circ R$ is a similarity of Σ_p onto $S_q(R)$. For very small $c_0 > 0$ these assertions are "almost true". Given $\eta \in B_\xi(\pi/4) \subset \Sigma_p$ let $U_\eta = \{\zeta \in S(\Sigma_\xi) : |\xi\zeta| \geq 3\pi/4, |\log_\xi(\eta)\log_\xi(\zeta)| < \pi/4\}$, and define $V_\eta = \{x \in S_q(R) : \log_p(x) \subset C_p$ projects into $\exp_\xi(U_\eta) \subset \Sigma_p\}$. The verification that V_η satisfies our requirements is easy. $\square$

5.7 Now, having prepared the necessary estimates, we can start the formal proof. First introduce some notation.

Let $M_A = \{p \in M : \mathrm{Vol}(\Sigma_p) > A\mathrm{Vol}(S^{n-1})\}$, $0 \leq A \leq 1$. M_A is an open set for each A, $M_1 = \emptyset$, $M_{A_0} = M$. Let $M_A(\delta) = \{x \in M : B_x(\delta) \subset M_A\}$, $\delta > 0$. $M_A(\delta)$ is a compact subset of M_A, and the open sets $\mathrm{int}M_A(\delta)$ form an exhaustion of M_A as $\delta \to 0$. A pre-QG γ is called an A-pre-QG if its entropy satisfies $\mu(\gamma^{-1}(M_A)) = 0$. Thus our goal is to construct infinite A_0-pre-QG with prescribed initial data. It will be achieved by an inductive argument: we already know (4.5) that 1-pre-QG can be constructed, and we will show how to construct A-pre-QG assuming the existence of $A(1+c_0)$-pre-QG.

Since the class of A-pre-QG is closed with respect to gluing along a pair of polar unit vectors, as well as taking limits, the problem again reduces to a local one. Furthermore, it suffices to prove the following statement

$$(*) \qquad \begin{aligned} &\text{For any } \nu > 0,\ \delta > 0 \text{ there exists an} \\ &A(1+c_0)\text{-pre-QG } \gamma : [0,\infty) \to M \text{ with } \gamma(0) = p_0,\ \gamma^+(0) = \xi_0, \\ &\text{such that } \mu(\gamma^{-1}(\mathrm{int}M_A(\delta)) \cap [0,\overline{a})) < \nu \end{aligned}$$

Proof of (*). All pre-QG γ appearing in the argument will be $A(1+c_0)$-pre-QG defined on $[0,\infty)$ and having initial data (p_0, ξ_0).

Choose a finite covering of $M_A(\delta)$ by balls $B_{q_j}(c_0 r_j)$, $1 \leq j \leq N$, such that $R_j^{1-n}\mathrm{Vol}(S_{q_j}(R_j)) > A(1-c_0)\mathrm{Vol}(S^{n-1})$ for each j, and consider (-1)-concave functions $f_j = (\mathrm{Vol}(S_{q_j}(R_i)))^{-1} \int_{x \in S_{q_j}(R_j)} \frac{1}{2}\mathrm{dist}_x^2$. Obviously the expressions $(f_j \circ \gamma)^+(t) - (f_j \circ \gamma)^-(t') + t' - t$ are positive and bounded independently of γ for $0 \leq t < t' \leq \overline{a}$. Therefore, it suffices to show that for any $\nu' > 0$ there exists γ satisfying

$$(1) \qquad \mu(\gamma^{-1}(\mathrm{int}M_A(\delta)) \cap [0,s)) \leq \nu'(s + \sum_{j=1}^{N}((f_j \circ \gamma)^+(0) - (f_j \circ \gamma)^-(s) + s)) \ ,$$

for $s = \overline{a}$. We will "construct" γ satisfying (1) for all $s \leq \overline{a}$ using Zorn's lemma. Obviously, if γ satisfies (1) for $s = s_i$ and $s_i \to \overline{s}-$, then γ satisfies (1) for $s = \overline{s}$. It remains to show that if γ satisfies (1) for some $s \in [0, \overline{a})$ then γ^s can be modified to make γ satisfy (1) for some $s' > s$.

Suppose that γ satisfies (1) for some $s \in [0, \overline{a})$. Modify γ^s to make $(\gamma^s)^+(0)$ a unit vector polar to $\gamma^-(s)/|\gamma^-(s)|$; clearly this makes $\mu\{s\} = 0$. (If $s = 0$ then $\gamma^+(0) = \xi_0$ and $\mu\{0\} = 0$.) If $\gamma(s) \notin M_A$ then $\gamma^{-1}(\mathrm{int} M_A(\delta)) \cap (s, s') = \emptyset$ for some $s' > s$ and (1) is trivially satisfied for such s'. If $\gamma(s, s') \subset M_{A(1+c_0)}$ for some $s' > s$, then again (1) is satisfied for such s' becuase γ is an $A(1 + c_0)$-pre-QG. In the remaining case we are in the conditions of 5.6, with $p = \gamma(s)$, $p_i = \gamma(s_i)$, $s_i \to s+$, $B_q(R) = B_{q_j}(R_j)$ for some j. Using Lemmas 5.5. 5.6 and (–1) concavity of functions $\frac{1}{2}\mathrm{dist}_x^2$, we get $\angle(\xi, \eta) \le 10c_0^{-2}R_j^{-1}((f_j \circ \gamma)^+(s) - (f_j \circ \gamma)^-(s') + 2(s' - s))$ for $s' \in (s, \overline{a})$. Thus, using 5.3(d), we conclude that (1) holds for s' slightly larger than s. $\square$

§6. Semiconcave functions and QG on Extremal Subsets

In this section we assume that our space M has no boundary.

6.1 Semiconcave functions and QG.

Proposition. *Let $\gamma : [a, b] \to U$ be a QG, and let f be a λ-concave function in U. Then*

$$(f \circ \gamma)'' \le -2\lambda \tag{1}$$

Proof.We'll show that (1) holds for every function g of the form $g(x) = \inf_{y \in U} (f(y) + A|xy|^2)$, where A is a large positive number, and then pass to the limit as $A \to \infty$.

Take any $t \in (a, b)$, let $p = \gamma(t)$ and let q satisfy $g(p) = f(q) + A|pq|^2$. Then according to 6.2(a) below, the derivative at q of the function $f + A\mathrm{dist}_p^2$ vanishes. It follows easily that there is only one shortest line $\overline{qp}$, and its direction at q is a pole of Σ_q (which turns out to be a spherical suspension). Now the estimate $(g \circ \gamma)'' \le -2\lambda$ can be proved by an argument similar to [Per,6.1] ,where it was shown that the distance function from the boundary of nonnegatively curved Alexandrov space is concave. $\square$

6.2 Lemma. *Let ϕ be a spherically concave function on Σ, $\partial\Sigma = \emptyset$. Then*

(a) *If $\phi \ge 0$ on Σ then $\phi \equiv 0$.*
(b) *If $\xi \in \Sigma$ is the point where ϕ attains its minimal value, then $\phi \le \phi(\xi) \cdot \cos \circ \mathrm{dist}_\xi$ on Σ.*
(c) *In particular, if f is a semiconcave function near $p \in M$, then there exists $v \in C_p$ such that $df(u) \le -\langle v, u \rangle$ for all $u \in C_p$.*

Proof. (a) We use induction. If dim $\Sigma = 0$, then Σ is a couple of points $\{\xi_1, \xi_2\}$ at distance π, and our assertion follows from the condition of spherical concavity $\phi(\xi_1) + \phi(\xi_2) \le 0$. To carry out the induction step, consider the derivative ϕ' on the space of directions Σ_ξ at the point ξ where ϕ attains its minimal value. Clearly $\phi' \ge 0$ on Σ_ξ, therefore, by the induction assumption, $\phi' \equiv 0$ on Σ_ξ. On the other hand, if $\eta \in \Sigma$ is such that $f(\eta) \geq f(\xi) \geq 0$ and at least one of these inequalities is strict, then the spherical concavity of ϕ implies that ϕ' is strictly positive at the direction of $\overline{\xi\eta}$ at ξ. Thus, $\phi \equiv 0$ on Σ.

(b) It follows from (a) that $\phi' \equiv 0$ on Σ_ξ. Now to prove the inequality $\phi(\eta) \leq \phi(\xi) \cdot \cos|\xi\eta|$ for some $\eta \in \Sigma$ it suffices to observe that the left- and the right-hand sides have the same values and equal derivatives at ξ, and the right-hand side is spherically linear on $\overline{\xi\eta}$, whereas the left-hand side is spherically concave.

(c) Apply (b) to $\Sigma = \Sigma_p$, $\phi = f'$ and let $v = -\phi(\xi) \cdot \xi$.

6.3 QG on extremal subsets.

Theorem. *Let $F \subset M$ be an extremal subset, $p \in F$, $\xi \in \Sigma_p F$. Then*

(a) *An f-gradient curve starting at p remains in F for any semiconcave f.*
(b) *There exists an infinite QG with initial data (p, ξ), contained in F.*

Proof. (a) The only new ingredient in the proof is the following.

6.3.1 Lemma.

(a) *$\nabla f(q) \in C_q F$ for any $q \in F$ and any semiconcave f.*
(b) *Let $\Phi \subset \Sigma$ be an extremal subset, $\partial\Sigma = \emptyset$, ϕ a spherically concave function on Σ, $x \in \Sigma$ — the point where ϕ attains its positive maximal value. Then $x \in \Phi$.*

Proof of the Lemma. (b) Let $y \in \Phi$ be the point of Φ closest to x, $y \neq x$, and let $\eta \in \Sigma_y$ denote the direction of $\overline{yx}$. Then $\phi(y) < \phi(x)$ and $|xy| \leq \pi/2$ (see [PP, 1.4.1]), therefore $\phi'(\eta) > 0$. Applying 6.2(b) to ϕ' on Σ_y, we find $\xi \in \Sigma_y$ such that $|\eta\xi| > \pi/2$; this contradicts extremality of Φ. (a) follows from (b) applied to f' on Σ_q, with $\Phi = \Sigma_q F$. □

(b) The arguments of §§4,5 work in our situation once we know that for any $q \in F$, $\eta \in \Sigma_q F$, there exists $\zeta \in \Sigma_q F$ polar to η. This again requires an inductive argument, as in 5.1, with the base of induction provided by

6.3.2 Lemma. *Let $\Phi \subset \Sigma$ be an extremal subset, $\eta \in \Phi$ be an isolated point of Φ. Then there exists $\zeta \in \Phi$ such that $|\xi\eta| + |\xi\zeta| \leq \pi$ for all $\xi \in \Sigma$.*

Proof. If $\mathrm{clos}(B_\eta(\pi/2)) = \Sigma$ then we can take $\zeta = \eta$. Otherwise let ζ be the point of Σ farthest from η, $|\eta\zeta| > \pi/2$. According to [PP, 1.6], $\zeta \in \Phi$. Now for any $\xi \in \Sigma$ we have $|\xi\eta| < |\zeta\eta|$ and $\angle\xi\eta\zeta \leq \pi/2$ (since η is isolated in Φ). Therefore, by comparison inequality, $\tilde{\angle}\eta\xi\zeta > \pi/2$. It follows that $\Phi = \{\eta, \zeta\}$, since otherwise we would have $\tilde{\angle}\eta\xi\zeta \leq \pi/2$ if ξ is the point of Φ closest to η. Since ζ also turned out to be isolated in Φ, we have $\angle\eta\zeta\xi \leq \pi/2$ and $\angle\zeta\eta\xi \leq \pi/2$ for any $\xi \in \Sigma$, whence by comparison inequality, $|\xi\eta| + |\xi\zeta| \leq \pi$. □

6.4 Spaces with boundary.

Strong QG in a space M with boundary can be defined by requiring $(f \circ \gamma)'' \leq -2\lambda$ to hold only for those λ-concave f whose tautological extension to the double of M remains λ-concave. (Note that distance functions satisfy this additional condition, as can easily be checked.) Proposition 6.1 generalizes easily (because it is not hard to check that any

QG in M can be obtained from some QG in the double by reflection), while Theorem 6.3(b) requires an additional observation that if F is a primitive extremal subset in M, not contained in ∂M, then its double is extremal in the double of M. This is not hard to prove. Finally, Theorem 6.3(a) and Lemma 6.3.1 hold true for semiconcave f whose tautological extension is semiconcave.

Appendix
Gradient Curves in Infinite-Dimensional Alexandrov Spaces

A.1 Let X be a complete Alexandrov space of curvature $\geq k$. We will pretend that every two points in X can be connected by a shortest line. (The general case requires minor modifications based on the fact that for any countable collection of points $x_i \in X$ there exists a dense set of points $x \in X$, which can be connected with each of x_i by a unique shortest line, see [Pl].) Define an absolute gradient $|\nabla f|$ of a semiconcave function f by

$$|\nabla f|(p) = \max\{0, \limsup_{p_i \to p} (f(p_i) - f(p))/|pp_i|\} .$$

A point p is critical for f if $|\nabla f|(p) = 0$.

Let f be a semiconcave function without critical points in a domain $U \subset X$. A curve $\gamma : (a, b) \to U$ is called an f-gradient curve if $f \circ \gamma(t) = t$ and $\lim_{t' \to t+} |\gamma(t')\gamma(t)|/|t' - t| = |\nabla f|^{-1}(\gamma(t))$ for all $t \in (a, b)$. An f-gradient curve is called complete if it has no accumulation points in U.

A.2 Proposition. *For each $p \in U$ there exists a unique complete f-gradient curve, starting at p.*

In the finite-dimensional case the proof was based on two facts: the semi-continuity of the absolute gradient (3.2.1(b)) and the inequality $\langle \nabla f, u \rangle \geq df(u)$, which implied 3.2.1(a). The first fact is valid in our situation, and the proof needs no changes. The second one is replaced by the first statement of the following

A.2.1 Lemma. *(a) Let f be λ-concave in $U \subset X$, $q, x, y \in U$, $|\nabla f|(x) > 0$, and $\frac{f(y)-f(x)}{|xy|} + \lambda|xy| \geq (1 - \delta^2)|\nabla f|(x)$ for some small $\delta > 0$. Then $\cos \tilde{\angle} qxy \geq \left(\frac{f(q)-f(x)}{|qx|} + \lambda|qx|\right) |\nabla f|^{-1}(x) + C\delta$, where C may depend on $|\nabla f|(x)$, λ, $\mathrm{diam} U$ and Lipschitz constant of f, but not on δ.*

(b) Let $p, q, x, y \in U$, $|\nabla \mathrm{dist}_p|(x) > 0$, and $-\cos \tilde{\angle} pxy \geq (1 - \delta^2)|\nabla \mathrm{dist}_p|(x)$ for some small $\delta > 0$. Then $\cos \tilde{\angle} qxy \geq -\cos \tilde{\angle} pxq \cdot |\nabla \mathrm{dist}_p|^{-1}(x) + C\delta$, where C may depend on $|\nabla \mathrm{dist}_p|(x)$, and the distances between p, q, x, but not on δ.

The proof of (a) goes as follows. Take a point z_1 on a shortest line $\overline{xy}$ very close to x, then take a point z_2 on $\overline{z_1 q}$ such that $|z_1 z_2| = \delta |x z_1|$, estimate $f(z_1)$ from below in terms of $f(x), f(y)$ using λ-concavity, then $f(z_2)$ in terms of $f(z_1)$ and $f(q)$, then estimate $|x z_2|$ from below in terms of $|\nabla f|(x)$, $f(x)$, $f(z_2)$ using λ-concavity, then apply comparison inequality to $\Delta x z_1 z_2$ to estimate $\angle x z_1 z_2$, and finally observe that $\tilde{\angle} qxy$ cannot be substantially bigger than $\angle y z_1 q = \pi - \angle x z_1 z_2$ if z_1 was chosen close enough to x. The proof of (b) is similar. The details are routine. $\square$

A.2.2 PROOF OF THE PROPOSITION. First of all observe that it suffices to construct unique gradient curves locally, since if a gradient curve γ is not complete in U, and q is its accumulation point, then $|\nabla f|^{-1}$ is bounded near q, and therefore it is easy to see that γ converges to q and can be extended by a local gradient curve starting at q. Thus we may assume that U is bounded, f is Lipschitz and λ-concave in U, and $|\nabla f|^{-1}$ is bounded in U.

Fix a small $\delta > 0$. A finite sequence $x_0, x_1, \ldots, x_N$ will be called admissible if $x_0 = p$, $f(x_{i+1}) > f(x_i)$, $|x_i x_{i+1}| < \delta^2$ and

$$(1) \qquad \sum_{i \in I_x} |x_i x_{i+1}| < \delta^2 (f(x_N) - f(x_0)) \ , \quad \text{where} \quad I_x = \{i : f(x_{i+1}) - f(x_i) < (1-\delta^2)|\nabla f|(x_i)|x_i x_{i+1}|\} \ .$$

Since $|\nabla f|^{-1}$ is bounded in U, we have an estimate

$$(2) \qquad \sum_{i=1}^{N} |x_{i-1} x_i| \leq C(f(x_N) - f(x_0)) \ .$$

An infinite sequence $x_0, x_1, \ldots$ will be called admissible if $x_0, x_1, \ldots, x_{N_i}$ are admissible for some $N_1 < N_2 < \ldots$. We can find an infinite admissible sequence $x_0, x_1, \ldots$ without accumulation points in U. Indeed, if $x_0, x_1, \ldots$ is such a sequence with an accumulation point $\overline{x} \in U$, then (2) implies that $x_j \to \overline{x}$; so if $x' \in U$ satisfies $|\overline{x} x'| < \delta^2$ and $f(x') - f(\overline{x}) > (1-\delta^2)|x' \overline{x}|$, then $x_0, x_1, \ldots, x_{N_i}, \overline{x}, x'$ is an admissible sequence for i large enough; thus our claim follows from Zorn's lemma.

Now let $x_0, y_1, \ldots$ and $y_0, y_1, \ldots$ be two admissible sequences. We are going to estimate the distances $|x_i y_j|$ for such pairs (i, j) that $|f(x_i) - f(y_j)| < \delta^2$.

Assume that for some such (i, j) we have $|x_i, y_j| \geq \delta$ and, say, $f(x_i) \leq f(y_j)$. Consider the pair (x_{i+1}, y_j). If $i \notin I_x$, then we can estimate

$$(3) \qquad |x_{i+1} y_j| - |x_i y_j| \leq C |x_i y_j| |x_i x_{i+1}|$$

according to A.2.1(a). If I_x were empty then we could easily "integrate" (3), using an inductive argument, to get $|x_i y_j| < C\delta$ for all pairs (i, j) such that $|f(x_i) - f(y_j)| < \delta^2$. In fact, using (1), it is easy to check that the same conclusion holds even if $I_x \neq \emptyset$.

Therefore, any family of admissible sequences with $\delta \to 0$ converges to some curve γ; this curve does not have accumulation points in U if those sequences did not. Using semicontinuity of the absolute gradieint, it is easy to see that γ is an f-gradient curve. Finally, to prove uniqueness it suffices to observe that any f-gradient curve γ starting at p can be easily approximated by admissible sequences with $x_i = \gamma(t_i)$. $\square$

A.3. Let $\gamma : (0, a) \to X$ be a complete dist_p-gradient curve. Then the monotonicity estimates 3.3 are valid under the assumption that $I_\gamma(t)$ converges at $t = 0$. The only change in the proofs is the derivation of the inequality $dh \leq \frac{t^2 + h^2 - t_1^2}{2th}\, dt + \frac{t_1^2 + h^2 - t^2}{2t_1 h}\, dt_1$; namely, in 3.3 we used inequalities $-\langle \nabla f, \xi \rangle \leq -df(\xi)$, and now we have to use an argument based on A.2.1(b) instead.

The convergence condition is obviously satisfied if γ starts as a shortest line. Moreover, the argument in the end of proof 3.3 justifies this condition for those γ which have a definite direction at p in the following sense. We say that γ has direction ξ at p if for any sequences $p_i \to p,\ \ t_i \to 0+$ such that $|p_i\gamma(t_i)| = o(|pp_i|)$, the direction of shortest lines $\overline{pp_i}$ converge to ξ. Applying 3.3(c) we immediately see that a complete dist_p-gradienit curve starting at p in direction ξ is unique. The existence also holds, but is a little more delicate than in the finite-dimensional case because we can no longer claim that $|\nabla \mathrm{dist}_p|^{-1}$ is bounded near p. Still, it is not hard to see from the monotonicity estimate 3.3(b) that if $\gamma_i : (0, a_i) \to X$ are complete dist_p-gradient curves, such that their directions at p form a relatively compact set, then there exists $a > 0$ such that all $a_i > a$ and $|\gamma_i^+(t)|$ are uniformly bounded on $(0, a)$. Therefore the argument in 3.4 proving existence goes through.

A.4 As a corollary to the previous discussion and the work of Plaut [Pl] we can now prove that if the Hausdorff dimension of X is infinite then its topological dimension is infinite as well. Indeed, suppose $\dim_H(X) > n$. Then, according to [Pl], there is a point $p \in X$, such that the space of directions at p contains an isometrically embedded standard unit sphere S^n. The dist_p-gradient curves γ_ξ starting at p in directions $\xi \in S^n$ are all defined on some interval $(0, a)$, and each map $\Gamma_t : S^n \to X$, given by $\Gamma_t(\xi) = \gamma_\xi(t)$, is Lipschitz for $t < a$, according to 3.3(c). On the other hand, it is easy to see, using compactness of S^n and monotonicity estimate 3.3(b), that the diameters of Γ_t-inverse images uniformly converge to zero at $t \to 0$. Therefore, topological dimension of X must be at least n, and letting $n \to \infty$ completes the proof.

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