Homework assignments

- HWA3, due Mon, Feb 2, 10:10:
  - Exercises: 4.3, 4.10, 5.4, 5.11, 5.18.
- HWA2, due Mon, Jan 26, 10:10:
  - Exercises: 2.8, 3.3, 3.11a, 3.12.
  - Read Chapter 4 and EITHER formulate one question OR solve Exercise 4.9.
- HWA1, due Fri, Jan 16, 10:10:
  - Exercises: 1.2, 1.12, 1.29 + one exercise of your choice in Chapter 0 (each part of 0.4 is counted as one exercise; you may upload multiple photos showing your solution).
  - Read Chapter 2 and EITHER formulate one question OR solve Exercise 2.6.

Extra-credit problems

These are challenging problems. They may improve your grade, but they are intended to be done for fun (with the exception of Problem 0). Solutions should be presented orally. Only the first solution will be graded. Solutions will not be accepted after Fri, Apr 17.

Problem 0: Find a mistake or misprint in the covered part of lecture notes. (The score depends on the type of mistake.)

Problems: 1.10, 3.5, 3.8, 3.16, 4.5, 4.6, 4.7, 4.11, 5.12.

Contents

Chapter 0

Puzzles and anecdotes

You have probably heard that for topologists, *two shapes are the same if one can be turned into the other by stretching and squeezing* (without ripping, piercing, gluing, and so on).

This statement is neither precise nor correct, but it helps build intuition. So, imagine an object made from endlessly stretchy material that can be twisted and pulled like chewing gum; if you can get one shape from another, then they have the same topology. Note that the size plays no role.

The picture above suggests that a square and a disc have the same topology. It might be more impressive to see that a donut is topologically equivalent to a coffee mug. On the other hand, a disc and an annulus are topologically different. It might be obvious, but this is a nontrivial statement (it will eventually be proved in our course). So, at least not all shapes are topologically equivalent.

0.1. Exercise. *In the morning, a topologist looks at his clothes; there should be a T-shirt, pants, and socks.*

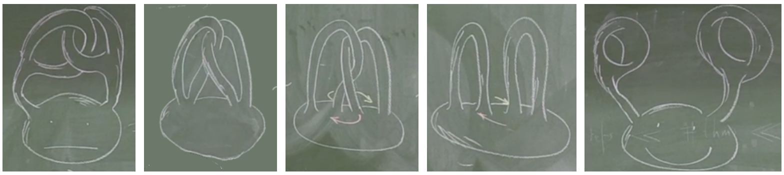
Help him to identify each piece of clothing. Which piece has a hole in it?

0.2. Exercise. *A topologist dropped his mug and noticed that the handle had snapped off. "Its topology has changed," he thaut, "but it can still be used for its intended purpose."*

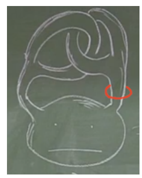
He picked the mug up, turned it over, looked at the bottom, and said: "Oh, I was wrong—the topology has not changed, but it is now impossible to use."

What did he see?

0.3. Exercise. *The topologist decided to fix the broken mug with superglue. Being a rather clumsy person, he somehow managed to glue his index finger to his thumb—in fact, on both hands. Worse still, the*



finger rings he'd made were linked together. But after some thinking and stretching he managed to separate the rings.



What would happen if he had a hair tie on his hand?

At this point, you might think that topology has no practical value, and you are almost right. Altho, thinking topologically, it is very easy to solve some puzzles:

0.4. Exercise.

(a) *Put a loop of cord thru the handles of a pair of scissors as shown. Have someone hold the ends. Try to get the scissors off without cutting the string or releasing the ends.*

(b) *Tie your hands together with a long cord; then tie the hands of your friend in the same way, linking your cords, as shown. Now try to get them apart without cutting the string or untying the knots.*

(c) *If there is no friend nearby, try undoing the knot in the middle of the cord shown in the last picture. Again, no cutting the string or untying the little knots.*

Chapter 1

Metric spaces

In this chapter we discuss *metric spaces* — a motivating example that will guide us toward the definition of our main object of study — *topological spaces.*

Examples of metric spaces were considered for thousands of years, but the first general definition was given only in 1906 by Maurice Fréchet [7].

A Definition

The following definition grabs together the most important properties of the intuitive notion of *distance.*

1.1. Definition. *Let $\mathcal{X}$ be a set with a function that returns a real number, denoted as $|x-y|$, for any pair $x,y\in\mathcal{X}$. Assume that the following conditions are satisfied for any $x,y,z\in\mathcal{X}$:*

(a) $|x-y|\geqslant 0$.

(b) $x=y$ *if and only if* $|x-y|=0$.

(c) $|x-y|=|y-x|$.

(d) $|x-y|+|y-z|\geqslant|x-z|$; *this property is called the triangle inequality.*

In this case, we say that $\mathcal{X}$ is a metric space and the function

$$(x,y)\mapsto|x-y|$$

is called a metric. The elements of $\mathcal{X}$ are called points of the metric space. Given two points $x,y\in\mathcal{X}$, the value $|x-y|$ is called the distance from x to y.

For two points x and y in a metric space the difference $x-y$ may have no meaning, but $|x-y|$ means the distance.

Typically, we consider only one metric on a set, but if a few metrics are needed, we can distinguish them by an index, say $|x-y|_\bullet$ or $|x-y|_{239}$. If we need to emphasize that the distance is taken in the metric space $\mathcal{X}$ we write $|x-y|_\mathcal{X}$ instead of $|x-y|$.

1.2. Exercise. *Show that*

$$|x-y|_\natural = (x-y)^2$$

is not a metric on the real line $\mathbb{R}$.

1.3. Exercise. *Show that if* $(x,y)\mapsto|x-y|$ *is a metric, then so is*

$$(x,y)\mapsto|x-y|_{\max} = \max\{1,|x-y|\}.$$

B Examples

Let us give a few examples of metric spaces.

- Discrete space. Let $\mathcal{X}$ be an arbitrary set. For any $x,y\in\mathcal{X}$, set $|x-y|=0$ if $x=y$ and $|x-y|=1$ otherwise. This metric is called the discrete metric on $\mathcal{X}$ and the obtained metric space is called discrete.
- Real line. The set $\mathbb{R}$ of all real numbers with the metric defined by $|x-y|$. (Unless it is stated otherwise, the real line $\mathbb{R}$ will be considered with this metric.)
- Metrics on the plane. Let us denote by $\mathbb{R}^2$ the set of all pairs (x,y) of real numbers. Consider two points $p=(x_p,y_p)$ and $q=(x_q,y_q)$ in $\mathbb{R}^2$. One can equip $\mathbb{R}^2$ with the following metrics:
  - Euclidean metric,

    $$|p-q|_2 = \sqrt{(x_p-x_q)^2+(y_p-y_q)^2}.$$

    (Unless it is stated otherwise, the real line $\mathbb{R}^2$ will be considered with the Euclidean metric.)
  - Manhattan metric,

    $$|p-q|_1 = |x_p-x_q|+|y_p-y_q|.$$

  - Maximum metric,

    $$|p-q|_\infty = \max\{|x_p-x_q|,|y_p-y_q|\}.$$

1.4. Exercise. *Prove that (a)* $|{*}-{*}|_1$*; (b)* $|{*}-{*}|_2$ *and (c)* $|{*}-{*}|_\infty$ *are metrics on* $\mathbb{R}^2$.

C Subspaces

Let us recall the set-builder notation: Given a set $\mathcal{X}$ and a property $P(x)$ that depends on $x \in \mathcal{X}$, we denote by

$$\{\, x \in \mathcal{X} \;:\; P(x) \,\}$$

the subset of all elements $x \in \mathcal{X}$ for which the property $P(x)$ holds. For example, $\{\, x \in \mathbb{R} \;:\; x > 0 \,\}$ denotes the set of positive reals.

Any subset $\mathcal{A}$ of a metric space $\mathcal{X}$ forms a metric space on its own; it is called a subspace of $\mathcal{X}$. This construction produces many more examples of metric spaces. For example, the disc

$$\mathbb{D}^2 = \left\{\, (x, y) \in \mathbb{R}^2 \;:\; x^2 + y^2 \leqslant 1 \,\right\}$$

and the circle

$$\mathbb{S}^1 = \left\{\, (x, y) \in \mathbb{R}^2 \;:\; x^2 + y^2 = 1 \,\right\},$$

are metric spaces with the metrics taken from the Euclidean plane. Similarly, the interval $[0, 1)$ is a metric space with metric taken from $\mathbb{R}$.

D Continuous maps

Recall that a real-to-real function f is called *continuous* if *for any $x_0 \in \mathbb{R}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x_0) - f(x)| < \varepsilon$, whenever $|x_0 - x| < \delta$.* It admits the following straightforward generalization to metric spaces:

1.5. Definition. *A map $f\colon \mathcal{X} \to \mathcal{Y}$ between metric spaces is called continuous if for any $x_0 \in \mathcal{X}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x_0) - f(x)|_{\mathcal{Y}} < \varepsilon$, for any $x \in \mathcal{X}$ such that $|x_0 - x|_{\mathcal{X}} < \delta$.*

1.6. Exercise. *Let $\mathcal{X}$ be a metric space and $z \in \mathcal{X}$ be a fixed point. Show that the function*

$$f(x) := |x - z|_{\mathcal{X}}$$

is continuous.

1.7. Exercise. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be metric spaces. Assume that the maps $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ are continuous, and*

$$h = g \circ f\colon \mathcal{X} \to \mathcal{Z}$$

is their composition; that is, $h(x) = g(f(x))$ *for any* $x \in \mathcal{X}$*. Show that* $h\colon \mathcal{X} \to \mathcal{Z}$ *is continuous at any point.*

1.8. Exercise. *Let* $f\colon \mathcal{X} \to \mathcal{Y}$ *be a distance-preserving map between metric spaces; that is,*

$$|x_0 - x_1|_{\mathcal{X}} = |f(x_0) - f(x_1)|_{\mathcal{Y}}$$

for any $x_0, x_1 \in \mathcal{X}$*.*

(a) *Show that* f *is continuous.*

(b) *Show that* f *is injective; that is, if* $x_0 \neq x_1$*, then* $f(x_0) \neq$ $\neq f(x_1)$*.*

1.9. Exercise. *Let* $\mathcal{X}$ *be a discrete metric space (defined in 1B) and* $\mathcal{Y}$ *be an arbitrary metric space. Show that any map* $f\colon \mathcal{X} \to \mathcal{Y}$ *is continuous.*

1.10. Advanced exercise. *Construct a continuous function*

$$f\colon [0,1] \to [0,1]$$

that takes every value in $[0,1]$ *an infinite number of times.*

E Balls

Let x be a point in a metric space $\mathcal{X}$, and $r > 0$. The set of points in $\mathcal{X}$ that lie at distance less than r is called the open ball of radius r centered at x. It is denoted as $\mathrm{B}(x, r)$ or $\mathrm{B}(x, r)_{\mathcal{X}}$; the latter notation is used if we need to emphasize that it is taken in the space $\mathcal{X}$.[1]

The ball $\mathrm{B}(x, r)$ is also called an r-neighborhood of x.

Analogously we may define closed balls

$$\bar{\mathrm{B}}[x, r] = \bar{\mathrm{B}}[x, r]_{\mathcal{X}} = \{\, y \in \mathcal{X} \ : \ |x - y| \leqslant r \,\}.$$

1.11. Exercise. *Sketch the unit balls for the metrics* $|{*} - {*}|_1$, $|{*} - {*}|_2$ *and* $|{*} - {*}|_\infty$ *defined in 1B.*

1.12. Exercise. *Consider two balls* $\mathrm{B}(x, r)$ *and* $\mathrm{B}(y, R)$ *in a metric space such that* $\mathrm{B}(x, r) \subsetneq \mathrm{B}(y, R)$*. Show that* $r < 2{\cdot}R$*.*

Give an example of a metric space and a pair of balls as above with $r > R$*.*

[1]Many authors use the notations $\mathrm{B}_r(x)$ and $\mathrm{B}_r(x)_{\mathcal{X}}$ as well.

Let us reformulate the definition of continuous map (1.5) using the introduced notion of ball.

1.13. Definition. *A map $f\colon \mathcal{X} \to \mathcal{Y}$ between metric spaces is called continuous if for every $x \in \mathcal{X}$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$f(\mathrm{B}(x,\delta)_{\mathcal{X}}) \subset \mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}};$$

that is, the image of the δ-ball centered at $x \in \mathcal{X}$ lies in the ε-ball centered at $f(x) \in \mathcal{Y}$.

1.14. Exercise. *Prove the equivalence of the definitions 1.5 and 1.13.*

F Open sets

1.15. Definition. *A subset V in a metric space $\mathcal{X}$ is called open if for any $x \in V$ there is $\varepsilon > 0$ such that* $\mathrm{B}(x,\varepsilon) \subset V$.

In other words, V is open if, together with each point, V contains its ε-neighborhood for some $\varepsilon > 0$.

For example, any set in a discrete metric space is open since together with any point it contains its 1-neighborhood.

Further, the set of positive real numbers

$$(0,\infty) = \{\, x \in \mathbb{R} \;:\; x > 0 \,\}$$

is an open subset of $\mathbb{R}$; indeed, for any $x > 0$ its x-neighborhood lies in $(0,\infty)$. On the other hand, the set of nonnegative reals

$$[0,\infty) = \{\, x \in \mathbb{R} \;:\; x \geqslant 0 \,\}$$

is not open since there are negative numbers in any neighborhood of 0.

1.16. Exercise. *Show that any open ball in a metric space is open.*[2]

1.17. Exercise. *Show that the union of an arbitrary collection of open sets is open.*

1.18. Exercise. *Show that the intersection of two open sets is open.*

1.19. Exercise. *Show that a set in a metric space is open if and only if it is a union of balls.*

[2]In other words, show that for any $y \in \mathrm{B}(x,r)$ there is $\varepsilon > 0$ such that $\mathrm{B}(y,\varepsilon) \subset$ $\subset \mathrm{B}(x,r)$.

1.20. Exercise. *Give an example of a metric space $\mathcal{X}$ and an infinite sequence of open sets $V_1, V_2, \dots$ such that their intersection $V_1 \cap V_2 \cap \dots$ is not open.*

1.21. Exercise. *Show that the metrics $|{*}-{*}|_1$, $|{*}-{*}|_2$ and $|{*}-{*}|_\infty$ (defined in 1B) give rise to the same open sets in $\mathbb{R}^2$. That is, if $V \subset \mathbb{R}^2$ is open for one of these metrics, then it is open for the others.*

G Gateway to topology

The following result is the main gateway to topology. It says that continuous maps can be defined entirely in terms of open sets.

1.22. Proposition. *A map $f\colon \mathcal{X} \to \mathcal{Y}$ between two metric spaces is continuous if and only if the inverse image of any open set is open; that is, for any open set $W \subset \mathcal{Y}$ its inverse image*

$$f^{-1}(W) = \{\, x \in \mathcal{X} \;:\; f(x) \in W \,\}$$

is open.

The following exercise emphasizes that the proposition says nothing about the images of open sets; it is instructive to solve it before going into the proof (see also 4B).

1.23. Exercise. *Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and an open set $V \subset \mathbb{R}$ such that the image $f(V) \subset \mathbb{R}$ is not open.*

Proof; only-if part. Let $W \subset \mathcal{Y}$ be an open set and $V = f^{-1}(W)$. Choose $x \in V$; note that $f(x) \in W$.

Since W is open,

$$\text{❶} \qquad \mathrm{B}(f(x), \varepsilon)_{\mathcal{Y}} \subset W$$

for some $\varepsilon > 0$.

Since f is continuous, by Definition 1.13, there is $\delta > 0$ such that

$$f(\mathrm{B}(x, \delta)_{\mathcal{X}}) \subset \mathrm{B}(f(x), \varepsilon)_{\mathcal{Y}}.$$

It follows that together with any point $x \in V$, the set V contains $\mathrm{B}(x, \delta)$; that is, V is open.

If part. Fix $x \in \mathcal{X}$ and $\varepsilon > 0$. According to Exercise 1.16,

$$W = \mathrm{B}(f(x), \varepsilon)_{\mathcal{Y}}$$

is an open set in $\mathcal{Y}$. Therefore its inverse image $f^{-1}(W)$ is open.

Clearly $x \in f^{-1}(W)$. By the definition of open set (1.15)

$$\mathrm{B}(x,\delta)_{\mathcal{X}} \subset f^{-1}(W)$$

for some $\delta > 0$. Or equivalently

$$f(\mathrm{B}(x,\delta)_{\mathcal{X}}) \subset W = \mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}.$$

Hence the if part follows. □

H Limits

1.24. Definition. *Let $x_1, x_2, \dots$ be a sequence of points in a metric space $\mathcal{X}$. We say that it converges to a point $x_\infty \in \mathcal{X}$ if*

$$|x_\infty - x_n|_{\mathcal{X}} \to 0 \quad \textit{as} \quad n \to \infty.$$

In this case, we say that the sequence $x_1, x_2, \dots$ is converging and x_∞ is its limit; it can be expressed by $x_n \to x_\infty$ as $n \to \infty$ or

$$x_\infty = \lim_{n\to\infty} x_n.$$

Note that we defined the convergence of points in a metric space using the convergence of real numbers $d_n = |x_\infty - x_n|_{\mathcal{X}}$, which we assume to be known.

1.25. Exercise. *Show that any sequence of points in a metric space has at most one limit.*

1.26. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map between metric spaces. Show that f is continuous if and only if the following condition holds:*

- *If $x_n \to x_\infty$ as $n \to \infty$ in $\mathcal{X}$, then the sequence $y_n = f(x_n)$ converges to $y_\infty = f(x_\infty)$ as $n \to \infty$ in $\mathcal{Y}$.*

I Closed sets

Let A be a set in a metric space $\mathcal{X}$. A point $x \in \mathcal{X}$ is a limit point of A if there is a sequence $x_n \in A$ such that $x_n \to x$ as $n \to \infty$.[3]

[3]Sometimes limit points are defined, assuming in addition that $x_n \neq x$ for any n — we do *not* follow this convention.

The set of all limit points of A is called the closure of A and denoted as $\bar{A}$. Note that $\bar{A} \supset A$; indeed, any point $x \in A$ is a limit point of the constant sequence $x_n = x$.

If $\bar{A} = A$, then the set A is called closed.

1.27. Exercise. *Give an example of a subset $A \subset \mathbb{R}$ that is neither closed nor open.*

1.28. Exercise. *Show that the closure of any set in a metric space is a closed set; that is, $\bar{\bar{A}} = \bar{A}$.*

1.29. Exercise. *Show that a subset Q in a metric space $\mathcal{X}$ is closed if and only if its complement $V = \mathcal{X} \setminus Q$ is open.*

Part I

Point-set topology

Chapter 2

Topological spaces

In the previous chapter we defined open sets in metric spaces and showed that continuity could be defined using only the notion of open sets. In this chapter we collect key properties of open sets and state them as axioms. It will give us a definition of a *topological space* as a set with a distinguished class of subsets called *open sets*.

The first definition of topological spaces was given by Felix Hausdorff in 1914 [10, VII § 1]. In 1922, the definition was generalized slightly by Kazimierz Kuratowski [12], and it is now standard.

A Definitions

We are about to define *abstract open sets* without referring to metric spaces; this definition is based on two properties in exercises 1.17 and 1.18.

2.1. Definition. *Suppose $\mathcal{X}$ is a set with a distinguished class of subsets, called open sets such that*

(a) *The empty set $\varnothing$ and the whole $\mathcal{X}$ are open.*

(b) *The union of any collection of open sets is an open set. That is, if V_α is open for any α in the index set $\mathcal{I}$, then the set*

$$W = \bigcup_{\alpha\in\mathcal{I}} V_\alpha = \{\, x \in \mathcal{X} \,:\, x \in V_\alpha \quad \text{for some} \quad \alpha \in \mathcal{I}\,\}$$

is open.

(c) *The intersection of two open sets is an open set. That is, if V_1 and V_2 are open, then the intersection $W = V_1 \cap V_2$ is open.*

In this case, $\mathcal{X}$ is called a topological space. The collection of all open sets in $\mathcal{X}$ is called a topology on $\mathcal{X}$.

Usually we consider a set with just one topology, therefore it is acceptable to use the same notation for the set and the corresponding topological space. Rarely we will need to consider different topologies, say $\mathscr{T}_1$ and $\mathscr{T}_2$, on the same set $\mathcal{X}$; in this case, the corresponding topological spaces will be denoted by $(\mathcal{X}, \mathscr{T}_1)$ and $(\mathcal{X}, \mathscr{T}_2)$.

From 2.1c, it follows that the intersection of a finite collection of open sets is open. That is, if V_1, V_2, $\dots, V_n$ are open, then the intersection

$$W = V_1 \cap \dots \cap V_n$$

is open. The latter is proved by induction on n using the identity

$$V_1 \cap \dots \cap V_n = (V_1 \cap \dots \cap V_{n-1}) \cap V_n.$$

B Examples

For any set $\mathcal{X}$, we can define the following topologies:

- The discrete topology — the topology consisting of all subsets of a set $\mathcal{X}$.
- The concrete topology (also known as trivial topology) — the topology consisting of just the whole set $\mathcal{X}$ and the empty set, $\varnothing$.
- The cofinite topology — the topology consisting of the empty set, $\varnothing$ and the complements of finite sets.

By 1.17 and 1.18 any metric space is a topological space if one defines open sets as in the definition 1.15. For example, the real line $\mathbb{R}$ comes with a natural metric which defines a topology on $\mathbb{R}$; if not stated otherwise, the real line $\mathbb{R}$ will be considered with this topology. As follows from Exercise 1.21, different metrics on one set might define the same topology.

A topological space is called metrizable if its topology can be defined by a metric — these examples are most important.

2.2. Exercise. *Assume an infinite set $\mathcal{X}$ equipped with the cofinite topology. Show that $\mathcal{X}$ is not metrizable.*

The so-called connected two-point space is a simple but non-trivial example of a topological space. This space consists of two points

$$\mathcal{X} = \{a, b\}$$

and it has three open sets:

$$\varnothing, \quad \{a\} \quad \text{and} \quad \{a, b\}.$$

It is instructive to check that this is indeed a topology.

2.3. Exercise. *Show that a finite topological space (that is, a finite set equipped with a topology) is metrizable if and only if it is discrete. In particular, the connected two-point space is not metrizable.*

Let us also mention the Zariski topology: it is a topology on $\mathbb{R}^n$ in which the open sets are complements of solution sets of systems of algebraic equations. (For $n = 1$, it is the same as the cofinite topology.)

This topology plays an essential role in algebraic geometry. Unfortunately, proving that this indeed defines a topology requires some commutative algebra, which is beyond the scope of this text.

C Comparison of topologies

Let $\mathscr{W}$ and $\mathscr{S}$ be two topologies on the same set. Suppose $\mathscr{W} \subset \mathscr{S}$; that is, any open set in the $\mathscr{W}$-topology is open in the $\mathscr{S}$-topology. In this case, we say that $\mathscr{W}$ is weaker than $\mathscr{S}$, or, equivalently, $\mathscr{S}$ is stronger than $\mathscr{W}$.[1]

Note that on any set, the concrete topology is the weakest and discrete topology is the strongest.

2.4. Exercise. *Let $\mathscr{W}$ and $\mathscr{S}$ be two topologies on one set. Suppose that for any point x and any $W \in \mathscr{W}$ such that $W \ni x$, there is $S \in \mathscr{S}$ such that $W \supset S \ni x$. Show that $\mathscr{W}$ is weaker than $\mathscr{S}$.*

D Continuous maps

Our challenge is to reformulate definitions from the previous chapter using only open sets. Continuous maps are first in line. The following definition is motivated by Proposition 1.22.

2.5. Definition. *A map between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called continuous if the inverse image of any open set is open. That is, if W is an open subset of $\mathcal{Y}$, then its inverse image*

$$V = f^{-1}(W) = \{\, x \in X \;:\; f(x) \in W \,\}$$

is an open subset of $\mathcal{X}$.

2.6. Exercise. *Let $\mathbb{R}$ be the real line with the standard topology, and let $\mathcal{X} = \{a, b\}$ be the connected two-point space described in 2B — it has only three open sets: $\varnothing$, $\{a\}$, and $\{a, b\}$.*

[1] Some authors use terms smaller or coarser for weaker topology and finer or larger for stronger topology.

(a) *Construct a nonconstant continuous map* $\mathbb{R} \to \mathcal{X}$.
(b) *Show that any continuous function* $\mathcal{X} \to \mathbb{R}$ *is constant.*

2.7. Exercise. *Show that the composition of continuous maps is continuous.*

2.8. Exercise. *Let* $\mathscr{T}$ *be a collection of subsets of* $\mathbb{R}$ *that consists of* $\varnothing$, $\mathbb{R}$ *and the intervals* $[a, \infty)$, (a, ∞) *for all* $a \in \mathbb{R}$.

(a) *Show that* $\mathscr{T}$ *is a topology on* $\mathbb{R}$.
(b) *Show that the topological space* $(\mathbb{R}, \mathscr{T})$ *is not metrizable.*
(c) *Show that a function* $f\colon \mathbb{R} \to \mathbb{R}$ *is nondecreasing if and only if it defines a continuous map* $(\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T})$.

In practice, continuity of maps between topological spaces is often verified indirectly. In particular, one can use 2.7 together with results from 1D and also apply the fact that *any differentiable map defined on a subset of* $\mathbb{R}^n$ *is continuous* (which should be familiar from calculus). Bit latter, we will be more flexible in constructing continuous maps; see 3.11 and 5.3.

2.9. Exercise. *Let* f *be a continuous real-valued function defined on a topological space. Show that the function* $x \mapsto |f(x)|$ *is continuous.*

Chapter 3

Subsets

This chapter starts with the definition of closed sets in a topological space. Further, we introduce constructions of interior, closure, and boundary. Finally, we define neighborhoods and discuss limits in a general topological space.

There is no particular reason why we define a topological space in terms of open sets — we could use closed sets instead. (In fact, closed sets were considered before open sets — the former were introduced by Georg Cantor in 1884 [4], and the latter by René Baire in 1899 [3].)

A Closed sets

Let $\mathcal{X}$ be a topological space. A subset $K \subset \mathcal{X}$ is called closed if its complement $\mathcal{X} \setminus K$ is open.

Sometimes it is easier to use closed sets; for example, the cofinite topology can be defined by declaring that the whole space and all its finite sets are closed.

From the definition of topological spaces, the following properties of closed sets follow.

3.1. Proposition. *Let $\mathcal{X}$ be a topological space.*

(a) The empty set and $\mathcal{X}$ are closed.

(b) The intersection of any collection of closed sets is a closed set. That is, if K_α is closed for any α in the index set $\mathcal{I}$, then the set

$$Q = \bigcap_{\alpha \in \mathcal{I}} K_\alpha = \{\, x \in \mathcal{X} \;:\; x \in K_\alpha \quad \textit{for any} \quad \alpha \in \mathcal{I} \,\}$$

is closed.

(c) *The union of two closed sets (or any finite collection of closed sets) is closed. That is, if* K_1 *and* K_2 *are closed, then the union* $Q = K_1 \cup K_2$ *is closed.*

The following proposition is completely analogous to the original definition of continuous maps via open sets (2.5).

3.2. Proposition. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be topological spaces. A map* $f\colon \mathcal{X} \to \to \mathcal{Y}$ *is continuous if and only if any closed set* $Q \subset \mathcal{Y}$ *has a closed inverse image* $f^{-1}(Q) \subset \mathcal{X}$.

Proof. In the proof, we will use the following set-theoretical identity. Suppose $A \subset \mathcal{Y}$ and $B = \mathcal{Y} \setminus A$ (or, equivalently, $A = \mathcal{Y} \setminus B$). Then

❶ $$f^{-1}(B) = \mathcal{X} \setminus f^{-1}(A)$$

for any map $f\colon \mathcal{X} \to \mathcal{Y}$. This identity is tautological; to prove it, observe that both sides can be spelled as

$$\{\, x \in \mathcal{X} \;:\; f(x) \notin A \,\}.$$

Only-if part. Let $B \subset \mathcal{Y}$ be a closed set. Then $A = \mathcal{Y} \setminus B$ is open. Since f is continuous, $f^{-1}(A)$ is open. By ❶, $f^{-1}(B)$ is the complement of $f^{-1}(A)$ in $\mathcal{X}$. Hence $f^{-1}(B)$ is closed.

The only-if part follows since B is an arbitrary closed set in $\mathcal{Y}$.

If part. Fix an open set B; its complement $A = \mathcal{Y} \setminus B$ is closed. Therefore $f^{-1}(A)$ is closed. By ❶, $f^{-1}(B)$ is the complement of $f^{-1}(A)$ in $\mathcal{X}$. Hence $f^{-1}(B)$ is open.

The if part follows since B is an arbitrary open set in $\mathcal{Y}$. □

3.3. Exercise.

(a) *Let* $\mathcal{X}$ *be a metrizable topological space. Show that any closed set in* $\mathcal{X}$ *is an intersection of a collection of open sets.*

(b) *Construct a topological space* $\mathcal{Y}$ *with a closed set* Q *that is not an intersection of any collection of open sets.*

B Interior and closure

Let A be an arbitrary subset of a topological space $\mathcal{X}$.

The union of all open subsets of A is called the interior of A and denoted as $\mathring{A}$ or $\operatorname{Int} A$.

Note that $\mathring{A}$ is open. Indeed, it is defined as a union of open sets and such union is open by the definition of a topology (2.1). So we can say that $\mathring{A}$ is the *maximal* open set in A, as any open subset of A lies in $\mathring{A}$.

The intersection of all closed subsets containing A is called the closure of A and denoted as $\bar{A}$ or $\operatorname{Cl} A$.

The set $\bar{A}$ is closed. Indeed, it is defined as an intersection of closed sets and such intersection has to be closed by 3.1. In other words, $\bar{A}$ is the *minimal* closed set that contains A, as any closed subset containing A contains $\bar{A}$.

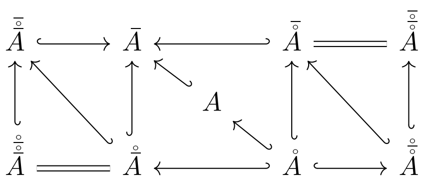
3.4. Exercise. *Assume A is a subset of a topological space $\mathcal{X}$; consider its complement $B = \mathcal{X} \setminus A$. Show that*

$$\bar{B} = \mathcal{X} \setminus \mathring{A}.$$

The following exercise is called the Kuratowski problem.

3.5. Advanced exercise.

(a) *Prove that all the inclusions and equalities shown on the diagram hold for any subset A in a topological space $\mathcal{X}$. Conclude that*



given a set A one can get at most 7 distinct subsets by repeatedly applying the set operations of closure and interior.

(b) *Give an example of a topological space $\mathcal{X}$ with a subset A such that all the following 7 subsets are distinct:*

$$\bar{\mathring{\bar{A}}},\ \mathring{\bar{A}},\ \bar{A},\ A,\ \mathring{A},\ \bar{\mathring{A}},\ \mathring{\bar{\mathring{A}}}.$$

C Boundary

Let A be an arbitrary subset of a topological space $\mathcal{X}$. The boundary of A (briefly ∂A) is defined as the complement

$$\partial A = \bar{A} \setminus \mathring{A}.$$

3.6. Exercise. *Show that the boundary of any set is closed.*

3.7. Exercise. *Show that the set A is closed if and only if $\partial A \subset A$.*

3.8. Advanced exercise. *Find three disjoint open sets on the real line that have the same nonempty boundary.*

D Subspaces

3.9. Proposition. *Let A be a subset of a topological space $\mathcal{Y}$. Then all subsets $V \subset A$ such that $V = A \cap W$ for some open set W in $\mathcal{Y}$ form a topology on A.*

The described topology is called the induced topology on A.

A subset A of a topological space $\mathcal{Y}$ equipped with the induced topology is called a subspace of $\mathcal{Y}$. It is straightforward to check that this notion agrees with the notion introduced in 1C; that is, if $\mathcal{Y}$ is a metric space, then any subset $A \subset \mathcal{Y}$ comes with a metric, and the topology defined by this metric coincides with the induced topology on A.

Proof. We need to check the conditions in 2.1.

First, the whole set A and the empty set are included; indeed, $\varnothing = A \cap \varnothing$ and $A = A \cap \mathcal{Y}$.

Assume $\{V_\alpha\}$ is a collection of open sets in A; here α runs in some index set, say $\mathcal{I}$. In other words, for each V_α there is an open set $W_\alpha \subset \mathcal{Y}$ such that $V_\alpha = A \cap W_\alpha$. Note that

$$\bigcup_\alpha V_\alpha = A \cap \left(\bigcup_\alpha W_\alpha\right).$$

Since the union of $\{W_\alpha\}$ is open in $\mathcal{Y}$ (2.1b), the union of $\{V_\alpha\}$ is open in the induced topology on A.

Assume V_1 and V_2 are open in A; that is, $V_1 = A \cap W_1$ and $V_2 = A \cap W_2$ for some open sets $W_1, W_2 \subset \mathcal{Y}$. Note that

$$V_1 \cap V_2 = A \cap (W_1 \cap W_2).$$

Since the intersection $W_1 \cap W_2$ is open in $\mathcal{Y}$ (2.1c), the intersection $V_1 \cap V_2$ is open in the induced topology on A. $\square$

3.10. Exercise. *Let $\mathcal{A}$ be a subspace in $\mathcal{X}$. Given a set $S \subset \mathcal{A}$ denote by $\operatorname{Int}_{\mathcal{A}} S$, $\operatorname{Cl}_{\mathcal{A}} S$, $\operatorname{Int}_{\mathcal{X}} S$, and $\operatorname{Cl}_{\mathcal{X}} S$ its interior and closure in $\mathcal{A}$ and $\mathcal{X}$, respectively. Show that $\operatorname{Int}_{\mathcal{A}} S \supset \operatorname{Int}_{\mathcal{X}} S$ and $\operatorname{Cl}_{\mathcal{A}} S \subset \operatorname{Cl}_{\mathcal{X}} S$. Provide examples showing that these inclusions might be strict.*

Let A be a subset of a topological space $\mathcal{X}$ and let $\mathcal{Y}$ be another topological space. A map $h\colon A \to \mathcal{Y}$ is called continuous if it is continuous with respect to the induced topology on A.

Recall the restriction of a map $f : \mathcal{X} \to \mathcal{Y}$ to $A \subset \mathcal{X}$ is obtained by keeping the same rule as f but only allowing inputs from A. It is usually denoted by $f|_A$; so $(f|_A)(a) = f(a)$ if $a \in A$ and otherwise it is undefined.

Observe that $(f|_A)^{-1}(W) = A \cap f^{-1}(W)$ for any $W \subset \mathcal{Y}$. It follows that if f is continuous, then so is $f|_A\colon A \to \mathcal{Y}$; here we consider set A with the induced topology. The following exercise provides a partial converse.

3.11. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map between topological spaces. Suppose $A, B \subset \mathcal{X}$ are sets such that $A \cup B = \mathcal{X}$ and either*

(a) both A and B are open sets, or

(b) both A and B are closed sets.

Show that f is continuous if and only if so are the restrictions $f|_A$ and $f|_B$.

(c) Show that this is no longer true without assumptions of (a) or (b).

E Neighborhoods

Let x be a point in a topological space $\mathcal{X}$. A neighborhood of x is any open set N containing x.

In topology, neighborhoods often replace balls, which makes sense only in metric spaces.

3.12. Exercise. *Let A be a set in a topological space $\mathcal{X}$. Show that $x \in \partial A$ if and only if any neighborhood of x contains points of A and of its complement $\mathcal{X} \setminus A$.*

Let A and B be subsets of a topological space $\mathcal{X}$. The set A is said to be dense in B if $\bar{A} \supset B$.

3.13. Exercise. *Show that A is dense in B if and only if any neighborhood of any point in B intersects A.*

F Limits

3.14. Definition. *Suppose x_n is a sequence of points in a topological space $\mathcal{X}$. We say that x_n converges to a point $x_\infty \in \mathcal{X}$ (briefly*

$x_n \to x_\infty$ as $n \to \infty$) if for any neighborhood N of x_∞, we have that $x_n \in N$ for all sufficiently large n.

Observe that the above definition agrees with 1.24. In other words, a sequence of points $x_1, x_2, \dots$ in a metric space converges to a point x_∞ in the sense of the definition 1.24 if and only if it converges in the sense of the definition 3.14.

3.15. Exercise. *Show that a convergent sequence of points in a topological space is also convergent for every weaker topology.*

Note that in a space with the concrete topology any sequence converges to any point. In particular, a sequence might have several different limits. Furthermore, if we equip $\mathbb{R}$ with the cofinite topology then any sequence of pairwise distinct numbers converges to every point in $\mathbb{R}$.

The following exercise shows that converging sequences do *not* adequately describe the topology of a space; namely, an analog of 1.26 does not hold.[1]

Recall that a set is called countable if it admits a bijection to a subset of the set of natural numbers. In particular, all finite sets are countable.

3.16. Advanced exercise. *Let $\mathcal{X}$ be $\mathbb{R}$ with the so-called cocountable topology; its closed sets are either countable or the whole $\mathbb{R}$.*

(a) Construct a map $f\colon \mathcal{X} \to \mathcal{X}$ that is not continuous.

(b) Describe all convergent sequences in $\mathcal{X}$.

(c) Show that if the sequence x_n converges to x_∞ in $\mathcal{X}$, then for any map $f\colon \mathcal{X} \to \mathcal{X}$ the sequence $y_n = f(x_n)$ converges to $y_\infty = f(x_\infty)$.

[1] So-called nets [14] provide an appropriate generalization of sequences that works well in topological spaces, but we are not going to consider them.

Chapter 4

Maps

Recall that continuous maps were defined in 2D; now we will discuss their relatives.

A Homeomorphisms

A bijection $f\colon\mathcal{X}\to\mathcal{Y}$ between topological spaces is called a homeomorphism if f and its inverse $f^{-1}\colon\mathcal{Y}\to\mathcal{X}$ are continuous.[1]

Topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are called homeomorphic (briefly, $\mathcal{X}\simeq\mathcal{Y}$) if there is a homeomorphism $f\colon\mathcal{X}\to\mathcal{Y}$.

A map $f\colon\mathcal{X}\to\mathcal{Y}$ is called an embedding if f defines a homeomorphism from $\mathcal{X}$ to the subspace $f(\mathcal{X})$ in $\mathcal{Y}$.

4.1. Exercise. *Show that any homeomorphism is a continuous bijection.*

Give an example of continuous bijection between topological spaces that is not a homeomorphism.

4.2. Exercise. *Show that $x\mapsto e^x$ is a homeomorphism $\mathbb{R}\to(0,\infty)$.*

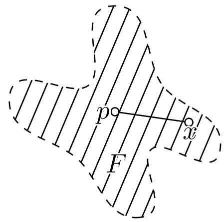
4.3. Exercise. *Construct a homeomorphism $f\colon\mathbb{R}\to(0,1)$.*

4.4. Exercise. *Show that $\simeq$ is an equivalence relation; that is, for any topological spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ we have the following:*

(a) $\mathcal{X}\simeq\mathcal{X}$;
(b) if $\mathcal{X}\simeq\mathcal{Y}$, then $\mathcal{Y}\simeq\mathcal{X}$;
(c) if $\mathcal{X}\simeq\mathcal{Y}$ and $\mathcal{Y}\simeq\mathcal{Z}$, then $\mathcal{X}\simeq\mathcal{Z}$.

[1] The term *homomorphism* from abstract algebra looks similar and it has a similar meaning but should not be confused with a *homeomorphism*.

4.5. Advanced exercise. *Prove that the complement of a circle in the Euclidean space is homeomorphic to the Euclidean space without a line ℓ and a point $p \notin \ell$.*



Recall that a figure F is called *star-shaped* if there exists a point $p \in F$ such that for all $x \in F$ the line segment px lies in F.

4.6. Advanced exercise. *Show that any nonempty open star-shaped set in the plane is homeomorphic to the open disc.*

4.7. Advanced exercise. *Show that the complements of two countable dense subsets of the plane are homeomorphic.*

B Closed and open maps

4.8. Definition. *A map between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called open if, for any open set $V \subset \mathcal{X}$, the image $f(V)$ is open in $\mathcal{Y}$.*

A map between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called closed if, for any closed set $Q \subset \mathcal{X}$, the image $f(Q)$ is closed in $\mathcal{Y}$.

Note that a homeomorphism can be defined as a *continuous open bijection.*

4.9. Exercise. *Show that a bijective map between topological spaces is closed if and only if it is open.*

4.10. Exercise. *Give an example of a map f between topological spaces such that*

(a) f is continuous and open, but not closed,
(b) f is continuous and closed, but not open,
(c) f is closed and open, but not continuous.

4.11. Advanced exercise. *Construct a map $\mathbb{R} \to \mathbb{R}$ that is open, but not continuous.*

Chapter 5

Constructions

Here we introduce several constructions that produce new topological spaces from the given ones.

A Product space

Recall that $\mathcal{X} \times \mathcal{Y}$ denotes the set of all pairs (x, y) such that $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Suppose that the sets $\mathcal{X}$ and $\mathcal{Y}$ are equipped with topologies. Let us construct the product topology on $\mathcal{X} \times \mathcal{Y}$ by declaring that a set is open in $\mathcal{X} \times \mathcal{Y}$ if it can be presented as a union of sets of the following type: $V \times W$ for open sets $V \subset \mathcal{X}$ and $W \subset \mathcal{Y}$. In other words, a subset U is open in $\mathcal{X} \times \mathcal{Y}$ if and only if there are collections of open sets $V_\alpha \subset \mathcal{X}$ and $W_\alpha \subset \mathcal{Y}$ such that

$$U = \bigcup_\alpha V_\alpha \times W_\alpha,$$

here α runs in some index set.

By default, we assume that $\mathcal{X} \times \mathcal{Y}$ is equipped with the product topology; in this case, $\mathcal{X} \times \mathcal{Y}$ is called the product space.

5.1. Proposition. *The product topology is indeed a topology.*

Proof. Parts *(a)* and *(b)* in 2.1 are evident. It remains to check *(c)*. Consider two sets

$$U = \bigcup_\alpha V_\alpha \times W_\alpha \quad \text{and} \quad U' = \bigcup_\beta V'_\beta \times W'_\beta.$$

where α and β run in some index sets, say $\mathcal{I}$ and $\mathcal{J}$ respectively. We need to show that $U \cap U'$ can be presented as a union of products of open sets; the latter follows from the next set-theoretical identity

❶ $$U \cap U' = \bigcup_{\alpha,\beta} (V_\alpha \cap V'_\beta) \times (W_\alpha \cap W'_\beta).$$

Checking ❶ is straightforward. Indeed, $(x, y) \in U \cap U'$ means that $(x, y) \in U$ and $(x, y) \in U'$; the latter means that $x \in V_\alpha$, $y \in W_\alpha$ and $x \in V'_\beta$, $y \in W'_\beta$ for *some* α and β. In other words, $x \in V_\alpha \cap V'_\beta$ and $y \in W_\alpha \cap W'_\beta$ for *some* α and β; the latter means that (x, y) belongs to the right-hand side in ❶. □

By spelling the definition of product topology, we get the following statement.

5.2. Observation. *Two maps $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{X} \to \mathcal{Z}$ are continuous if and only if the $x \mapsto (f(x), g(x))$ defines a continuous map $\mathcal{X} \to \mathcal{Y} \times \mathcal{Z}$.*

5.3. Exercise. *Let f and g be continuous real-valued functions on the topological space $\mathcal{X}$. Show that (a) $h_1 = f + g$, (b) $h_2 = f \cdot g$, and (c) $h_3(x) := \max\{ f(x), g(x) \}$ are continuous.*

5.4. Exercise. *Construct a function $f\colon [0, 1] \times [0, 1] \to [0, 1]$ such that f is not continuous, but the functions $x \mapsto f(a, x)$ and $x \mapsto f(x, b)$ are continuous for any $a, b \in [0, 1]$.*

B Base

5.5. Definition. *A collection $\mathscr{B}$ of open sets in a topological space $\mathcal{X}$ is called its b a s e if every open set in $\mathcal{X}$ is a union of sets in $\mathscr{B}$.*

By 1.19, *open balls form a base of a metric space.*

A base completely defines its topology, but typically a topology has many different bases. In metric spaces, for example, the set of all balls with rational radii is a base; another example is the set of all balls with radii smaller than 1.

In many cases, it is convenient (and also economical) to describe the topology by specifying its base. For example, the product topology on $\mathcal{X} \times \mathcal{Y}$ can be redefined as a *topology with a base formed by all products $V \times W$, where V is open in $\mathcal{X}$, and W is open in $\mathcal{Y}$.*

5.6. Exercise. *Let $\mathscr{B}$ be a base for the topology on $\mathcal{Y}$. Show that a map $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if and only if $f^{-1}(B)$ is open for any set B in $\mathscr{B}$.*

5.7. Exercise. *Let $\mathscr{B}$ be a collection of open sets in a topological space $\mathcal{X}$. Show that $\mathscr{B}$ is a base in $\mathcal{X}$ if and only if for any point $x \in \mathcal{X}$ and any neighborhood $N \ni x$ there is $B \in \mathscr{B}$ such that $x \in B \subset N$.*

5.8. Proposition. *Let $\mathscr{B}$ be a set of subsets of some set $\mathcal{X}$. Then $\mathscr{B}$ is a base of some topology on $\mathcal{X}$ if and only if it satisfies the following conditions:*

(a) $\mathscr{B}$ covers $\mathcal{X}$; that is, every point $x \in \mathcal{X}$ lies in some set $B \in \mathscr{B}$.
(b) For each pair of sets $B_1, B_2 \in \mathscr{B}$ and each point $x \in B_1 \cap B_2$ there exists a set $B \in \mathscr{B}$ such that $x \in B \subset B_1 \cap B_2$.

Proof. Denote by $\mathscr{O}$ the set of all unions of sets in $\mathscr{B}$. We need to show that $\mathscr{O}$ is a topology on $\mathcal{X}$.

Evidently, the union of any collection of sets in $\mathscr{O}$ is in $\mathscr{O}$. Further, $\mathcal{X}$ is in $\mathscr{O}$ by *(a)*. The empty set is in $\mathscr{O}$ since it is a union of the empty collection.

It remains to check 2.1c; suppose

$$O = \bigcup_{\alpha} B_\alpha \quad \text{and} \quad O' = \bigcup_{\beta} B'_\beta,$$

where α and β run in some index sets, and B_α, $B'_\beta \in \mathscr{B}$ for any α and β. Then $x \in O \cap O'$ if and only if for some α and β we have $x \in B_\alpha$ and $x \in B'_\beta$. By *(b)*, we can choose $B \in \mathscr{B}$ so that $x \in B \subset B_\alpha \cap B'_\beta$. Since $B_\alpha \cap B'_\beta \subset O \cap O'$, it follows that

for any $x \in O \cap O'$ there is $B_x \in \mathscr{B}$ such that $x \in B_x \subset O \cap O'$.

Observe that

$$O \cap O' = \bigcup_{x \in O \cap O'} B_x.$$

It follows that $O \cap O' \in \mathscr{O}$ if $O, O' \in \mathscr{O}$. □

5.9. Exercise. *Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of natural numbers, and let $\mathscr{B}$ be a set of all arithmetic progressions in $\mathbb{N}$; that is, $\mathscr{B}$ includes $\{a, a + d, a + 2{\cdot}d, \dots\}$ for any $a, d \in \mathbb{N}$.*

Show that $\mathscr{B}$ is a base of some topology on $\mathbb{N}$. Is it true that in this topology the set $\{1\}$ is open and/or closed?

C Prebase

Suppose $\mathscr{P}$ is a collection of subsets of $\mathcal{X}$ that covers the whole space; that is, $\mathcal{X}$ is a union of all sets in $\mathscr{P}$. By 5.8, the set of all finite intersections of sets in $\mathscr{P}$ is a base for *some* topology on $\mathcal{X}$. The set $\mathscr{P}$ is called a prebase for this topology (also known as a subbase).

5.10. Exercise. *Let $\mathscr{P}$ be a prebase for the topology on $\mathcal{Y}$. Show that a map $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if and only if $f^{-1}(P)$ is open for any set P in $\mathscr{P}$.*

There are almost no restrictions on a prebase — we may start with any collection $\mathscr{P}$ of subsets that covers the whole space $\mathcal{X}$ and define a topology by declaring that *$\mathscr{P}$ is a prebase for the topology.* It defines the weakest topology on $\mathcal{X}$ such that every set of $\mathscr{P}$ is open.

5.11. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces.*

(a) Show that the product topology on $\mathcal{X} \times \mathcal{Y}$ can be redefined as a topology with a prebase formed by all products $\mathcal{X} \times W$ and $V \times \mathcal{Y}$, where V is open in $\mathcal{X}$ and W is open in $\mathcal{Y}$.

(b) Apply part (a) to prove the following: given a map $f\colon \mathcal{X} \to \mathcal{Y}$, consider the map $F\colon \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ defined by $F\colon x \mapsto (x, f(x))$. Show that f is continuous if and only if F is an embedding.

D Initial topology

Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. Note that the product topology on $\mathcal{X} \times \mathcal{Y}$ is the weakest topology such that the following two projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$ respectively are continuous.

Indeed, these projections are continuous if the inverse images of all open sets in $\mathcal{X}$ and $\mathcal{Y}$ are open in $\mathcal{X} \times \mathcal{Y}$. In other words, the topology on $\mathcal{X} \times \mathcal{Y}$ must contain all sets of the form $V \times \mathcal{Y}$ and $\mathcal{X} \times W$ for open sets $V \subset \mathcal{X}$ and $W \subset \mathcal{Y}$. By 5.11a, these sets form a prebase in $\mathcal{X} \times \mathcal{Y}$ and its topology is the weakest topology that contains these sets.

More generally, given a collection of maps $f_\alpha\colon S \to \mathcal{Y}_\alpha$ from a set S to topological spaces $\mathcal{Y}_\alpha$, we can introduce a topology on S by stating that the inverse images $f_\alpha^{-1}(W_\alpha)$ for open sets $W_\alpha \subset \mathcal{Y}_\alpha$ form its prebase. It defines the topology on S induced by the maps f_α; it is the weakest topology on S that makes all maps f_α continuous.

This construction produces a topology on the source space; for that reason it is also called the initial topology.

Note that induced topology on a subset $A \to \mathcal{Y}_\alpha$ discussed in 3D is the initial topology for the inclusion map $A \to \mathcal{Y}_\alpha$. Furthermore, this construction can be used to define a topology on an infinite product of spaces, as the induced topology for all its projections. This topology is called the product topology, or the Tychonoff topology.

5.12. Advanced exercise. *Let $\mathcal{O}$ be the topology on $\mathbb{R}$ induced by the maps $x \mapsto (\cos(t{\cdot}x), \sin(t{\cdot}x))$ for all $t \in \mathbb{R}$. Show that the space $(\mathbb{R}, \mathcal{O})$ is not metrizable.*

E Final topology

In the previous section, we defined a natural way to move a topology from the target space to the source of a map. Namely, suppose $f\colon \mathcal{X} \to \mathcal{Y}$ is a map between two sets. If $\mathcal{Y}$ is equipped with a topology, then we can declare a subset $V \subset \mathcal{X}$ to be open if there is an open subset $W \subset \mathcal{Y}$ such that $V = f^{-1}(W)$.

The following exercise describes an analogous construction that moves a topology from source to target. It can be solved by checking the conditions in 2.1 as we did in 3D.

5.13. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map between two sets. Assume $\mathcal{X}$ is equipped with a topology. Declare a subset $W \subset \mathcal{Y}$ to be open if the subset $V = f^{-1}(W)$ is open in $\mathcal{X}$. Show that it defines a topology on $\mathcal{Y}$.*

The constructed topology on $\mathcal{Y}$ is called the final topology for f.

5.14. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map between topological spaces.*

(a) Show that the initial topology for f on $\mathcal{X}$ is weaker than its own topology.

(b) Show that the final topology for f on $\mathcal{Y}$ is stronger than its own topology.

5.15. Exercise. *Let $g\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map.*

(a) Suppose $\mathcal{X}$ is equipped with the initial topology. Show that a map $f\colon \mathcal{W} \to \mathcal{X}$ is continuous if and only if the composition $g \circ f\colon \mathcal{W} \to \mathcal{Y}$ is continuous.

(b) Suppose $\mathcal{Y}$ is equipped with the final topology. Show that a map $h\colon \mathcal{Y} \to \mathcal{Z}$ is continuous if and only if the composition $h \circ g\colon \mathcal{X} \to \to \mathcal{Z}$ is continuous.

5.16. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous surjective map. Assume f is (a) open or (b) closed. Show that $\mathcal{Y}$ is equipped with the final topology.*

F Quotient topology

The initial topology is used mostly for injective maps; in this case, it is essentially the same as the induced topology. Similarly, the final topology is mostly used for surjective maps. This particular case of the construction is called the quotient topology, which we are about to describe.

Let $\sim$ be an equivalence relation on a topological space $\mathcal{X}$; that is, for any $x, y, z \in \mathcal{X}$ the following conditions hold:

- $x \sim x$;
- if $x \sim y$, then $y \sim x$;
- if $x \sim y$ and $y \sim z$, then $x \sim z$.

Recall that the set

$$[x] = \{\, y \in \mathcal{X} \;:\; y \sim x \,\}$$

is called the equivalence class of x. The set of all equivalence classes in $\mathcal{X}$ will be denoted by $\mathcal{X}/{\sim}$.

Observe that $x \mapsto [x]$ defines a surjective map $\mathcal{X} \to \mathcal{X}/{\sim}$. The corresponding final topology on $\mathcal{X}/{\sim}$ is called the quotient topology on $\mathcal{X}/{\sim}$. By default, $\mathcal{X}/{\sim}$ is equipped with the quotient topology; in this case, it is called a quotient space.

The following exercise ties equivalence relations with maps.

5.17. Exercise. *Show that an arbitrary map $f\colon \mathcal{X} \to \mathcal{Y}$ defines the following equivalence relation on $\mathcal{X}$:*

$$x \sim x' \quad \text{if and only if} \quad f(x) = f(x').$$

Moreover,

$$y = f(x) \quad \text{if and only if} \quad [x] = f^{-1}\{f(x)\}.$$

Given a set A in a topological space $\mathcal{X}$, the space $\mathcal{X}/A$ is defined as the quotient space $\mathcal{X}/{\sim}$ for the minimal equivalence relation such that $a \sim b$ for any $a, b \in A$.

5.18. Exercise. *Describe the quotient space $[0,1]/(0,1)$, where $[0,1]$ and $(0,1)$ are real intervals; that is, list the points and the open sets of the quotient space.*

One may also check by hand that the quotient space $[0,1]/\{0,1\} \simeq \simeq \mathbb{S}^1$; that is, the unit segment with identified ends is homeomorphic to the circle. Theorem 8.12 will provide a very general tool that helps to prove such statements.

G Actions

Let $\mathcal{X}$ be a set. A subset G of bijections $\mathcal{X} \to \mathcal{X}$ that includes the identity map $\mathrm{id}_{\mathcal{X}}$ and is closed under composition and inversion is called a group of symmetries of $\mathcal{X}$. In this case we may say that G acts on $\mathcal{X}$, briefly $G \curvearrowright \mathcal{X}$.

Typically we are interested in group actions that preserve some structure of the set $\mathcal{X}$. If $\mathcal{X}$ is a topological space, then it is natural to assume that each $g \in G$ is a continuous bijection $\mathcal{X} \to \mathcal{X}$. Such an action $G \curvearrowright \mathcal{X}$ is called continuous. For such an action each $g \in G$ defines a homeomorphism $\mathcal{X} \to \mathcal{X}$. Indeed, if $g \in G$, then $g^{-1} \in G$; therefore g as well as its inverse must be continuous. Actions on topological spaces will always be assumed to be continuous.

Given an action $G \curvearrowright \mathcal{X}$, we can equip the set G with product defined as composition; that is, if $g, h \in G$ then its product $g \cdot h \in G$ is defined as $(g \cdot h)(x) := g(h(x))$. With this operation, G becomes a group; that is, it meets the following properties:

- (associativity) for any $f, g, h \in G$ we have

$$(f \cdot g) \cdot h = f \cdot (g \cdot h);$$

- (existence of neutral element) there exists $1 \in G$ such that for every $g \in G$ we have

$$1 \cdot g = g \cdot 1 = g;$$

- (inverse) for each $g \in G$ there exists $g^{-1} \in G$ such that

$$g \cdot g^{-1} = g^{-1} \cdot g = 1.$$

Checking these properties is straightforward. Associativity follows since it holds for composition, and one can take $1 = \mathrm{id}_{\mathcal{X}}$.

Finally note that an abstract group (that is, a set G with product that meets the three properties above) acts on itself by $g \colon h \mapsto g \cdot h$, so any group acts on some set. Furthermore, we may equip G with discrete topology and this way we obtain a continuous action $G \curvearrowright G$. So you may think that actions are more fundamental than groups. A group should be defined by its action, not the other way around.

For an action $G \curvearrowright \mathcal{X}$, it is common to denote $g(x)$ by $g \cdot x$, where $g \in G$ and $x \in \mathcal{X}$. Note that associativity implies that the expression

$$g \cdot h \cdot x$$

makes sense for any $g, h \in G$ and $x \in \mathcal{X}$; that is, it does not depend on parentheses.

H Orbit spaces

Choose a continuous action $G \curvearrowright \mathcal{X}$. The set

$$G\cdot x := \{\, g\cdot x \in \mathcal{X} \;:\; g \in G \,\}$$

is called the G-orbit of x (or, briefly, orbit).

Set $x \sim y$ if there is $g \in G$ such that $y = g\cdot x$. Observe that $\sim$ is an equivalence relation on $\mathcal{X}$. Indeed, $x \sim x$ since $x = 1\cdot x$. Further, if $y = g\cdot x$, then

$$x = 1\cdot x = g^{-1}\cdot g\cdot x = g^{-1}\cdot y;$$

since $g^{-1} \in G$ we get that $x \sim y \Longrightarrow y \sim x$. Finally, suppose $x \sim y$ and $y \sim z$; that is, $y = g\cdot x$ and $z = h\cdot y$ for some $g, h \in G$, then $z = (h\cdot g)\cdot x$; therefore $x \sim z$.

For the described equivalence relation, the quotient space $\mathcal{X}/{\sim}$ can also be denoted by $\mathcal{X}/G$; it is called the quotient of $\mathcal{X}$ by the action of G.

Note that $[x] = G\cdot x$; that is, the orbit of x coincides with its equivalence class. For that reason, $\mathcal{X}/G$ is also called the orbit space.

5.19. Exercise. *Positive real numbers $\mathbb{R}_+$ act on $\mathbb{R}$ by multiplication. Describe the quotient space $\mathbb{R}/\mathbb{R}_+$; that is, list the points and the open sets of the quotient space.*

5.20. Exercise. *Suppose a group G acts on a topological space $\mathcal{X}$ and $f\colon \mathcal{X} \to \mathcal{X}/G$ is the quotient map.*

(a) Show that f is open.

(b) Assume G is finite. Show that f is closed.

Chapter 6

Compactness

In this chapter we define compact topological spaces — especially nice spaces that behave in many ways like finite sets, yet are far more general.

In practice, compactness is a powerful replacement for "closed and bounded" in $\mathbb{R}^n$, but the general definition does not resemble it at first glance. On compact spaces continuous functions behave well: they are bounded and attain their maxima and minima (6.9). Moreover, many limiting and convergence arguments become simpler and more robust.

A Definition

We will denote by $\{V_\alpha\} = \{V_\alpha\}_{\alpha\in\mathcal{I}}$ a collection of sets, where α runs in an arbitrary index set $\mathcal{I}$.

6.1. Definition. *A collection $\{V_\alpha\}$ of open subsets of a topological space $\mathcal{X}$ is called its open cover if it covers the whole $\mathcal{X}$; that is, if every $x \in \mathcal{X}$ belongs to some V_α.*

More generally, $\{V_\alpha\}$ is an open cover of a subset $S \subset \mathcal{X}$ if any $s \in S$ belongs to some V_α.

A subset of cover $\{V_\alpha\}$ that is also a cover is called its subcover of $\{V_\alpha\}$.

6.2. Exercise. *Let $\{V_\alpha\}$ be an open cover of a topological space $\mathcal{X}$. Show that $W \subset \mathcal{X}$ is open if and only if for any α the intersection $W \cap V_\alpha$ is open.*

Conclude that a map $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if for any α the restriction $f|_{V_\alpha}\colon V_\alpha \to \mathcal{Y}$ is continuous.

6.3. Definition. *A topological space $\mathcal{X}$ is called* compact *if any cover $\{V_\alpha\}$ of $\mathcal{X}$ contains a finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$.*

Analogously, a subset S of a topological space $\mathcal{X}$ is called compact *if any cover of S contains a finite subcover of S.*

6.4. Exercise. *Show that a subset S of a topological space is compact if and only if S equipped with the induced topology is a compact space.*

Clearly, *any finite topological space is compact.* In fact, compact spaces in topology, in many ways, resemble finite sets in set theory. The next exercise provides a source of examples of infinite compact spaces. More interesting examples are given in Section 6C.

6.5. Exercise. *Show that any set equipped with the cofinite topology is compact.*

6.6. Exercise. *Let S be an unbounded subset of the real line; that is, for any $c \in \mathbb{R}$ there is $s \in S$ such that $|s| > c$. Show that S is not compact.*

6.7. Exercise. *Let $S \subset \mathbb{R}$ be a subset that is not closed. Show that S is not compact.*

6.8. Exercise. *Construct a topological space with two compact sets such that their intersection is not compact.*

6.9. Exercise. *Let f be a continuous real-valued function that is defined on a compact space $\mathcal{K}$.*

(a) Show that f is bounded; that is there is a constant C such that $|f(x)| \leqslant C$ for any $x \in \mathcal{K}$.

(b) Show that f attains its maximum and minimum; that is there are points $x_{\min}, x_{\max} \in \mathcal{K}$ such that

$$f(x_{\min}) \leqslant f(x) \leqslant f(x_{\max})$$

for any $x \in \mathcal{K}$.

B Finite intersection property

6.10. Proposition. *A space $\mathcal{X}$ is compact if for any collection of closed sets $\{Q_\alpha\}$ in $\mathcal{X}$ such that*

$$\bigcap_\alpha Q_\alpha = \varnothing$$

There is a finite collection $\{Q_{\alpha_1}, \dots, Q_{\alpha_n}\}$ *such that*

$$Q_{\alpha_1} \cap \dots \cap Q_{\alpha_n} = \varnothing.$$

The condition in the above proposition is called the finite intersection property; it redefines compactness via closed sets.

Proof. By the definition of closed sets, the complements $V_\alpha = \mathcal{X} \setminus Q_\alpha$ are open. Note that

$$\bigcup_\alpha V_\alpha = \mathcal{X} \setminus \left(\bigcap_\alpha Q_\alpha \right) = \mathcal{X};$$

that is, $\{V_\alpha\}$ is an open cover of $\mathcal{X}$.

Choose a finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$; so, $V_{\alpha_1} \cup \dots \cup V_{\alpha_n} = \mathcal{X}$. Observe that

$$Q_{\alpha_1} \cap \dots \cap Q_{\alpha_n} = \mathcal{X} \setminus (V_{\alpha_1} \cup \dots \cup V_{\alpha_n}) = \varnothing. \qquad \square$$

Cantor set is constructed the following way: start with the unit interval $[0,1]$, subdivide it into three equal inervals and remove the interior of the middle one. Repeat this procedure recursively for each of the remaining closed intervals. Observe that the following exercise imlies that the Cantor set is not empty.

6.11. Exercise. *Let* $Q_1 \supset Q_2 \supset \dots$ *be a nested sequence of closed nonempty sets in a compact space* $\mathcal{K}$. *Show that there is a point* $q \in \mathcal{K}$ *such that* $q \in Q_n$ *for any* n.

6.12. Advanced exercise. *Let* $Q_1, Q_2, \dots$ *be a sequence of nonempty disjoint closed sets in* $[0,1]$. *Show that* $Q_1 \cup Q_2 \cup \dots \neq [0,1]$.

C Real interval

6.13. Theorem. *Any closed interval* $[a,b]$ *is a compact subset of* $\mathbb{R}$.

Proof. Set $a_0 = a$ and $b_0 = b$, so $[a,b] = [a_0, b_0]$.

Arguing by contradiction, assume that there is an open cover $\{V_\alpha\}$ of $[a_0, b_0]$ that has no finite subcovers.

Note that $\{V_\alpha\}$ is also a cover for two intervals

$$[a_0, \tfrac{a_0+b_0}{2}] \quad \text{and} \quad [\tfrac{a_0+b_0}{2}, b_0].$$

Furthermore, if $\{V_\alpha\}$ has a finite subcover of each of these two subintervals, then these two subcovers together give a finite cover of $[a,b]$. Thus $\{V_\alpha\}$ must have no finite subcovers of *at least one* of these subintervals; denote it by $[a_1,b_1]$; so either $a_1 = a_0$ and $b_1 = \frac{a_0+b_0}{2}$ or $a_1 = \frac{a_0+b_0}{2}$ and $b_1 = b_0$.

Continuing in this manner we get a sequence of intervals

$$[a_0,b_0] \supset [a_1,b_1] \supset \dots$$

such that no finite collection of sets from $\{V_\alpha\}$ covers any of the intervals $[a_n,b_n]$. In particular,

❶ $$[a_n,b_n] \not\subset V_\alpha \quad \text{for any } n \text{ and } \alpha.$$

Observe that

$$a_0 \leqslant a_1 \leqslant \dots \leqslant b_1 \leqslant b_0 \quad \text{and} \quad b_n - a_n = \tfrac{b-a}{2^n}.$$

Denote by x the least upper bound of $\{a_n\}$. Note that $x \in [a_n,b_n]$ for any n.[1]

Since $\{V_\alpha\}$ is a cover, we can choose $V_\alpha \ni x$. Since V_α is open, it contains an interval $(x-\varepsilon, x+\varepsilon)$ for some $\varepsilon > 0$. Choose a large n so that $\frac{b-a}{2^n} < \varepsilon$. Clearly, $V_\alpha \supset (x-\varepsilon, x+\varepsilon) \supset [a_n,b_n]$; the latter contradicts ❶. □

D Images

6.14. Proposition. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map between topological spaces and K is a compact set in $\mathcal{X}$. Then the image $Q = f(K)$ is compact in $\mathcal{Y}$.*

Proof. Choose an open cover $\{W_\alpha\}$ of Q. Since f is continuous, $V_\alpha = f^{-1}(W_\alpha)$ is open for each α. Note that $\{V_\alpha\}$ covers K.

Since K is compact, there is a finite subcover $\{V_{\alpha_1},\dots,V_{\alpha_n}\}$. It remains to observe that $\{W_{\alpha_1},\dots,W_{\alpha_n}\}$ covers Q. □

6.15. Exercise. *Show that the circle*

$$\mathbb{S}^1 = \{\, (x,y) \in \mathbb{R}^2 \,:\, x^2+y^2 = 1 \,\}$$

is compact.

---

[1] In fact, $a_n \to x$ and $b_n \to x$ as $n \to \infty$, but we will not use it directly.

E Closed subsets

6.16. Proposition. *A closed set in a compact space is compact.*

Proof. Let Q be a closed set in a compact space $\mathcal{K}$. Since Q is closed, its complement $W = \mathcal{K} \setminus Q$ is open.

Consider an open cover $\{V_\alpha\}_{\alpha \in \mathcal{I}}$ of Q. Add to it W; that is, consider the collection of sets that includes W as well as V_α for all $\alpha \in \mathcal{I}$. Note that we get an open cover of $\mathcal{K}$. Indeed, W covers all points in the complement of Q and any point of Q is covered by some V_α.

Since $\mathcal{K}$ is compact, we can choose a finite subcover, say $\{W, V_{\alpha_1}, \dots$ $\dots, V_{\alpha_n}\}$ — without loss of generality, we can assume that it includes W. Observe that $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$ is a cover of Q, hence the result. □

In the proof, we add an extra open set to the cover, use it, and take it away.[2]

6.17. Exercise. *Show that any closed bounded subset of the real line is compact.*

F Product spaces

6.18. Theorem. *Assume $\mathcal{X}$ and $\mathcal{Y}$ are compact topological spaces. Then their product space $\mathcal{X} \times \mathcal{Y}$ is compact.*

The following exercise provides a partial converse.

6.19. Exercise. *Suppose that a product space $\mathcal{X} \times \mathcal{Y}$ is nonempty and compact. Show that its factors $\mathcal{X}$ and $\mathcal{Y}$ are compact.*

In the proof, we will need the following definition.

6.20. Definition. *Let $\{V_\alpha\}$ and $\{W_\beta\}$ be two covers of a topological space $\mathcal{X}$. We say that $\{V_\alpha\}$ is inscribed in $\{W_\beta\}$ if for any α there is β such that $V_\alpha \subset W_\beta$.*

[2] This type of reasoning is useful in all branches of mathematics; sometimes it is called the 17 camels trick [1]. The name comes from the following mathematical parable: *A father left 17 camels to his three sons and, according to the will, the eldest son should be given half of all camels, the middle son the 1/3 part, and the youngest son the 1/9. It was impossible to follow his will until a wise man appeared. He added his own camel, the oldest son took* $18/2 = 9$ *camels, the second son took* $18/3 = 6$ *camels, and the third son* $18/9 = 2$ *camels, the wise man took his camel and went away.*

6.21. Exercise. *Let $\mathscr{B}$ be a base in a topological space $\mathcal{X}$. Show that for any cover $\{V_\alpha\}$ of $\mathcal{X}$ there is an inscribed cover made from sets in $\mathscr{B}$.*

Suppose that $\{V_\alpha\}$ is inscribed in $\{W_\beta\}$. If $\{V_\alpha\}$ has a finite subcover $\{V_{\alpha_1},\dots,V_{\alpha_n}\}$, then for each α_i we can choose β_i such that $V_{\alpha_i}\subset W_{\beta_i}$. Note that $\{W_{\beta_1},\dots,W_{\beta_n}\}$ is a finite subcover of $\{W_\beta\}$. It proves the following:

6.22. Observation. *A space $\mathcal{X}$ is compact if and only if any cover of $\mathcal{X}$ has a finite inscribed cover.*

It is instructive to solve the following exercise before reading the proof of 6.18.

6.23. Exercise. *Find a flaw in the following argument.*

Fake proof of 6.18. Fix an open cover $\{U_\beta\}$ of $\mathcal{X}\times\mathcal{Y}$. Consider all product sets $V_\alpha\times W_\alpha$ such that $V_\alpha\times W_\alpha\subset U_\beta$ for some β (as before, V_α and W_α are open in $\mathcal{X}$ and $\mathcal{Y}$ respectively). Note that $\{V_\alpha\times W_\alpha\}$ is a cover of $\mathcal{X}\times\mathcal{Y}$ that is inscribed in $\{U_\beta\}$. By the observation, it is sufficient to find a finite subcover of $\{V_\alpha\times W_\alpha\}$.

Note that $\{V_\alpha\}$ is a cover of $\mathcal{X}$. Since $\mathcal{X}$ is compact, we can choose its finite subcover $\{V_{\alpha_1},\dots,V_{\alpha_n}\}$. Similarly, $\{W_\alpha\}$ is a cover of $\mathcal{Y}$. So we can choose its finite subcover $\{W_{\alpha'_1},\dots,W_{\alpha'_m}\}$.

Finally observe that

$$\{V_{\alpha_1}\times W_{\alpha_1},\dots,V_{\alpha_n}\times W_{\alpha_n},V_{\alpha'_1}\times W_{\alpha'_1},\dots,V_{\alpha'_m}\times W_{\alpha'_m}\}$$

is a finite cover of $\mathcal{X}\times\mathcal{Y}$. □

Proof of 6.18. Recall that by definition of the product topology, any open set in $\mathcal{X}\times\mathcal{Y}$ is a union of product sets $V_\alpha\times W_\alpha$, where V_α is open in $\mathcal{X}$ and W_α is open in $\mathcal{Y}$.

Fix an open cover $\{U_\beta\}$ of $\mathcal{X}\times\mathcal{Y}$. Consider all product sets $V_\alpha\times W_\alpha$ such that $V_\alpha\times W_\alpha\subset U_\beta$ for some β (as before, V_α and W_α are open in $\mathcal{X}$ and $\mathcal{Y}$ respectively). Note that $\{V_\alpha\times W_\alpha\}$ is a cover of $\mathcal{X}\times\mathcal{Y}$ that is inscribed in $\{U_\beta\}$. By the observation, it is sufficient to find a finite subcover of $\{V_\alpha\times W_\alpha\}$.

Fix $x\in\mathcal{X}$. Note that the subspace $\{x\}\times\mathcal{Y}$ is homeomorphic to $\mathcal{Y}$; see 5.11. In particular, the set $\{x\}\times\mathcal{Y}$ is compact. Therefore, $\{x\}\times\mathcal{Y}$ has a finite cover $\{V_{\alpha_1}\times W_{\alpha_1},\dots,V_{\alpha_n}\times W_{\alpha_n}\}$; that is,

$$(V_{\alpha_1}\times W_{\alpha_1})\cup\dots\cup(V_{\alpha_n}\times W_{\alpha_n})\supset\{x\}\times\mathcal{Y}$$

Consider the set

$$N_x=V_{\alpha_1}\cap\dots\cap V_{\alpha_n};$$

note that N_x is open in $\mathcal{X}$. Since $N_x \subset V_{\alpha_i}$ for any i, we have

$$N_x \times \mathcal{Y} \subset (V_{\alpha_1} \times W_{\alpha_1}) \cup \dots \cup (V_{\alpha_n} \times W_{\alpha_n}).$$

We get that

❷ *every point $x \in \mathcal{X}$ admits an open neighborhood N_x such that $N_x \times \mathcal{Y}$ can be covered by finitely many product sets from $\{V_\alpha \times W_\alpha\}$.*

The sets $\{N_x\}_{x \in \mathcal{X}}$ form a cover of $\mathcal{X}$. Since $\mathcal{X}$ is compact, there is a finite subcover $\{N_{x_1}, \dots, N_{x_m}\}$. Note that

$$\mathcal{X} \times \mathcal{Y} = (N_{x_1} \times \mathcal{Y}) \cup \dots \cup (N_{x_m} \times \mathcal{Y});$$

that is, $\mathcal{X} \times \mathcal{Y}$ can be covered by a finite set of sets from $\{N_x \times \mathcal{Y}\}_{x \in \mathcal{X}}$. Applying ❷, we get that $\mathcal{X} \times \mathcal{Y}$ can be covered by a finite number of product sets from $\{V_\alpha \times W_\alpha\}$. □

6.24. Advanced exercise. *Let $f\colon \mathcal{X} \to \mathcal{K}$ be a map between topological spaces. Assume $\mathcal{K}$ is compact. Show that f is continuous if and only if its graph $\Gamma = \{\,(x, f(x)) \,:\, x \in \mathcal{X}\,\}$ is a closed set in $\mathcal{X} \times \mathcal{K}$.*

G Remarks

We omit the proof of the following theorem, but it is worth knowing this result.

6.25. Alexander prebase theorem. *A topological space $\mathcal{X}$ is compact if and only if for some (and therefore any) prebase $\mathscr{P}$ in $\mathcal{X}$, every cover of $\mathcal{X}$ by elements of $\mathscr{P}$ admits a finite subcover.*

6.26. Exercise. *Use the Alexander prebase theorem to give a short proof of 6.18.*

Chapter 7

Metric spaces revisited

Recall that any metric space comes with a natural topology. In particular, we may talk about compact metric spaces. In this chapter we discuss specific properties of metric spaces that are related to compctness.

A Lebesgue number

7.1. Lebesgue number. *Let* $\{V_\alpha\}$ *be an open cover of a compact metric space* $\mathcal{M}$*. Then there is* $\varepsilon > 0$ *such that for every* $x \in \mathcal{M}$ *there is* α *such that* $V_\alpha \supset \mathrm{B}(x, \varepsilon)$*.*

The number ε in the lemma is called the Lebesgue number of the cover; this is a very useful characteristic of an open cover.

Proof. Given a point $p \in \mathcal{M}$ we can choose $r = r(p) > 0$ such that the ball $\mathrm{B}(p, 2{\cdot}r)$ lies in V_α for some α. Observe that all balls $\mathrm{B}(p, r(p))$ form an open cover of $\mathcal{M}$. Since $\mathcal{M}$ is compact, we can choose a finite subcover $\{\mathrm{B}(p_1, r_1), \dots, \mathrm{B}(p_n, r_n)\}$.

Let $\varepsilon = \min\{r_1, \dots, r_n\}$. For any $p \in \mathcal{M}$ we can choose a ball $\mathrm{B}(p_i, r_i) \ni p$. Observe that $\mathrm{B}(p, \varepsilon) \subset \mathrm{B}(p_i, 2{\cdot}r_i)$. Since $\mathrm{B}(p_i, 2{\cdot}r_i)$ lies in some V_{α_i}, so is $\mathrm{B}(p, \varepsilon)$. $\square$

7.2. Exercise. *Construct a noncompact metric space* $\mathcal{M}$ *such that* 1 *is a Lebesgue number for any cover of* $\mathcal{M}$*.*

B Compactness ⇒ sequential compactness

A topological space is called sequentially compact if every sequence in it has a converging subsequence. For general topological spaces sequential compactness does not imply compactness, and the other way around. The following proposition states that these two notions are equivalent for metric spaces.

7.3. Exercise. *Show that the product of two sequentially compact spaces is sequentially compact.*

7.4. Proposition. *A metric space $\mathcal{M}$ is compact if and only if it is sequentially compact.*

In this section, we prove the only-if part. The if part requires deeper diving into metric spaces; it will be done in 7F after proving auxiliary statements in the following two sections.

Proof of the only-if part in 7.4. Choose a sequence $x_1, x_2, \ldots \in \mathcal{M}$.

Note that a point $p \in \mathcal{M}$ is a limit of a subsequence of x_n if and only if for any $\varepsilon > 0$, the ball $\mathrm{B}(p, \varepsilon)$ contains infinitely many elements x_n. Indeed, if this property holds, then we can choose i_1 such that $x_{i_1} \in \mathrm{B}(p, 1)$, further $i_2 > i_1$ such that $x_{i_2} \in \mathrm{B}(p, \frac{1}{2})$ and so on; on the n^{th} step we get $i_n > i_{n-1}$ such that $x_{i_n} \in \mathrm{B}(p, \frac{1}{n})$. The obtained subsequence $x_{i_1}, x_{i_2}, \ldots$ converges to p.

Assume the sequence x_n has no converging subsequence. Then for any point p there is $\varepsilon_p > 0$ such that $\mathrm{B}(p, \varepsilon_p)$ contains only finitely many elements of x_n. Note that all balls $\mathrm{B}(p, \varepsilon_p)$ form a cover of $\mathcal{M}$. Since the sequence is infinite, this cover does not have a finite subcover. That is, if a sequence x_n has no converging subsequence, then $\mathcal{M}$ is not compact. □

C Complete spaces

A sequence $x_1, x_2, \ldots$ of points in a metric space is called Cauchy if for any $\varepsilon > 0$ there is n such that $|x_i - x_j| < \varepsilon$ for all $i, j > n$. It is easy to prove that any converging sequence is Cauchy, the converse does not hold in general. A metric space $\mathcal{M}$ is called complete if any Cauchy sequence in $\mathcal{M}$ converges to a point in $\mathcal{M}$.

For example, as it follows from the Cauchy test, the real line $\mathbb{R}$ with the standard metric is a complete space. On the other hand, an open interval $(0, 1)$ forms a noncomplete subspace of $\mathbb{R}$; indeed, the sequence $x_n = \frac{1}{2 \cdot n}$ is Cauchy, it also converges to zero in $\mathbb{R}$ which not a point of the subspace $(0, 1)$.

7.5. Exercise. *Show that any compact metric space is complete.*

D Nets and separability

Let $\mathcal{M}$ be a metric space. A subset $A \subset \mathcal{M}$ is called an ε-net of $\mathcal{M}$ if for any $p \in \mathcal{M}$ there is $a \in A$ such that $|p - a|_{\mathcal{M}} < \varepsilon$.

7.6. Lemma. *Let $\mathcal{M}$ be a sequentially compact metric space. Then for any $\varepsilon > 0$ there is a finite ε-net in $\mathcal{M}$.*

Proof. Choose $\varepsilon > 0$. Consider the following recursive procedure.

Choose a point x_1 in $\mathcal{M}$. Further, choose a point x_2 so that $|x_1 - x_2| > \varepsilon$. Further, choose a point x_3 so that $|x_1 - x_3| > \varepsilon$ and $|x_2 - x_3| > \varepsilon$; and so on. On the n^{th} step we choose a point x_n such that $|x_i - x_n| > \varepsilon$ for all $i < n$.

Suppose that the procedure terminates at some n; that is, there is no point x_n such that $|x_i - x_n| > \varepsilon$ for all $i < n$. In this case, the set $\{x_1, \dots, x_{n-1}\}$ is an ε-net in $\mathcal{M}$ — the lemma is proved.

If the procedure does not terminate, we get an infinite sequence of points $x_1, x_2, \dots$ such that $|x_i - x_j| > \varepsilon$ for all $i \neq j$. Any of its subsequence has the same property; in particular none of its subsequences converges — a contradiction. ☐

A topological space is called separable if it contains a countable dense subset.

7.7. Corollary. *Sequentially compact metric spaces are separable.*

Proof. Let $\mathcal{M}$ be a sequentially compact metric space.

By 7.6, for each positive integer n, we can choose a finite ε-net $N_n \subset \mathcal{M}$. It remains to observe that the union $N_1 \cup N_2 \cup \dots$ is countable and dense. ☐

E Countable base

7.8. Proposition. *Any sequentially compact metric space has a countable base.*

Topological spaces that admit a countable base are called second-countable. So the proposition states that *any sequentially compact metric space is second-countable.*

Proof. Let $\mathcal{M}$ be a sequentially compact metric space. By 7.7, we can choose a countable dense subset $A \subset \mathcal{M}$.

Consider the set of all balls $\mathrm{B}(a, \frac{1}{n})$ for $a \in A$ and positive integers n. Note that this set is countable; it remains to show that it forms a base in $\mathcal{M}$.

Let x be a point in an open set V. Then $\mathrm{B}(x, \varepsilon) \subset V$ for some $\varepsilon > 0$. Choose n so that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since A is everywhere dense, we can choose $a \in A$ so that $|a - x| < \frac{1}{n}$. By the triangle inequality, $x \in \mathrm{B}(a, \frac{1}{n}) \subset \mathrm{B}(x, \varepsilon)$; in particular,

$$x \in \mathrm{B}(a, \tfrac{1}{n}) \subset V.$$

It remains to apply 5.8. □

7.9. Lemma. *Let $\mathcal{X}$ be a topological space with a countable base. Then any open cover of $\mathcal{X}$ has a countable subcover.*

Proof. Choose an open cover $\{V_\alpha\}$.

Let $\{B_1, B_2, \dots\}$ be a countable base of $\mathcal{X}$. By 5.7, for any $x \in \mathcal{X}$ we can choose $i = i(x)$ such that $x \in B_i \subset V_\alpha$ for some α. Denote by S all integers that appear as $i(x)$ for some x. Then $\{B_i\}_{i \in S}$ is a countable open cover that is inscribed in $\{V_\alpha\}$. That is for every B_i there is α_i such that $B_i \subset V_{\alpha_i}$. It remains to observe that $\{V_{\alpha_i}\}_{i \in S}$ is a countable cover of $\mathcal{X}$. □

F Sequential compactness ⇒ compactness

Proof of the if part in 7.4. Choose an open cover of $\mathcal{M}$. By 7.9, we can assume that the cover is countable; denote it by $\{V_1, V_2, \dots\}$.

Assume $\{V_1, V_2, \dots\}$ does not have a finite subcover. Then we can choose a sequence of points $x_1, x_2, \dots \in \mathcal{M}$ such that

$$x_n \notin V_1 \cup \dots \cup V_n$$

for any n.

Since $\mathcal{M}$ is sequentially compact, a subsequence of $x_1, x_2, \dots$ has a limit, say x; we have that $x \in V_n$ for some n. It follows that $x_i \in V_n$ for an infinite set of indices i, but by construction, $x_i \notin V_n$ for all $i > n$ — a contradiction. □

Chapter 8

Hausdorff spaces

Hausdorff spaces are especially nice topological spaces that share many features of metric spaces; for example, any convergent sequence in a Hausdorff space has a unique limit.

Historically, Hausdorff included an extra property in his definition of a topological space, and thus he defined what we now call a Hausdorff space. Later it became clear that this convention is too restrictive: dropping this extra property leads to a broader and more useful notion of a topological space. Nevertheless, many important examples of topological spaces are Hausdorff, so it makes sense to study them.

A Definition

8.1. Definition. *A topological space $\mathcal{X}$ is called Hausdorff if for each pair of distinct points $x, y \in \mathcal{X}$ there are disjoint neighborhoods $V \ni x$ and $W \ni y$.*

8.2. Observation. *Any metrizable space is Hausdorff.*

Proof. Assume that the topology on the space $\mathcal{X}$ is induced by a metric $|*-*|$.

If the points $x, y \in \mathcal{X}$ are distinct, then $|x-y| > 0$. By the triangle inequality $\mathrm{B}(x, \frac{r}{2}) \cap \mathrm{B}(y, \frac{r}{2}) = \varnothing$. Hence the statement follows □

Recall that in general, a sequence of points in a topological space might have different limits; see 3F. For example, for the $\mathbb{R}$ with the cofinite topology most sequences of points converge to *every* point in $\mathbb{R}$.

8.3. Exercise. *Show that any converging sequence in Hausdorff space has a unique limit.*

8.4. Exercise. *Show that topological space $\mathcal{X}$ is Hausdorff if and only if for any two distinct points $x, y \in \mathcal{X}$ there is an open set $V \ni x$ such that $\bar{V} \not\ni y$.*

8.5. Exercise. *Show that a topological space $\mathcal{X}$ is Hausdorff if and only if the diagonal*

$$\Delta = \{\, (x, x) \in \mathcal{X} \times \mathcal{X} \,\}$$

is a closed set in the product space $\mathcal{X} \times \mathcal{X}$.

B Observations

8.6. Observation. *Any one-point set in a Hausdorff space is closed.*

If every one-point set is closed in a topological space is closed, then the space is called T_1-space or sometimes Tikhonov space. Therefore the observation above states that *any Hausdorff space is T_1*.

Proof. Let $\mathcal{X}$ be a Hausdorff space and $x \in \mathcal{X}$. By 8.1, given a point $y \neq x$, there are disjoint open sets $V_y \ni x$ and $W_y \ni y$. In particular $W_y \not\ni x$.

Note that

$$\mathcal{X} \setminus \{x\} = \bigcup_{y \neq x} W_y.$$

It follows that $\mathcal{X} \setminus \{x\}$ is open, and therefore $\{x\}$ is closed. □

8.7. Observation. *Any subspace of a Hausdorff space is Hausdorff.*

Proof. Choose two points x, y in a subspace A of a Hausdorff space $\mathcal{X}$. Since $\mathcal{X}$ is Hausdorff, we can choose neighborhoods $V \ni x$ and $W \ni y$ such that $V \cap W = \varnothing$. Then $A \cap V$ and $A \cap W$ are neighborhoods of x and y in A. Clearly,

$$(A \cap V) \cap (A \cap W) \subset V \cap W = \varnothing.$$

Whence the observation follows. □

C Hausdorff meets compactness

8.8. Proposition. *Any compact subset of a Hausdorff space is closed.*

Note that any one-point set is compact. Therefore the proposition generalizes Observation 8.6. The proof is similar but requires an extra step. It is instructive to solve the following exercise before reading the proof.

8.9. Exercise. *Describe a topological space $\mathcal{X}$ with a nonclosed, but compact subset K.*

The proof of the proposition is based on the following theorem.

8.10. Theorem. *Let $\mathcal{X}$ be a Hausdorff space and $K \subset X$ be a compact subset. Then for any point $y \notin K$ there are open sets $V \supset K$ and $W \ni y$ such that $V \cap W = \varnothing$*

Proof of 8.8 modulo 8.10. For $y \notin K$, let us denote by W_y the open set provided by 8.10; in particular, $W_y \ni y$ and $W_y \cap K = \varnothing$. Note that

$$\mathcal{X} \setminus K = \bigcup_{y \notin K} W_y.$$

It follows that $\mathcal{X} \setminus K$ is open; therefore, K is closed. □

Proof of 8.10. By definition of Hausdorff space, for any point $x \in K$ there is a pair of disjoint open sets $V_x \ni x$ and $W_x \ni y$. Note that $\{V_x\}_{x \in K}$ is a cover of K. Since K is compact we can choose a finite subcover $\{V_{x_1}, \dots, V_{x_n}\}$. Set

$$V = V_{x_1} \cup \dots \cup V_{x_n} \qquad \text{and} \qquad W = W_{x_1} \cap \dots \cap W_{x_n}.$$

It remains to observe that V and W are open, $y \in W$, $K \subset V$, and

$$V \cap W \subset \bigcup_i (V_{x_i} \cap W_{x_i}) = \varnothing.$$ □

8.11. Exercise. *Let $\mathcal{X}$ be a Hausdorff space and $K, L \subset X$ be two compact subsets. Assume $K \cap L = \varnothing$. Show that there are open sets $V \supset K$ and $W \supset L$ such that $V \cap W = \varnothing$.*

D Compact-to-Hausdorff maps

8.12. Observation. *Let f be a continuous map from a compact space $\mathcal{K}$ to a Hausdorff space $\mathcal{Y}$. Then f is a closed map.*

If in addition, the map f is onto, then $\mathcal{Y}$ is equipped with the quotient topology induced by f.

Recall that every homeomorphism is a continuous bijection, but not the other way around; see 4.1.

8.13. Corollary. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proof of 8.12. Since $\mathcal{K}$ is compact, any closed subset $Q \subset \mathcal{K}$ is compact (6.16). Since the image of a compact set is compactl (6.14), we have that $f(Q)$ is a compact subset of $\mathcal{Y}$. Since $\mathcal{Y}$ is Hausdorff, $f(Q)$ is closed (8.8). Hence the first statement follows.

The second statement follows from 5.16. □

Recall that $\mathbb{D}^2$ denotes the unit disc and $\mathbb{S}^1$ denotes the unit circle; that is,

$$\begin{aligned}\mathbb{D}^2 &= \{\, (x,y) \in \mathbb{R}^2 \;:\; x^2 + y^2 \leqslant 1 \,\}, \\ \mathbb{S}^1 &= \{\, (x,y) \in \mathbb{R}^2 \;:\; x^2 + y^2 = 1 \,\}.\end{aligned}$$

8.14. Exercise. *Show that $\mathbb{S}^1$ is homeomorphic to the quotient space $[0,1]/\{0,1\}$. (In other words, $\mathbb{S}^1$ is homeomorphic to the unit interval with glued ends.)*

8.15. Exercise. *Show that the quotient space $\mathbb{D}^2/\mathbb{S}^1$ is homeomorphic to the unit sphere*

$$\mathbb{S}^2 = \{\, (x,y,z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 = 1 \,\}.$$

Recall that a convex body is a compact convex set with non-empty interior in $\mathbb{R}^3$.

8.16. Exercise. *Show that the boundary or a convex body is homeomorphic to $\mathbb{S}^2$.*

Chapter 9

Connected spaces

A Definitions

A subset of a topological space is called clopen if it is closed and open at the same time.

9.1. Definition. *A topological space $\mathcal{X}$ is called connected if it has exactly two clopen sets $\varnothing$ and the whole space $\mathcal{X}$.*

According to our definition, *the empty space is not connected.* Not everyone agrees with this convention.[1]

Suppose V is a proper clopen subset of a topological space $\mathcal{X}$; that is, $V \neq \varnothing$ and $V \neq \mathcal{X}$. Note that its complement $W = \mathcal{X} \setminus V$ is also a proper clopen subset. In particular, there are two open sets $V, W \subset \mathcal{X}$ such that $V \neq \varnothing$, $W \neq \varnothing$, $V \cup W = \mathcal{X}$ and $V \cap W = \varnothing$.

9.2. Exercise. *Let $f\colon \mathcal{X} \to \mathbb{R}$ be a continuous function defined of a connected space. Suppose that for $f(x_1) > 0$ and $f(x_2) < 0$ for some $x_1, x_2 \in \mathcal{X}$. Show that there is $x_3 \in \mathcal{X}$ such that $f(x_3) = 0$.*

A subset of a topological space is called connected or disconnected if the corresponding subspace is connected or disconnected, respectively. Spelling out the notion of subspace we get the following definition.

9.3. Definition. *A subset A of a topological space is called disconnected if it is empty or there are two open sets V and W such that*

$$V \cap W \cap A = \varnothing, \quad V \cap A \neq \varnothing, \quad W \cap A \neq \varnothing, \quad \text{and} \quad V \cup W \supset A.$$

[1]This convention is similar in spirit to saying that 1 is not prime — if prime meant *no nontrivial divisors*, then 1 would be prime, *but it is not.* Similarly, if a connected space meant *no proper clopen sets*, then the empty set would be connected, but *it is not* (at least in this course).

Otherwise, we say that A is connected.

A pair of open sets V and W as in the definition will be called an open splitting of A. So we can say that *a nonempty set A is disconnected if and only if it admits an open splitting.*

9.4. Proposition. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map between topological spaces. Suppose $A \subset \mathcal{X}$ is a connected set. Then the image $B = f(A)$ is a connected set in $\mathcal{Y}$.*

Proof. We can assume that $B \neq \varnothing$; otherwise the statement is trivial.

Assume that $B = f(A)$ is disconnected. Choose an open splitting V and W of B; that is,

❶ $V \cap W \cap B = \varnothing, \quad V \cap B \neq \varnothing, \quad W \cap B \neq \varnothing, \quad \text{and} \quad V \cup W \supset B.$

Since f is continuous, $V' = f^{-1}(V)$ and $W' = f^{-1}(W)$ are open sets in $\mathcal{X}$. Note that ❶ implies that V' and W' form an open splitting of A; that is,

$$V' \cap W' \cap A = \varnothing, \quad V' \cap A \neq \varnothing, \quad W' \cap A \neq \varnothing, \quad \text{and} \quad V' \cup W' \supset A.$$

Therefore A is disconnected — a contradiction. □

9.5. Exercise. *Let $\mathcal{X}$ be a connected space. Show that the quotient space $\mathcal{X}/\sim$ is connected for any equivalence relation $\sim$ on $\mathcal{X}$.*

9.6. Proposition. *Assume $\{A_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of connected subsets of a topological space. Suppose that $\bigcap_\alpha A_\alpha \neq \varnothing$. Then*

$$A = \bigcup_\alpha A_\alpha$$

is connected.

Proof. Assume that A is disconnected; choose its open splitting V, W. Since $\bigcap_\alpha A_\alpha \neq \varnothing$, we can choose $p \in \bigcap_\alpha A_\alpha$. Without loss of generality, we can assume that $p \in V$.

In particular, $V \cap A_\alpha \neq \varnothing$ for any α. Since A_α is connected, we have that $W \cap A_\alpha = \varnothing$ for each α; otherwise V and W form an open splitting of A_α. Therefore

$$\begin{aligned} W \cap A = W \cap \left(\bigcup_\alpha A_\alpha\right) &= \\ = \bigcup_\alpha (W \cap A_\alpha) &= \\ = \varnothing & \end{aligned}$$

— a contradiction. □

9.7. Exercise. *Let A be a connected set in a topological space $\mathcal{X}$. Suppose that $A \subset B \subset \bar{A}$. Show that B is connected.*

B Real interval

9.8. Proposition. *The real interval* $[0, 1]$ *is connected.*

The proof reuses the construction in 6C.

Proof. Assume the contrary; let V and W be an open splitting of $[0, 1]$. Fix a $a_0 \in V$ and $b_0 \in W$; without loss of generality, we can assume that $a_0 < b_0$.

Let us construct a nested sequence of closed intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset \dots$$

such that

❷ $$b_n - a_n = \tfrac{1}{2^n}(b_0 - a_0), \quad a_n \in V, \quad \text{and} \quad b_n \in W$$

for any n.

The construction is recursive. Assume $[a_{n-1}, b_{n-1}]$ is already constructed. Set $c = \frac{1}{2}\cdot(a_{n-1} + b_{n-1})$. If $c \in V$, then set $a_n = c$ and $b_n = b_{n-1}$; if $c \in W$, then set $a_n = a_{n-1}$ and $b_n = c$. In both cases, ❷ holds.

Observe that

$$a_0 \leqslant a_1 \leqslant \dots \leqslant b_1 \leqslant b_0.$$

In particular, the sequence a_n is nondecreasing and bounded above by b_0. Therefoe, it converges; denote its limit by x. Since $b_n - a_n = \frac{1}{2^n}\cdot(b_0 - a_0)$, the sequence b_n also converges to x. The point x has to belong to V or W. Since both V and W are open, one of them contains a_n and b_n for all large n, which contradicts ❷. □

9.9. Exercise. *Show that the real line $\mathbb{R}$ is a connected space. Conclude that any continuos map $\mathbb{R} \to \mathbb{Z}$ is constant; here $\mathbb{Z} \subset \mathbb{R}$ denotes the set of integer numbers.*

C Connected components

Let x be a point in a topological space $\mathcal{X}$. The intersection of all clopen sets containing x is called the connected component of x. Note that the space $\mathcal{X}$ is connected if and only if $\mathcal{X}$ is a connected component of some (and therefore of any) point in $\mathcal{X}$.

9.10. Exercise. *Show that any connected component is a closed set.*

Construct an example of a topological space $\mathcal{X}$ and a point $x \in \mathcal{X}$ such that the connected component of x is not open.

9.11. Exercise. *Show that two connected components either coincide or are disjoint.*

9.12. Exercise. *Suppose that a space $\mathcal{X}$ has a finite number of connected components. Show that each connected component of $\mathcal{X}$ is open.*

9.13. Advanced exercise. *Let us denote by $\mathbb{N}$ the set of positive integers. Show that that arithmetic progressions $\{a, a+d, a+2\cdot d, \dots\}$ for relatively prime all realtivley prime positive integers $a \leqslant d$ form a basis of some topology, and $\mathbb{N}$ equipped with this topology is a connected Hausdorff space.*

D Cut points

Evidently, the number of connected components is a topological invariant; that is, *if two spaces are homeomorphic, then they have the same number of connected components.* In particular, a connected space is not homeomorphic to a disconnected space.

Let us describe a more refined way to apply this observation. Suppose $\mathcal{X}$ is a connected space. A point $x \in \mathcal{X}$ is called a cut point if removing x from $\mathcal{X}$ produces a disconnected space; that is, the subset $\mathcal{X} \setminus \{x\}$ is disconnected.

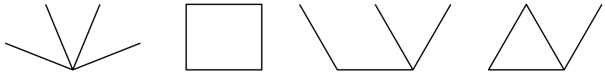
Note that if $f\colon \mathcal{X} \to \mathcal{Y}$ is a homeomorphism, then a point $x \in \mathcal{X}$ is a cut point of $\mathcal{X}$ if and only if $y = f(x)$ is a cut point of $\mathcal{Y}$. Indeed, the restriction of f defines a homeomorphism $\mathcal{X} \setminus \{x\} \to \mathcal{Y} \setminus \{y\}$. In particular, we get that the spaces $\mathcal{X} \setminus \{x\}$ and $\mathcal{Y} \setminus \{y\}$ have the same number of connected components.

These observations can be used to solve the following exercises.

9.14. Exercise. *Show that the circle $\mathbb{S}^1$ is not homeomorphic to the line segment* $[0, 1]$.

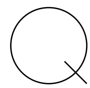
9.15. Exercise. *Show that the plane* $\mathbb{R}^2$ *is not homeomorphic to the real line* $\mathbb{R}$.

9.16. Exercise. *Show that no two of the following four closed connected sets in the plane are homeomorphic. (Each set is a union of four line segments, and if it looks like they share an end point, then this is ideed so.)*

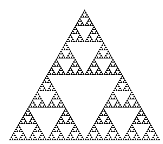


(a) (b) (c) (d)

9.17. Exercise. *Let* Q *be the set shown in the picture; it is a union of a cicle and a closed line segment. Consider that action on* Q *of the group* H *of all homeomorphisms* $Q \to Q$. *Describe the quotient space* Q/H*; that is, list its points and the open sets.*



Sierpiński gasket is constructed the following way: start with a solid equilateral triangle, subdivide it into four smaller congruent equilateral triangles and remove the interior of the central one. Repeat this procedure recursively for each of the remaining solid triangles.



9.18. Advanced exercise.

(a) Prove that the Sierpiński triangle is connected (in particular, nonempty).

(b) Describe all the homeomorphisms from the Sierpiński triangle to itself.

E Open-close argument

The followowing exercise gives an example of application of the so-called open-close argument.

9.19. Exercise. *Let* $f\colon [0,1]\times[0,1]\to[0,1]\times[0,1]$ *is an arbitrary map that increases the firs coordinate; that is, if* $(x_1,y_1)=f(x_0,y_0)$ *then* $x_1>x_0$. *Show that there is a path* $h\colon[0,1]\to[0,1]\times[0,1]$ *that connects the point* $(0,0)$ *to some point* $(1,y)$ *such that for any any point* $h(t)$ *for* $t\in[0,1)$ *lies on a line segment from* (x,y) *to* $f(x,y)$

Part II

Fundamental group

Chapter 10

Path-conneced spaces

Here we discuss paths and path-connected spaces,

A Paths

A continuous map $f\colon [0,1] \to \mathcal{X}$ is called a path. If $x = f(0)$ and $y = f(1)$ we say that f is a path from x to y.

Let us start with several examples and constructions of paths

- A path f is called constant if stays at one point. In other words, $f(t) = p$ for some fixed point p; in this case, the path f can be denoted by ε_p.
- Given a path $f\colon [0,1] \to \mathcal{X}$, one can consider the time-reversed path $\bar{f}$, defined by

  $$\bar{f}(t) = f(1-t).$$

  Note that $\bar{f}$ is continuous since f is.
- Let f and h be paths in the topological space $\mathcal{X}$. If $f(1) = h(0)$ we can join these two paths into one $g\colon [0,1] \to \mathcal{X}$ that follows f and then h; so it is defined as

  $$g(t) = \begin{cases} f(2{\cdot}t) & \text{if } t \leqslant \frac{1}{2} \\ h(2{\cdot}t - 1) & \text{if } t \geqslant \frac{1}{2} \end{cases}$$

  The path g is called the product (or concatenation) of paths f and h; it is denoted as $g = f * h$. Note that 3.11*b* implies that $f * h$ is continuous; in other words $f * h$ is indeed a path.

Consider the following relation on the set of points of a topological space:

$$x \sim y \quad \Leftrightarrow \quad \text{there is a path from } x \text{ to } y.$$

From above we have that $\sim$ is an equivalence relation on the set of points in $\mathcal{X}$; in other words, the following properties hold for any points $x, y, z \in \mathcal{X}$:

- $x \sim x$ (since ε_x is a path from x to x);
- if $x \sim y$, then $y \sim x$ (indeed, if f is a path from x to y, then $\bar{f}$ is a path from y to x);
- if $x \sim y$ and $y \sim z$, then $x \sim z$ (indeed, if f is a path from x to y, and h is a path from y to x then $f * h$ is a path from x to z).

The equivalence class of a point x for the equivalence relation $\sim$ is called the path-connected component of x; this is the set of all point in $\mathcal{X}$ that can be jointed with y by a path.

B Path-connected spaces

A space $\mathcal{X}$ is called path-connected if it is nonempty and any two points in $\mathcal{X}$ can be connected by a path; that is, for any $x, y \in \mathcal{X}$ there is a path f from x to y. One can say that $\mathcal{X}$ is path-connected if $\mathcal{X}$ is a connected component of one (and therefore every) point $x \in \mathcal{X}$.

10.1. Exercise. *Show that the connected two-point space (defined in 2B) is path-connected.*

10.2. Theorem. *Any path-connected space is connected; the converse does not hold.*

Proof; main part. Let $\mathcal{X}$ be a path-connected space.

By Proposition 9.8, the unit interval $[0,1]$ is connected. By Proposition 9.4, for any path $f\colon [0,1] \to \mathcal{X}$ the image $f([0,1])$ is connected.

Fix $x \in \mathcal{X}$. Since $\mathcal{X}$ is path-connected,

$$\mathcal{X} = \bigcup_f f([0,1]),$$

where the union is taken for all paths f starting from x. It remains to apply 9.6.

Second part. We need to present an example of a space that is connected, but not path-connected.

Denote by I the closed line segment from $(0,0)$ to $(1,0)$ in $\mathbb{R}^2$. Further, denote by J_n the closed line segment from $(\frac{1}{n}, 0)$ to $(\frac{1}{n}, 1)$. Consider the union of all these segments

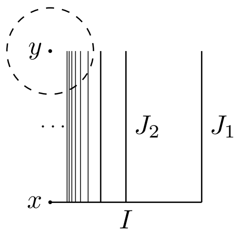
$$W = I \cup J_1 \cup J_2 \cup \dots$$

and set

$$W' = W \cup \{y\},$$

where $y = (0,1)$. The space W' is called the flea and comb; the set W is called the comb, and the point y is called the flea.

Note that $W \subset W' \subset \bar{W}$. Therefore, by 9.7, W' is connected.



It remains to show that W' is not path-connected. Assume the contrary. Let f be a path from $x = (0,0)$ to $y = (0,1)$.

Note that $f^{-1}(\{y\})$ is closed subset of compact space $[0,1]$. Therefore $f^{-1}(\{y\})$ is compact (6.16). In particular, the set $f^{-1}(\{y\})$ has the minimal element, denote it by b. Note that $b > 0$, $f(b) = y$ and $f(t) \neq y$ for any $t < b$.

Choose a positive $\varepsilon < 1$. Since f is continuous, there is $a < b$ such that $|f(t) - y| < \varepsilon$ for any $t \in [a,b]$. Note that $f(a) \in J_n$ for some n.

Denote by N the intersection of the ε-neighborhood of y with the comb. Note that the intersection of J_n with ε-neighborhood of y forms a connected component of N. By 9.4, $f(t) \in J_n$ for any $t \in [a,b]$; in particular, $f(b) \neq y$ — a contradiction. □

10.3. Exercise. *Show that the image of a path-connected set under a continuous map is path-connected.*

10.4. Exercise. *Show that the product of path-connected spaces is path-connected.*

10.5. Exercise. *Describe path-connected components in the flea and comb.*

10.6. Exercise. *Assume every path-connected component in a topological space $\mathcal{X}$ is open. Show that $\mathcal{X}$ is connected if and only if $\mathcal{X}$ is path-connected.*

10.7. Advanced exercise. *Consider the lexicographical order $\prec$ on $\mathbb{R}^2$:*

$$(x_1, y_1) \prec (x_2, y_2) \quad \Leftrightarrow \quad x_1 < x_2 \quad \text{or} \quad x_1 = x_2 \text{ and } y_1 < y_2.$$

Show that the following sets for all pairs $(x_1, y_1) \prec (x_2, y_2)$ in $\mathbb{R}^2$

$$I_{(x_1,y_1),(x_2,y_2)} := \{\, (x,y) \in \mathbb{R}^2 \;:\; (x_1,y_1) \prec (x,y) \prec (x_2,y_2) \,\}$$

form a base of some topology $\mathscr{T}$ on $\mathbb{R}^2$.

Furthermore, show that the squre $([0,1] \times [0,1]$ with the topology induced by $\mathscr{T}$ is compact, Hausdorff, connected, but not path-connected, *and each path-conencted component in* $([0,1]\times[0,1],\mathscr{T})$ *is a veritcal interval* $\{x\}\times[0,1]$.

10.8. Advanced exercise. *Recall that* $\mathbb{Q}$ *denotes the set of rational numbers. Consider the following sets in the plane:*

$$A=\left\{\,(x,y)\in\mathbb{R}^2\;:\;x,y\in\mathbb{Q}\,\right\}\quad\text{and}\quad B=\left\{\,(x,y)\in\mathbb{R}^2\;:\;x,y\notin\mathbb{Q}\,\right\}.$$

Show that $A\cup B$ *is path-connected.*

C Sets of Euclidean space

The following theorem provides a class of topological spaces for which connectedness implies path-connectedness.

10.9. Theorem. *An open set in a Euclidean space* $\mathbb{R}^n$ *is path-connected if and only if it is connected.*

Proof. The only-if part follows from 10.2; it remains to prove the if part.

Let $\Omega\subset\mathbb{R}^n$ be an open subset. Choose a point $p\in\Omega$; denote by $P\subset\Omega$ the path-connected component of p.

Let us show that for any point $q\in\Omega$ there is $\varepsilon>0$ such that either $\mathrm{B}(q,\varepsilon)\subset P$, or $\mathrm{B}(q,\varepsilon)\cap P=\varnothing$.

Indeed, since Ω is open, we can choose $\varepsilon>0$ such that the ball $\mathrm{B}(q,\varepsilon)$ lies in Ω. Note that $\mathrm{B}(q,\varepsilon)$ is convex, in particular path-connected. Therefore $\mathrm{B}(q,\varepsilon)\cap P\neq\varnothing$ if and only if $\mathrm{B}(q,\varepsilon)\subset P$.

It follows that P and its complement $\Omega\setminus P$ are open. Since Ω is connected, we get that $\Omega\setminus P=\varnothing$ — hence the result. □

A topological space $\mathcal{X}$ is called locally path-connected if for any point $p\in\mathcal{X}$ and any open set $V\ni p$ there is a path-connected open set W such that $V\supset W\ni p$. For instance, Euclidean space is locally path-connected; it follows since any open ball in a Euclidean space is path-connected. Therefore the following exercise generalizes the theorem above.

10.10. Exercise. *Show that a connected open set in a locally path-connected space is path-connected.*

10.11. Advaneced exercise. *Construct a bounded open set* V *in the plane such its boundary is connected, but totally path-disconnected; that is each connected component in* ∂V *consists of a sinle point.*

Chapter 11

Homotopy

A Homotopy

Two continuous maps $f,g\colon \mathcal{X}\to\mathcal{Y}$ are called homotopic (briefly, $f\sim g$) if f can be continuously deformed into g. Formally, this means that there exists a continuous map

$$H\colon \mathcal{X}\times[0,1]\to\mathcal{Y}$$

such that $H(x,0)=f(x)$ and $H(x,1)=g(x)$ for all $x\in\mathcal{X}$. The map $H\colon \mathcal{X}\times[0,1]\to\mathcal{Y}$ is called homotopy from f to g.

It is often convinient to think of the homotopy H as a one-parameter family of maps $h_t\colon \mathcal{X}\to\mathcal{Y}$ defined by $h_t(x)=H(x,t)$.

We say that map is null-homotopic if it is homotopic to a constant map.

11.1. Exercise. *Show that any two continuous maps $\mathcal{X}\to\mathbb{R}^2$ are homotopic.*

Recall that

$$\mathbb{S}^2:=\{\,(x,y,z)\in\mathbb{R}^3\;:\;x^2+y^2+z^2=1\,\}.$$

11.2. Exercise. *Suppose that for two continuous maps $f_0,f_1\colon\mathcal{X}\to\mathbb{S}^2$, we have that $f_0(x)+f_1(x)\ne0$ for any $x\in\mathcal{X}$. Show that $f_0\sim f_1$.*

11.3. Exercise. *Let $f_0,f_1\colon\mathcal{X}\to\mathcal{Y}$ and $g_0,g_1\colon\mathcal{Y}\to\mathcal{Z}$ be four maps between topological spaces. Assume that $f_0\sim f_1$ and $g_0\sim g_1$. Show that $g_0\circ f_0\sim g_1\circ f_1$.*

B Relative homotopy

Let $H\colon \mathcal{X}\times[0,1]\to\mathcal{Y}$ be a homotopy, and let $A\subset\mathcal{X}$. Suppose that for any $a\in A$ the point $H(a,t)$ does not depend on t. Then we say that H is a homotopy *relative to* A. Two maps $f,g\colon\mathcal{X}\to\mathcal{Y}$ are called *homotopic relative to* A, briefly, $f\sim g$ (rel A) if there is a homotopy relative to A from f to g.

Note also that if $f\sim g$ (rel A), then f agrees with g on A; that is, $f(a)=g(a)$ for any $a\in A$.

The properties of homotopies that we are about to describe admit straightforward generalizations to its relative version.

C Operations on homotopies

Observe that

- the *constant homotopy* $H(x,t)=f(x)$ joins f with itself;
- if H is a homotopy from f with g, then the *time-reversed* homotopy defined by

$$\bar H(x,t)=H(x,1-t)$$

  is a homotopy from g to f;
- Consider two homotopies H and G from f to g and from g to h, respectively. Then their *concatenation* $F=H*G$ defined by

$$F(x,t)=\begin{cases}H(x,2{\cdot}t), & t\leqslant\tfrac12,\\ G(x,2{\cdot}t-1), & t\geqslant\tfrac12,\end{cases}$$

  defines a homotopy from f with h. (By 3.11*b*, F is continuous.)

Recall that

$$\mathbb{S}^1:=\left\{\,(x,y)\in\mathbb{R}^2\;:\;x^2+y^2=1\,\right\}.$$

11.4. Exercise. *Let $f\colon\mathbb{S}^1\to\mathbb{S}^1$, be a continuous map such that*
(a) $f(x)\ne -x$ for any x, or
(b) $f(x)\ne x$ for any x.
Show that f is homotopic to the identity map.

D Homotopy classes

The previous section implies that $\sim$ is an equivalence relation on the set of continuous maps $\mathcal{X}\to\mathcal{Y}$; in other words, the following properties hold for any continuous maps $f,g,h\colon\mathcal{X}\to\mathcal{Y}$:

- $f \sim f$;
- if $f \sim g$, then $g \sim f$;
- if $f \sim g$, and $g \sim h$, then $f \sim h$.

The equivalence classes for $\sim$ are called homotopy classes; the equivalence class of a map $f\colon \mathcal{X} \to \mathcal{Y}$ will be denoted by $[f]$. Given two spaces $\mathcal{X}$ and $\mathcal{Y}$, the set of all homotopy classes will be denoted by $\pi(\mathcal{X}, \mathcal{Y})$.

Often the set $\pi(\mathcal{X}, \mathcal{Y})$ admits a direct description; this is precisely its advantage over the vast set of all continuous maps $\mathcal{X} \to \mathcal{Y}$. Note that any homeomorphism $\mathcal{Y} \to \mathcal{Y}'$ defines a bijection between homotopy classes $\pi(\mathcal{X}, \mathcal{Y}) \to \pi(\mathcal{X}, \mathcal{Y}')$ for any given space $\mathcal{X}$. So, stydying the homotopy classes may help to distiguish topological spaces.

This construction is meaningful even for very simple choices of $\mathcal{X}$. If $\mathcal{X}$ is the one-point space, then $\pi(\mathcal{X}, \mathcal{Y})$ is also denoted by $\pi_0(\mathcal{Y})$. Note that $\pi_0(\mathcal{Y})$ is the set of path-connected components in $\mathcal{Y}$ — this was the main subject of the previous chapter. The set $\pi(\mathbb{S}^1, \mathcal{Y})$ is closely related to the fundamental group of $\mathcal{Y}$, which will be soon introduced.

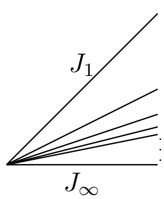
The idea to use homotopy classes to distiguish topological spaces will guide us in the right direction. However, the true motivation to consider homotopy classes is different; it will described in Capter 13.

E Retracts

Let A be a subset in a topological space $\mathcal{X}$. A continuous map $r\colon \mathcal{X} \to$ $\to A$ is called a retraction if $r(a) = a$ for any $a \in A$. In this case A is called a retract of $\mathcal{X}$. If in addition, $r\colon \mathcal{X} \to A$ is homotopic to the identity map $\mathrm{id}_{\mathcal{X}}$, then A is called deformation retract of $\mathcal{X}$. If $r \sim \mathrm{id}_{\mathcal{X}}$ (rel A), then A is called strong deformation retract of $\mathcal{X}$.

11.5. Exercise. *Show that a retract of a Hausdorff space has to be a closed subset.*

11.6. Exercise. *Given a positive integer n, denote by J_n the closed line segment from the origin to the point $(1, \frac{1}{n})$ in $\mathbb{R}^2$, and let J_∞ be the closed line segment from the origin to the point $(1, 0)$. Let F be the union of all the these line segments; it is called fan.*



(a) Show that J_1 is a strong deformation retract of F.

(b) Show that J_∞ is a deformation retrtact of F, but not strong deformation retract of F.

F Contractible spaces

A topological space $\mathcal{X}$ is called contactable if the identity map $\mathrm{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ is homotopic to a constant map; that is, there is a homotopy $h_t\colon \mathcal{X} \to \mathcal{X}$ such that $h_0(x) = x$ and $h_1(x) = p$ for some fixed point $p \in \mathcal{X}$ and any t.

Soon we will show that the circle $\mathbb{S}^1$ is an example of path-connected space that is not contractible; in particular, there are noncontratible path-connected spaces.

11.7. Exercise. *Show that any convex subset of the Euclidean space is contractible.*

11.8. Exercise. *Show that any contractible space is path-connected.*

11.9. Exercise. *Let $\mathcal{X}$ be a contractible space.*

(a) Show that any two continuous maps from a topological space to $\mathcal{X}$ are homotopic.

(b) Show that any two continuous maps from $\mathcal{X}$ to a path-connected space are homotopic.

G Homotopy type

Two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ have the same homotopy type (briefly $\mathcal{X} \sim \mathcal{Y}$) if there are continuous maps $f\colon \mathcal{X} \to \mathcal{Y}$ and $h\colon \mathcal{Y} \to \mathcal{X}$ such that $h \circ f \sim \mathrm{id}_{\mathcal{X}}$ and $f \circ h \sim \mathrm{id}_{\mathcal{Y}}$. In this case f (as well as h) is called a homotopy equivalence and g is called its homotopy inverse of f.

11.10. Exercise. *Show that $\sim$ defines an equivalence relation on topological spaces. (The corresponding class of equivalence is called homotopy type)*

11.11. Exercise. *Show that a topological space $\mathcal{X}$ is contractible if and only if it has the same homotopy type with the one-point space.*

11.12. Proposition. *If A is a deformation retract of $\mathcal{X}$, then $\mathcal{X}$ and A have the same homotopy type.*

Proof. Let $r\colon \mathcal{X} \to A$ be the retraction and $\iota\colon A \hookrightarrow \mathcal{X}$ the inclusion. By definition, we have a homotopy H from $\mathrm{id}_{\mathcal{X}}$ to $\iota \circ r$, hence $\mathrm{id}_{\mathcal{X}} \sim \iota \circ r$. On the other hand, $r \circ \iota = \mathrm{id}_A$ since r is a retraction. Therefore ι and r define a homotopy equivalence between A and $\mathcal{X}$. □

This proposition can be used together with 11.10 to prove that given spaces have the same homotopy type; here is one example:

11.13. Exercise. *Show that the two closed connected sets in the picture have the same homotopy type. (Each set is a union of several line segments.)*

11.14. Advaned exercise. *Show that two topological spaces have the same homotopy type if and only if they homeomorphic to two deformation reatracts of one topological space.*

Chapter 12

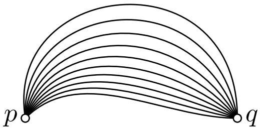
Fundamental group

Let $\mathcal{X}$ be a topological space with marked point $x_0 \in \mathcal{X}$. Choose a point s_0 on the circle $\mathbb{S}^1$. Consider all continuous maps $\mathbb{S}^1 \to \mathcal{X}$ that send u_0 to x_0. Let us denote by $\pi_1(\mathcal{X}, x_0)$ the set of homotopy classes of these maps within the described class of maps. In this chpter we will equip this set with multiplication and show that it is indeed a group, after that we will have right to call it the fundamental group $\mathcal{X}$ with marked point x_0.

A Homotopy of paths

Any path is a continuous map defined on the unit interval $[0,1]$. By homotopy of paths we will understand homotopy realive to the ends of $[0,1]$; in the notations of 11B, we have $\mathcal{X} = [0,1]$ and $A = \{0,1\}$. If we need to talk about general homotopy of paths we say free homotopy.

Let p and q be two points in a topological sapce $\mathcal{X}$ and $f_\tau \colon [0,1] \to \mathcal{X}$ be a one-parameter family of paths from p to q; here $\tau \in [0,1]$. If the map $[0,1] \times [0,1] \to \mathcal{X}$, defined as $(\tau, t) \mapsto f_\tau(t)$ is continuous, then f_τ is called a homotopy of paths in $\mathcal{X}$.



Intuitively, homotopy of paths is a path in the space of paths with fixed ends. To make this statement precise, one has to introduce an appropriate topology on the space of all paths with fixed ends; the so-called compact-open topology provides a right choice, but we are not going to touch this subject.

Two paths $g, h \colon [0,1] \to \mathcal{X}$ are called homotopic (briefly $g \sim h$) if there is a homotopy $f_\tau \colon [0,1] \to \mathcal{X}$ such that $g = f_0$ and $h = f_1$.

Recall that $\sim$ is an equivalence relation. Therefore, we can talk about the equivalence class of a path f that will be called its homotopy class; it will be denoted by $[f]$.

12.1. Exercise. *Suppose that f and g are two paths in $\mathbb{R}$ with common ends; that is, $f(0) = g(0)$ and $f(1) = g(1)$. Show that $f \sim g$.*

B Technical claims

12.2. Claim. *Suppose f_0 is a path from p to q, and g_0 is a path from q to r. Suppose $f_0 \sim f_1$ and $g_0 \sim g_1$, then*

$$f_0 * g_0 \sim f_1 * g_1.$$

Proof. Choose homotopies f_τ from f_0 to f_1, and g_τ from g_0 to g_1. Observe that 3.11*b* implies that $f_\tau * g_\tau$ is a homotopy from $f_0 * g_0$ to $f_1 * g_1$. □

Each of the following claims proved by explicit construction of the needed homotopy. Each time the homotopy constructed as a composition $h \circ s_\tau(t)$, where h is a fixed path and s_τ is a one-parameter family of functions $[0,1] \to [0,1]$. The graphs of s_τ provide more intuitive descriptions of the families; the formulas presented just to make it formally correct.

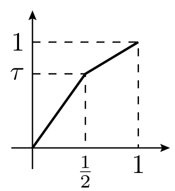
Recall that ε_p is the constant path with image p; that is, $\varepsilon_p(t) = p$ for any t.

12.3. Claim. *Suppose f is a path from p to q, then*

$$\varepsilon_p * f \sim f * \varepsilon_q \sim f.$$

Proof. Consider the function

$$s_\tau(t) = \begin{cases} 2\cdot\tau\cdot t & \text{if } t \leqslant \tfrac{1}{2}, \\ 2\cdot\tau - 1 + 2\cdot(1-\tau)\cdot t & \text{if } t \geqslant \tfrac{1}{2}. \end{cases}$$



Observe that $(\tau, t) \mapsto s_\tau(t)$ and therefore $(\tau, t) \mapsto f(s_\tau(t))$ are continuous maps. Therefore $h_\tau(t) = f(s_\tau(t))$ is a homotopy.

Further,

$$\begin{aligned} f(s_0(t)) &= \varepsilon_p * f(t), \\ f(s_{\frac{1}{2}}(t)) &= f(t), \\ f(s_1(t)) &= f(t) * \varepsilon_q \end{aligned}$$

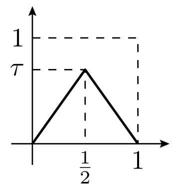
for any t. Whence the claim follows. □

12.4. Exercise. *Suppose that f is a path in a Hausdorff space. Assume $f(0) = p$, $f(1) = q$, and $\varepsilon_p * f = f$. Show that $f = \varepsilon_p$; in particular, $p = q$.*

12.5. Claim. *Suppose f is a path from p to q, then*

$$f * \bar{f} \sim \varepsilon_p \quad \textit{and} \quad \bar{f} * f \sim \varepsilon_q.$$

Proof. Consider the function



$$s_\tau(t) = \begin{cases} 2{\cdot}\tau{\cdot}t & \text{if } t \leqslant \frac{1}{2}, \\ 1 - 2{\cdot}\tau{\cdot}t & \text{if } t \geqslant \frac{1}{2}. \end{cases}$$

Observe that $(\tau, t) \mapsto s_\tau(t)$ and therefore $(\tau, t) \mapsto f(s_\tau(t))$ are continuous maps.

Note that $f(s_1(t)) = f * \bar{f}(t)$ for any t. Therefore $h_\tau(t) = f(s_\tau(t))$ is a homotopy from ε_p to $f * \bar{f}$.

It proves the first statement. The second statement follows from the first one since $\bar{\bar{f}} = f$. □

The product of paths is not associative; that is, in general,

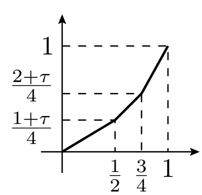
$$(f * g) * h \neq f * (g * h)$$

for paths f, g, h such that both products are defined. In other words, we have to specify the order of product. The following claim says that *product of paths is not associative up to homotopy.*

12.6. Claim. *Suppose f, g, and h are paths such that $f(1) = g(0)$ and $g(1) = h(0)$. Then*

$$(f * g) * h \sim f * (g * h).$$

Proof. Consider the function s_τ defined by



$$s_\tau(t) = \begin{cases} \frac{1+\tau}{2}{\cdot}t & \text{if } t \leqslant \frac{1}{2}, \\ \frac{\tau-1}{4} + t & \text{if } \frac{3}{4} \geqslant t \geqslant \frac{1}{2}, \\ \tau - 1 + (2-\tau){\cdot}t & \text{if } t \geqslant \frac{3}{4}. \end{cases}$$

Observe that $(\tau, t) \mapsto s_\tau(t)$ and therefore $(\tau, t) \mapsto f * (g * h)(s_\tau(t))$ are continuous maps.

Note that $s_1(t) = t$, and therefore

$$f * (g * h)(t) = f * (g * h)(s_1(t))$$

for any t. Further,

$$(f * g) * h(t) = f * (g * h)(s_0(t))$$

for any t. It remains to observe that $f * (g * h)(s_\tau(t))$ is the needed homotopy. □

12.7. Exercise. *Let f and g be paths from p to q. Show that $f \sim g$ if and only if $f * \bar{g} \sim \varepsilon_p$.*

12.8. Advanced exercise. *Let f, g, and h be paths in a Hausdorff space. Suppose that $(f * g) * h = f * (g * h)$ and both sides of the equation are defined. Show that $f = g = h = \varepsilon_p$ for some point p.*

C Fundamental group

Let $\mathcal{X}$ be a topological space. A path $f\colon [0,1] \to \mathcal{X}$ is called a loop with base at $p \in \mathcal{X}$ if $f(0) = f(1) = p$;

Note that if f and g are loops based at p, then their products $f * g$, $g * f$ are defined and they are loops based at p as well; see Section 11C. Moreover, the time-reversed paths $\bar{f}$, $\bar{g}$ are also loops based at p.

Recall that $[f]$ denotes the homotopy class of f; recall that homotopy of paths and loops does not move their ends. The multiplication of homotopy classes of loops based at p is defined by

$$[f]\cdot[g] = [f * g];$$

that is, *the product of homotopy classes of loops f and g is the homotopy class of the product $f * g$.*

Observe that the product is well defined; that is, if $[f_0] = [f_1]$ and $[g_0] = [g_1]$, then $[f_0 * g_0] = [f_1 * g_1]$. In other words, if $f_0 \sim f_1$ and $g_0 \sim g_1$, then $f_0 * g_0 \sim f_1 * g_1$. The latter is stated in Claim 12.2.

Denote by $\pi_1(\mathcal{X}, p)$ the set of all homotopy classes of loops at p.

12.9. Theorem. *$\pi_1(\mathcal{X}, p)$ with the introduced multiplication is a group.*

The group $\pi_1(\mathcal{X}, p)$ is called the fundamental group of $\mathcal{X}$ with base point p.

Proof. Recall that ε_p denotes the constant loop at p in $\mathcal{X}$; that is, $\varepsilon_p(t) = p$ for any t. We will show that the homotopy class $[\varepsilon_p]$ is the neutral element of $\pi_1(\mathcal{X}, p)$ and $[\bar{f}\,] = [f\,]^{-1}$, where $\bar{f}$ denoted the time-reversed f.

Note that conditions in the definition of group follow from the next three conditions for any loops f, g, and h based at p in $\mathcal{X}$.

(i) $f * \varepsilon_p \sim \varepsilon_p * f \sim f$;
(ii) $f * \bar{f} \sim \bar{f} * f \sim \varepsilon_p$;
(iii) $(f * g) * h \sim f * (g * h)$.

These statements are provided by 12.3, 12.5, and 12.6. □

12.10. Exercise. *Suppose that V and W are open subsets of topological space $\mathcal{X}$ such that $\mathcal{X} = V \cup W$, and the set $V \cap W$ is path-connected. Let $p \in V \cap W$. Show that any loop in $\mathcal{X}$ based at p is homotopic to a product of loops in V or W with the same base.*

D Simply-connected spaces

Recall that a group is called trivial if it contains only one element which is necessary the neutral element.

A path connected topological space with trivial fundamental group is called simply-connected.

If the fundamental group $\pi_1(\mathcal{X}, p)$ is trivial, it is common to write $\pi_1(\mathcal{X}, p) = 0$ despite that this equality does not have much sense — in general the group $\pi_1(\mathcal{X}, p)$ is not commutative and so it would be more reasonable to write $\pi_1(\mathcal{X}, p) = \{1\}$, meaning that 1 is the only element of $\pi_1(\mathcal{X}, p)$.

12.11. Exercise. *Show that any contractible space is simply-connected.*

Chapter 13

Meeting algebra

In the previous chapter we gave a construction of the fundamental group for a given topological space with marked point. This construction gives a bridge between topology and abstract algebra.

In this chapter we explain how to use this bridge. In this chapter we describe In the next chapter, we will calculate fundamental group of the circle — this will be the first example of a space with nontrivial fundamntal group. Once this group is calculated, the connections of this chapter will lead to many applications.

A Induced homomorphism

13.1. Claim. *Let* $\varphi\colon \mathcal{X} \to \mathcal{Y}$ *be a continuous map and* $\varphi(p) = q$. *Suppose that* f_0 *and* f_1 *are loops bases at* p *in* $\mathcal{X}$. *Then*

(a) $\varphi \circ (f_0 * f_1) = (\varphi \circ f_0) * (\varphi \circ f_1)$,

(b) if $f_0 \sim f_1$, *then* $\varphi \circ f_0 \sim \varphi \circ f_1$.

Proof; (a). Applying the definition of the product of paths and composition of maps to $\varphi \circ (f_0 * f_1)$ and $(\varphi \circ f_0) * (\varphi \circ f_1)$ we get exactly the same expression:

$$\begin{cases} \varphi \circ f_0(t) & \text{if } t \leqslant \frac{1}{2}, \\ \varphi \circ f_1(t) & \text{if } t \geqslant \frac{1}{2}. \end{cases}$$

Hence *(a)* follows.

(b). Observe that if f_τ is a homotopy from f_0 to f_1, then $\varphi \circ f_\tau$ is a homotopy from $\varphi \circ f_0$ to $\varphi \circ f_1$. Hence *(b)* follows. □

13.2. Corollary. *Let $\varphi\colon \mathcal{X}\to\mathcal{Y}$ be a continuous map between topological spaces; suppose $\varphi(p)=q$. Given a loop f with base at p in $\mathcal{X}$, the composition $\varphi\circ f$ is a loop with base at q in $\mathcal{Y}$. Moreover, $\varphi_*\colon [f]\mapsto[\varphi\circ f]$ defines a homomorphism $\varphi_*\colon \pi_1(\mathcal{X},p)\to\pi_1(\mathcal{Y},q)$.*

13.3. Exercise. *Consider continuous maps $\mathcal{X}\xrightarrow{\varphi}\mathcal{Y}\xrightarrow{\psi}\mathcal{Z}$ between topological spaces. Show that $\psi_*\circ\varphi_*=(\psi\circ\varphi)_*$.*

13.4. Exercise. *Let A be a retract of $\mathcal{X}$ with retraction $r\colon\mathcal{X}\to A$. Choose $a\in A$. Show that $r_*\colon\pi_1(\mathcal{X},a)\to\pi_1(A,a)$ is a surjective homomorphism.*

B Base point

13.5. Theorem. *Let p and q be two points in a topological space $\mathcal{X}$. Suppose there is a path h from p to q, then the fundamental groups $\pi_1(\mathcal{X},p)$ and $\pi_1(\mathcal{X},q)$ are isomorphic.*

Recall that product of paths is not associative; that is, we might have that $(f*g)*h\ne f*(g*h)$ for some paths f,g,h such that both sides of the equation are defined. In other words, we have to fully parenthesize the products of paths. If the product is not parenthesized we assume that the product is taken in the usual order; that is,

$$f*g*h*k:=((f*g)*h)*k.$$

Proof. Suppose f is a loop based at p. Note that $\bar h*f*h$ is a loop at q. Moreover, the map $f\mapsto\bar h*f*h$ induces a homomorphism $u_h\colon\pi_1(M,p)\to\pi_1(M,q)$.

Indeed, suppose f_τ is a homotopy of loops at p. Then $\bar h*f_\tau*h$ is a homotopy of loops at q. It follows that the map

$$u_h\colon[f]\mapsto[\bar h*f*h]$$

is defined; that is, the right-hand side does not depend on the choice of loop f in the homotopy class $[f]$.

Further, if f and g are loops based at p, then 12.3, 12.5, and 12.6 imply that

$$\begin{aligned}(\bar h*f*h)*(\bar h*g*h)&\sim\bar h*f*(h*\bar h)*g*h\sim\\&\sim\bar h*(f*\varepsilon_p)*g*h\sim\\&\sim\bar h*(f*g)*h.\end{aligned}$$

Whence the map $u_h\colon \pi_1(M,p) \to \pi_1(M,q)$ is a homomorphism; that is,

$$u_h([f]\cdot[g]) = u_h[f]\cdot u_h[g] \quad \text{for any} \quad [f],[g] \in \pi_1(\mathcal{X},p).$$

The same argument shows that $u_{\bar h}\colon \pi_1(M,q) \to \pi_1(M,p)$ defined by

$$u_{\bar h}\colon [k] \mapsto [h * (k * \bar h)]$$

is a homomorphism.

Finally note that

$$\begin{aligned} h * (\bar h * f * h) * \bar h &\sim (h * \bar h) * f * (h * \bar h) \sim \\ &\sim \varepsilon_p * f * \varepsilon_p \sim \\ &\sim f \end{aligned}$$

for any loop f based at p. Therefore, $u_{\bar h}$ is a left inverse of u_h. The same way we show that u_h is a left inverse of $u_{\bar h}$. It follows that u_h is an isomorphism. □

According to the theorem, the fundamental group (more precisely its *isomorphism class*) of path-connected space does not depend on its base point. Therefore, for a path-connected space $\mathcal{X}$ we do not need to specify the base point of its fundamental group; so we could write $\pi_1(\mathcal{X})$ instead of $\pi_1(\mathcal{X},p)$ (if we think about the group up to isomorphism).

13.6. Exercise. *Let $\varphi_\tau\colon \mathcal{X} \to \mathcal{Y}$ be a homotopy. Suppose that $q_0 = \varphi_0(p)$ and $q_1 = \varphi_1(p)$; consider the path from q_0 to q_1 defined by $h(t) = \varphi_\tau(p)$. Show that*

$$u_h \circ \varphi_{0*} = \varphi_{1*}.$$

13.7. Exercise. *Suppose that path-connected topological spaces $\mathcal{X}$ and $\mathcal{Y}$ have the same homotopy type. Use 13.6 to show that their fundamental groups are isomorphic.*

C Product space

13.8. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two path-connected topological spaces. Choose points $p \in \mathcal{X}$ and $q \in \mathcal{Y}$. Consider the projections $\varphi\colon \mathcal{X}\times\mathcal{Y} \to \mathcal{X}$ and $\psi\colon \mathcal{X}\times\mathcal{Y} \to \mathcal{Y}$ and their induced homomorphisms $\varphi_*\colon \pi_1(\mathcal{X}\times\mathcal{Y},(p,q)) \to \pi_1(\mathcal{X},p)$ and $\psi_*\colon \pi_1(\mathcal{X}\times\mathcal{Y},(p,q)) \to \pi_1(\mathcal{Y},q)$. Define $\Phi\colon \pi_1(\mathcal{X}\times\mathcal{Y},(p,q)) \to \pi_1(\mathcal{X},p)\times\pi_1(\mathcal{Y},q)$ by*

$$\Phi\colon \alpha \mapsto (\varphi_*(\alpha), \psi_*(\alpha))$$

for any $\alpha \in \pi_1(\mathcal{X}\times\mathcal{Y},(p,q))$*. Note that the map* Φ *is a homomorphism.*

(a) *Show that* Φ *is a monomorphism; that is, if* $\Phi(\alpha) = \Phi(\beta)$ *for some* $\alpha, \beta \in \pi_1(\mathcal{X} \times \mathcal{Y},(p,q))$*, then* $\alpha = \beta$*.*

(b) *Show that* Φ *is an epimorphism; that is, for any* $\gamma \in \pi_1(\mathcal{X}, p) \times \times \pi_1(\mathcal{Y}, q)$ *there is* $\alpha \in \pi_1(\mathcal{X} \times \mathcal{Y},(p,q))$ *such that* $\Phi(\alpha) = \gamma$*.*

Conclude that $\pi_1(\mathcal{X} \times \mathcal{Y})$ *is isomorphic to* $\pi_1(\mathcal{X}) \times \pi_1(\mathcal{Y})$*.*

13.9. Advanced exercise. *Let* $\mathcal{X}$ *be a topological space. Consider the quotient space* $\mathcal{Y} = (\mathcal{X}\times\mathcal{X})/\sim$ *where* $\sim$ *is the minimal equivalence relation such that* $(x, y) \sim (y, x)$*. Show that the fundamental group* $\pi_1(\mathcal{Y},(x, x))$ *is commutative for any* $x \in \mathcal{X}$*.*

Chapter 14

Fundamental group of a circle

A The statement

14.1. Theorem. *The fundamental group of the circle $\mathbb{S}^1$ is isomorphic to the additive group of integers $\mathbb{Z}$.*

Given a loop $f\colon[0,1]\to\mathbb{S}^1$, denote by $\deg f$ the number of times it goes aroond $\mathbb{S}^1$ counteclockwise (each clockwise turn is counded with minus). In the following sections we will define deg formally, prove that it depends only on the homotopy class of f (in other words, if $f_0\sim f_1$, then $\deg f_0=\deg f_1$) and then show that $\deg\colon\pi_1(\mathbb{S}^1)\to\mathbb{Z}$ is an isomorphism.

Let us show how to apply this theorem before diving into the proof.

14.2. Exercise. *Apply the theorem to show the following.*

(a) The space $\mathbb{R}^3$ is not homeomorphic to the plane $\mathbb{R}^2$.

(b) The circle $\mathbb{S}^1=\partial\mathbb{D}^2$ is not a retract of the disc $\mathbb{D}^2$.

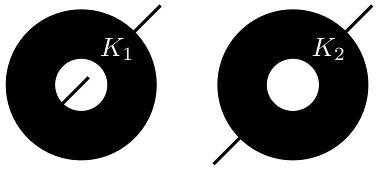
(c) Given $x\in\mathbb{D}^2$ the complement $\mathbb{D}^2\setminus\{x\}$ is simply connected if and only if $x\in\mathbb{S}^1=\partial\mathbb{D}^2$. Conclude that any homeomorphism $\mathbb{D}^2\to\mathbb{D}^2$ maps $\partial\mathbb{D}$ to itself.

(d) The complement of any finite nonempty set $F\subset\mathbb{R}^2$ is not contractible.

Part *(b)* in the last exercise has the following application.

14.3. Brouwer fixed point theorem in the plane. *Any continuous map $f\colon\mathbb{D}^2\to\mathbb{D}^2$ has a fixed point.*

Proof. Arguing by contradiction, assume $x \neq f(x)$ for all $x \in \mathbb{D}^2$. Consider given $x \in \mathbb{D}^2$ consider the half-line H from $f(x)$ to x. Note that H intesects $\mathbb{S}^1 = \partial\mathbb{D}^2$ at a sinle point; denote it by $h(x)$. Note that h is continuous and $h(x) = x$ if $x \in \partial\mathbb{D}^2$; that is, h defines a retruction $\mathbb{D}^2 \to \mathbb{S}^1$. The later contradicts 14.2*b*. ☐



14.4. Advanced exercise. *Consider two sets K_1 and K_2 in the plane, shown in the picture. Each one being a closed annulus with two attached line segments. In K_1 one segment is attached from the inside and another from the outside; in K_2 both segments are attached from the outside.*

(a) *Show that $K_1 \not\simeq K_2$.*

(b) *Show that $K_1 \times [0,1] \simeq K_2 \times [0,1]$.*

B Liftings

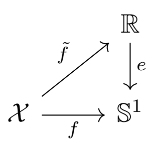
Let us identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$; namely, we will encode a point $(x,y) \in \mathbb{R}^2$ by the complex number $z = x + i\cdot y$; here i denotes the imaginary unit. In particular, $\mathbb{S}^1 = \{\, z = x + i\cdot y \in \mathbb{C} \,:\, |z|^2 = x^2 + y^2 = 1 \,\}$.

Consider the map $e\colon \mathbb{R} \to \mathbb{S}^1 \subset \mathbb{C}$ defined by

$$e\colon x \mapsto \exp(2\cdot\pi\cdot i\cdot x) = \cos(2\cdot\pi\cdot x) + i\cdot \sin(2\cdot\pi\cdot x)).$$

(The last equality is called Euler's idenity, it can be taken as a definition of $\exp(2\cdot\pi\cdot i\cdot x)$.)

Let $f\colon \mathcal{X} \to \mathbb{S}^1$ be a continuous map. A continuous map $\tilde f\colon \mathcal{X} \to \mathbb{R}$ will be called l i f t i n g of f if $f = e \circ \tilde f$; in other words if the following diagram c o m m u t e s; the latter means that following directed paths in



the diagram with the same start and endpoints lead to the same result. Even in this simple case, using a commutative diagram is helpful, and later on you simply can't do without them.

14.5. Proposition. *Let $f_0\colon [0,1] \to \mathbb{S}^1$ be a path and let $x_0 \in \mathbb{R}$ be a point such that $e(x_0) = f_0(0)$. Then there is unidue lifting $\tilde f_0$ of f such that $\tilde f_0(0) = x_0$. Moreover, if $f_0 \sim f_1$ then $\tilde f_0 \sim \tilde f_1$.*

14.6. Lemma. *Let $W = \mathbb{S}^1 \setminus \{u_0\}$ for some $u_0 = e(x_0) \in \mathbb{S}^1$. Then $e^{-1}(W)$ is a disjoint union of open intervals $V_n = (x_0 + n, x_0 + n + 1)$ for $n \in \mathbb{Z}$ and e defines a homeomorphism $V_n \to U$ for each n.*

In particular, given a connected space $\mathcal{X}$, continuous map $f\colon \mathcal{X} \to \to W$, and a point $q \in \mathbb{R}$ such that $e(q) = f(p)$ for some $p \in \mathcal{X}$ there is a unique lifting $\tilde{f}\colon \mathcal{X} \to \mathbb{R}$ of f such that $\tilde{f}(p) = q$.

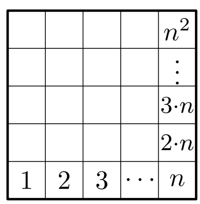
Proof. Observe that $e_n = e|_{V_n}$ is homeomorphism $V_n \to W$. Indeed, e_n is a continuous bijection. Furthermore, by 8.12 $e|_{\bar{V}_n}$ is a closed map. Therefore $e_n = e|_{V_n}$ is a closed continuous bijection $V_n \to W$ and hence a homeomorphism.

Note that the intervals V_n are connected components of $e^{-1}W$. In particular, $q \in V_n$ for some n. Since $\mathcal{X}$ is connected, $\tilde{f}(\mathcal{X}) \subset V_n$ for any lifting $\tilde{f}$ lifting of f such that $e(q) = f(p)$. Since $e_n = e|_{V_n}$ is a homeomorpism, we have $\tilde{f} = e_n^{-1} \circ f$, which proves existance and uiqueness. □

Proof of 14.5. Consider two open subsets $W_1 = \mathbb{S}^1 \setminus (1,0)$ and $W_2 = \mathbb{S}^1 \setminus (-1,0)$. Note that $\mathbb{S}^1 = W_1 \cup W_2$. Therefore, $[0,1] = f^{-1}(W_1) \cup \cup f^{-1}(W_2)$.

Let $\varepsilon > 0$ be the Lebesgue number of this covering; see 7.1. Consider a particition $0 = t_0 < t_1 < \dots < t_n = 1$ of $[0,1]$ into equal intervals shorter than ε. Note that $f([t_i, t_{i+1}]) \subset W_1$ or $f([t_i, t_{i+1}]) \subset W_2$ for each i.

By 14.6, there is unique lifting of $f|_{[t_0,t_1]}$ such that $\tilde{f}(0) = x_0$; let $x_1 = \tilde{f}(t_1)$. By the same argument, there is unique lifting of $f|_{[t_1,t_2]}$ such that $\tilde{f}(t_1) = x_1$. Repeating this argument finitely many times produces a lift of f, which is unique by construction. The continuity of f follows from 3.11b.



The proof of the last statement is similar. Choose a homotopy $F\colon [0,1] \times [0,1] \to \mathbb{S}^1$ from f_0 to f_1. Applying 7.1 again, we can subdivide $[0,1] \times [0,1]$ into small squares by horizontal and vertical lines so that F maps each small square in W_1 or in W_2. Then we can build the lifting $\tilde{F}$ square by square, from button up in the order shown in the picture.

When we extending $\tilde{F}$ over a small square □ the value of $\tilde{F}$ is already defined at its lower left corner, say v. So we can apply 14.6. The part of □ on which $\tilde{F}$ was already defined is one or two sides adjacent to v. In both cases this set is connected and by 14.6 the extension agrees with the already constructed part. Again, the continuity of $\tilde{F}$ follows from 3.11b. □

Degree

A loop in $\mathbb{S}^1$ based at 1 is a continuous map $f\colon [0,1] \to \mathbb{S}^1$ such that $f(0) = f(1) = 1$. Let $\tilde{f}\colon [0,1] \to \mathbb{R}$ be the unique lift with $\tilde{f}(0) = 0$. Since $e^{-1}(1) = \mathbb{Z}$, we have $\tilde{f}(1) \in \mathbb{Z}$. The integer $\tilde{f}(1)$ will be called degree of the loop f, it will be denoted by $\deg f$.

14.7. Theorem. *The map* $[f] \mapsto \deg f$ *defines an isomorphism* $\pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$.

Proof. First let us show that $\deg f$ depends only on the homotopy class of f. In other words the map $[f] \mapsto \deg f$ is well defined.

Suppose f_0 and f_1 are two homotopic loops in $\mathbb{S}^1$. ...

It remains to show that $\deg f_0 * f_1 = \deg f_0 + \deg f_1$ for any two loops f_0 and f_1. Let $\tilde{f}_0$ and $\tilde{f}_1$ be liftings of f_0 and f_1 such that $\tilde{f}_0(0) = \tilde{f}_1(0) = 0$. Recall that $\deg f_0 = \tilde{f}_0(1)$ and $\deg f_1 = \tilde{f}_1(1)$. Observe that $t \mapsto \deg f_0 + \tilde{f}_1$ is a lifting of f_1 s

We proved that degree defines a homomorphism $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$; it remains to show that this homomorphism is injective and surjective. Surjectivity follows ???. □

C Fundamental theorem of algebra

14.8. Theorem. *Every non-constant complex polynomial has a complex root.*

Proof. If a polynomial p has no roots, then for each $r \geqslant 0$ the loop $t \mapsto p(r\exp(2\pi i t))/|p(r\exp(2\pi i t))|$ lies in $\mathbb{S}^1$. For large r this loop is homotopic to $t \mapsto \exp(2\pi i k t)$, where k is the degree of p, while for $r = 0$ it is constant; this forces a change of degree, which is impossible under a homotopy rel endpoints.

Chapter 15

Jordan curve theorem

15.1. Theorem. *Suppose $\Gamma \subset \mathbb{R}^2$ is a closed set homeomorphic to $\mathbb{R}$. Then $\mathbb{R}^2 \setminus \Gamma$ has at least two connected components.*

Note that the assumption that Γ is closed is necessary; indeed, a finite open interval I of a line in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}$, but its complement $\mathbb{R}^2 \setminus I$ is connected.

The theorem follows from 15.2, 15.5, and 15.7.

15.2. Proposition. *Suppose $\Gamma \subset \mathbb{R}^2$ is a closed set such that the complement $X = \mathbb{R}^2 \setminus \Gamma$ is connected. Let us identify $\mathbb{R}^2$ with the (x, y)-plane in $\mathbb{R}^3$. Then the complement $Y = \mathbb{R}^3 \setminus \Gamma$ is simply-connected.*

The proof is based on the following partial case of the Van Kampen theorem.

15.3. Exercise. *Suppose that V and W are open simply-connected subsets of a topological space $\mathcal{X}$ such that $\mathcal{X} = V \cup W$, and the set $V \cap W$ is path-connected. Show that $\mathcal{X}$ is simply-connected.*

Conclude that the sphere $\mathbb{S}^2$ is simply-connected.

Proof of 15.2. Denote by A (respectively B) the sets that include Γ and the points below (respectively above) Γ; that is,

$$A = \{ (x, y, z) : (x, y) \in \Gamma \text{ and } z \leqslant 0 \},$$
$$B = \{ (x, y, z) : (x, y) \in \Gamma \text{ and } z \geqslant 0 \}.$$

Consider their complements $V = \mathbb{R}^3 \setminus A$ and $W = \mathbb{R}^3 \setminus B$. Note that $Y = V \cup W$.

The sets V and W are simply-connected. Indeed, the horizontal plane $z = 1$ is a deformation retract of V; a retraction can be defined by $(x, y, z) \mapsto (x, y, 1)$ and the following homotopy shows that it is homotopic to the identity map:

$$h_t(x, y, z) = (x, y, (1-t)\cdot z + t).$$

The plane is contractible, in particular simply-connected; therefore so is V. Similarly, one proves that W is simply-connected.

Since X is an open and connected set in $\mathbb{R}^2$, by 10.9, X is path-connected. Further, note that $V \cap W = X \times \mathbb{R}$. Therefore $V \cap W$ is path-connected as well.

Summarizing, V and W are open simply-connected sets, $Y = V \cup \cup W$, and $V \cap W$ is path-connected. Applying 15.3, we get that Y is simply-connected. □

15.4. Exercise. *Observe that 15.2 above does not hold without assuming that Γ is closed. Spot the place in the proof that breaks in this case.*

15.5. Proposition. *Suppose $\Gamma \subset \mathbb{R}^2$ is a closed subset homeomorphic to $\mathbb{R}$. Then there is a homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ that maps Γ to the z-axis.*

The following exercise provides a partial case of the so-called Tietze–Urysohn extension theorem.

15.6. Exercise. *Let φ be a continuous map from a closed subset $\Gamma \subset \mathbb{R}^2$ to the open interval $(1, 2)$. Given $x \in \mathbb{R}^2$, set*

$$a(x) = \inf_{y\in\Gamma}\{\,|x-y|\,\} \quad \text{and} \quad b(x) = \inf_{y\in\Gamma}\{\,\varphi(y)\cdot|x-y|\,\}.$$

Show that

$$f(x) = \begin{cases} \varphi(x) & \text{if } x \in \Gamma, \\ \frac{b(x)}{a(x)} & \text{if } x \notin \Gamma \end{cases}$$

defines a continuous function $f\colon \mathbb{R}^2 \to (1, 2)$.

Conclude that if there is a homeomorphism $h\colon \mathbb{R} \to \Gamma$, then there is a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f \circ h(t) = t$ for any $t \in \mathbb{R}$.

The following proof uses the so-called Klee trick, which is quite useful in many topological problems.

Proof of 15.5. Let $h\colon t \mapsto (a(t), b(t))$ be a homeomorphism $\mathbb{R} \to \Gamma$. By 15.6, there is a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(a(t), b(t)) = f \circ h(t) = t$$

for any $t \in \mathbb{R}$.

Note that the map

$$F\colon (x,y,z)\mapsto (x,y,z+f(x,y))$$

is a homeomorphism. Indeed, this map is continuous and its inverse

$$F^{-1}\colon (x,y,z)\mapsto (x,y,z-f(x,y))$$

is continuous as well.

Similarly, the map

$$G\colon (x,y,z)\mapsto (x-a(z),y-b(z),z)$$

is a homeomorphism as well. Indeed, G is continuous and it has an inverse

$$G^{-1}\colon (x,y,z)\mapsto (x+a(z),y+b(z),z)$$

that is continuous as well.

It follows that the composition $G\circ F\colon \mathbb{R}^3\to\mathbb{R}^3$ is a homeomorphism. Since $f(a(t),b(t))=t$,

$$G\circ F(a(t),b(t),0)=G(a(t),b(t),t)=(0,0,t).$$

It follows that $G\circ F$ sends Γ to the z-axis as required. $\square$

15.7. Exercise. *Show that the complement of the z-axis in $\mathbb{R}^3$ is not simply-connected.*

15.8. Theorem. *Let $J\subset\mathbb{S}^2$ be a subset homeomorphic to $\mathbb{S}^1$. Then $\mathbb{S}^2\setminus J$ has at least two connected components.*

This theorem is a partial case of the famous Jordan theorem; it is known for its simple formulation and annoyingly tricky proofs. The presented proof is due to Patrick Doyle [6]; it is among the shortest proofs, but it uses quite a bit of topology.

Proof. Remove a point p from J to get a closed line $\Gamma=J\setminus\{p\}$ in $\mathbb{S}^2\setminus\{p\}\simeq\mathbb{R}^2$. It remains to apply 15.1. $\square$

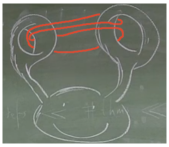
Appendix A

Semisolutions

0.1. Topologically speaking, a T-shirt is a disc with three holes, two for the hands and one for the head; pants have two holes, one for each leg. In an ideal universe socks should not have holes, but apparently our universe is not ideal.

0.2. He saw a hole in the bottom, so the topology of the mug is the same as a donut (topologically speaking, a solid torus).

0.3. Draw the hair tie on each picture, the last one is shown.



Source. Suggested by Rostislav Matveev.

0.4; *(a).* Push the loop up thru the other handle, bring it around over the points and back over the handles; the string will come off.

(b). Take the center of the string holding your wrists, push it up thru one of the loops on your friend's wrist and bring it down over his hand.

(c). Use the same trick as in *(b).*

Source. Part *(a)* and *(b)* appear in Harry Houdini's book [11]. A version of part *(a)* appears earlier in the collection of Sam Loyd [13] under the name "The Gordian knot". Part *(b)* under the name "The prisoners' release puzzle" appears in Cassell's book [5] published in 1881. Both puzzles should be much older.

1.2. Check the triangle inequality for 0, $\frac{1}{2}$, and 1.

1.3. Check the conditions in 1.1.

1.4. Check all the conditions in Definition 1.1. Further we discuss the triangle inequality — the remaining conditions are nearly evident.

Let $a = (x_a, y_a)$, $b = (x_b, y_b)$, and $c = (x_c, y_c)$. Set

$$x_1 = x_b - x_a, \qquad y_1 = y_b - y_a,$$
$$x_2 = x_c - x_b, \qquad y_2 = y_c - y_b.$$

(a). The inequality

$$|a - c|_1 \leqslant |a - b|_1 + |b - c|_1$$

can be written as

$$|x_1 + x_2| + |y_1 + y_2| \leqslant |x_1| + |y_1| + |x_2| + |y_2|.$$

The latter follows since $|x_1 + x_2| \leqslant |x_1| + |x_2|$ and $|y_1 + y_2| \leqslant |y_1| + |y_2|$.

(b). The inequality

❶ $$|a - c|_2 \leqslant |a - b|_2 + |b - c|_2$$

can be written as

$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \leqslant$$
$$\leqslant \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}.$$

Take the square of the left and the right-hand sides, simplify, take the square again and simplify again. You should get the following inequality:

$$0 \leqslant (x_1 \cdot y_2 - x_2 \cdot y_1)^2,$$

which is equivalent to ❶ and evidently true.

(c). The inequality

$$|a - c|_\infty \leqslant |a - b|_\infty + |b - c|_\infty$$

can be written as

❷ $$\max\{|x_1+x_2|,|y_1+y_2|\} \leqslant \leqslant \max\{|x_1|,|y_1|\}+\max\{|x_2|,|y_2|\}.$$

Without loss of generality, we may assume that

$$\max\{|x_1+x_2|,|y_1+y_2|\} = |x_1+x_2|.$$

Further,

$$|x_1+x_2| \leqslant |x_1|+|x_2| \leqslant \leqslant \max\{|x_1|,|y_1|\}+\max\{|x_2|,|y_2|\}.$$

Hence ❷ follows.

1.6. Show that the triangle inequality implies that $|f(x)-f(y)|<\varepsilon$ if $|x-y|_{\mathcal{X}}<\varepsilon$; make a conclusion.

1.7. Fix $x\in\mathcal{X}$ and $y\in\mathcal{Y}$ such that $f(x)=y$.

Fix $\varepsilon>0$. Since g is continuous at y, there is a positive value δ_1 such that

$$|g(y')-g(y)|_{\mathcal{Z}}<\varepsilon \quad \text{if} \quad |y'-y|_{\mathcal{Y}}<\delta_1.$$

Since f is continuous at x, there is $\delta_2>0$ such that

$$|f(x'-f(x)|_{\mathcal{Y}}<\delta_1 \quad \text{if} \quad |x'-x|_{\mathcal{X}}<\delta_2.$$

Since $f(x)=y$, we get that

$$|h(x')-h(x)|_{\mathcal{Z}}<\varepsilon \quad \text{if} \quad |x'-x|_{\mathcal{X}}<\delta_2.$$

Hence the result.

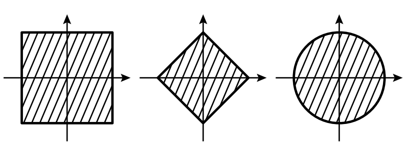
1.8. *(a)* Show that the triangle inequality implies that $|f(x)-f(y)|_{\mathcal{Y}}<\varepsilon$ if $|x-y|_{\mathcal{X}}<\varepsilon$; make a conclusion.

(b). Apply 1.1*b*.

1.9. Show and use that in 1.5 one can take $\delta=1$ for any $\varepsilon>0$.

1.10. Learn about space-filling curves and think.

1.11. Figure out which is which.



1.12. Apply the triangle inequality to x, y, and $z\in B(y,R)\setminus B(x,r)$. For the second part, consider the balls $B(2,3)$ and $B(0,4)$ in $[0,\infty)$.

Comment. Note that we used that $B(y,R)\neq \neq B(x,r)$, without this condition there are no general restrictions on r in terms of R. For example the inclusion $B(x,1000)\subset B(y,1)$ holds in the discrete space.

1.14. Spell the definitions.

1.16. If $y\in B(x,R)$, then $r=R-|x-y|> >0$. Use the triangle inequality to show that $B(y,r)\subset B(x,R)$. Make a conclusion.

1.17. Apply 1.15.

1.18. Apply 1.15.

1.19. The if part follows from 1.16 and 1.17. It remains to prove the only-if part.

Let V be an open set. By 1.15, for any $x\in V$ there is $r_x>0$ such that $B(x,r_x)\subset V$. Observe that

$$V=\bigcup_{x\in V} B(x,r_x).$$

1.20. Consider the open segments $(-\varepsilon,\varepsilon)$ for all $\varepsilon>0$ in $\mathbb{R}$. Note that

$$\{0\}=\bigcap_{\varepsilon>0}(-\varepsilon,\varepsilon)$$

and the one-point set $\{0\}$ is not open.

1.21. Show and use that

$$B(x,r)_1\subset B(x,r)_2\subset B(x,r)_\infty\subset B(x,2\cdot r)_1;$$

here $B(x,r)_1$, $B(x,r)_2$, and $B(x,r)_\infty$ denote the balls in the metrics $|{*}-{*}|_1$, $|{*}-{*}|_2$, and $|{*}-{*}|_\infty$ respectively.

1.23. Look at the image of $\mathbb{R}$ under the function $x\mapsto|x|$.

1.25. Assume the contrary; that is, a sequence $x_1,x_2,\dots$ has two limits y and z. Set $r=|y-z|$. Note that $B(y,\frac{r}{2})$ contains all but finitely many elements of the sequence $x_1,x_2,\dots$; the same holds for $B(z,\frac{r}{2})$. Observe that $B(y,\frac{r}{2})\cap \cap B(z,\frac{r}{2})=\varnothing$ and arrive at a contradiction.

1.26. Suppose f is not continuous. This means that there is a point x_∞ and $\varepsilon>0$ such that there is a point $x_n\in B(x_\infty,\frac{1}{n})$ such that $|f(x_n)-f(x_\infty)|>\varepsilon$. In particular, $y_n=f(x_n)$ does not converge to $y_\infty=f(x_\infty)$. It proves the if part of the exercise.

To prove the only-if part, suppose that there is a sequence $x_n\to x_\infty$ such that $y_n\not\to y_\infty$ as $n\to\infty$. Note that in this case we can pass to a subsequence so that $x_n\in B(x_\infty,\frac{1}{n})$ and $|y_n-y_\infty|>\varepsilon$ for some fixed $\varepsilon>0$. From above, f is not continuous.

1.27. Show that the semiopen interval $[0, 1)$ is neither open nor closed in $\mathbb{R}$.

1.28. Choose a point $z \in \bar{\bar{A}}$. It means that there is a sequence $y_1, y_2, \dots \in \bar{A}$ such that $y_n \to z$ as $n \to \infty$. The latter means that for each y_i there is a sequence $x_{i,1}, x_{i,2}, \dots \in A$ such that $x_{i,n} \to y_i$ as $n \to \infty$. Try to choose a sequence of integers m_n such that $x_{n,m_n} \to z$ as $n \to \infty$. Make a conclusion.

1.29. Show that Q is closed if $x \in Q$ if and only if $\mathrm{B}(x, \varepsilon) \cap Q \neq \varnothing$ for any $\varepsilon > 0$. Show that the latter is equivalent to $y \in V$ if and only if $\mathrm{B}(y, \varepsilon) \subset V$ for some $\varepsilon > 0$. Make a conclusion.

2.2. Let $\mathcal{X}$ be an infinite set with cofinite topology. Show that any two nonempty open sets in $\mathcal{X}$ have nonempty intersection. Show that the latter does not hold for open balls in a metric space with at least two points.

2.3. Let $\mathcal{F}$ be a finite metric space. Observe that there is $\varepsilon > 0$ such that $|x - y| > \varepsilon$ for any two distinct points $x, y \in \mathcal{F}$. It follows that $\{x\} = \mathrm{B}(x, \varepsilon)$ for any $x \in \mathcal{F}$; in particular, each one-point set is open. By 2.1*b*, any set in $\mathcal{F}$ is open.

2.4. Choose $W \in \mathscr{W}$. By assumption, for any $w \in W$ there is $S_w \in \mathscr{S}$ such that $W \supset S_w \ni w$. Observe that

$$W = \bigcup_{w \in W} S_w.$$

It follows that $\mathscr{W} \subset \mathscr{S}$; in other words, any $\mathscr{W}$-open set is $\mathscr{S}$-open.

2.6; *(a).* Consider the function defined by

$$f(x) = \begin{cases} a & \text{if } x < 0, \\ b & \text{if } x \geqslant 0. \end{cases}$$

(b). Suppose $f\colon \mathcal{X} \to \mathbb{R}$ is nonconstant; that is, $f(a) \neq f(b)$. Note that $W = \mathbb{R} \setminus \{f(a)\}$ is an open set containing b. Assume f is continuous. Then $\{b\} = f^{-1}(W)$ is an open set — a contradiction.

2.7. Apply the definitions.

2.8. Note that $S \in \mathscr{T}$ if and only for any $x_0 \leqslant x_1$, if $x_0 \in S$, then $x_1 \in S$.

(a). Check the conditions in 2.1 using the property above.

(b). Show that every two nonempty open sets in $(\mathbb{R}, \mathscr{T})$ intersect. Show that the latter statement does not hold in a metric space with at least two points. Make a conclusion.

(c). To do the only-if part check the condition in 2.5.

To do the if part, suppose f is not nondecreasing; that is, we can find $x_0 < x_1$ such that $f(x_0) > f(x_1)$. Note that the inverse image $V = f^{-1}([f(x_0), \infty))$ contains x_0 but does not contain x_1. Conculde that V is not open in $(\mathbb{R}, \mathscr{T})$, so $f\colon (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T})$ is not continuous.

2.9. Apply 2.7 together with the fact that $x \mapsto |x|$ is a continuous real-to-real function (the latter is assumed to be known from calculus).

3.3; *(a)* Choose a metric that induces the topology on $\mathcal{X}$. Let Q be a closed set in $\mathcal{X}$. Given $\varepsilon > 0$, let W_ε be the union of ε-balls centered at points in Q. Show that W_ε is open and that $Q = \bigcap W_\varepsilon$.

(b). Try to find the needed closed set in the connected two-point space; see 2B.

3.4. Suppose V is an open subset and Q is its complement. Recall that Q is closed; see 3A. Show and use that $V \subset A$ if and only if $Q \supset B$.

3.5; Show and use the following for any two subsets A and B:

- $\mathring{\mathring{A}} = \mathring{A} \subset A \subset \bar{A} = \bar{\bar{A}}$.
- if $A \subset B$, then $\bar{A} \subset \bar{B}$ and $\mathring{A} \subset \mathring{B}$.

For the second part, try to choose a subset A in $\mathbb{R}$ so that it meets the following conditions:

- A and $\mathbb{R} \setminus A$ contain isolated points,
- A and $\mathbb{R} \setminus A$ contain intervals,
- A and $\mathbb{R} \setminus A$ are dense in some interval.

3.6–3.7. Apply the definitions of boundary and closed set.

3.8. Read about the Cantor set and think.

3.10. Spell the definitions. The needed examples can be found for $\mathcal{A} = [0, 1) \subset \mathbb{R} = \mathcal{X}$.

3.11. Set $a = f|_A$ and $b = f|_B$. Note that for any set $S \subset \mathcal{Y}$, we have

$$\begin{aligned} a^{-1}(S) &= f^{-1}(S) \cap A, \\ b^{-1}(S) &= f^{-1}(S) \cap B, \\ f^{-1}(S) &= a^{-1}(S) \cup b^{-1}(S). \end{aligned}$$

(a). Choose an open set $W\subset\mathcal{Y}$. Show that $a^{-1}(W)$ and $b^{-1}(W)$ are open in $\mathcal{X}$. Conclude that $f^{-1}(W)$ is open.

(b). Choose a closed set $Q\subset\mathcal{Y}$. Use 3.2 to show that $a^{-1}(Q)$ and $b^{-1}(Q)$ are closed. Apply 3.1*c* to show that $f^{-1}(Q)$ is closed. Apply 3.2 again.

(c). Consider function f that is constant on disjint sets A and B such that $\bar{A}\cap B\ne\varnothing$.

3.12. Let A be a subset of a topological space $\mathcal{X}$. Denote by B its complement; that is $B = \mathcal{X}\setminus A$. Show and use that the statement is equivalent to the following

$$\mathcal{X}\setminus\partial A=\mathring{A}\cup\mathring{B}.$$

3.13. Apply the definitions of neighborhood and dense set.

3.15+3.16. Apply the definition of the convergence.

4.1. Let $f\colon\mathcal{X}\to\mathcal{Y}$ be a homeomorphism. Recall that by the definition of inverse, we have $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$ for any $x\in\mathcal{X}$ and $y\in\mathcal{Y}$. It remains to show that the existence of $f^{-1}\colon\mathcal{Y}\to\mathcal{X}$ implies that f is a bijection $\mathcal{X}\leftrightarrow\mathcal{Y}$.

For the second part, try to find such bijection for the subspaces $A=[0,1)\cup\{2\}$ and $B=[0,1]$ of $\mathbb{R}$.

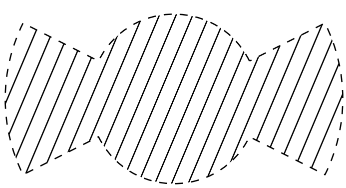
4.2. Observe that $y\mapsto\ln y$ is the inverse of $x\mapsto e^x$, and show that both functions are continuous. (You can use that differentiable functions are continuous.)

4.3. Try to build the needed function from $x\mapsto\arctan x$ or $x\mapsto e^{-e^x}$

4.4. Apply the definitions of homeomorphism and 2.7.

4.5. Learn about inversion and try to apply it.

4.6. Let Ω be an open star-shaped set with respect to the origin. Assume Ω can be described by the inequality $r<f_n(\theta)$ in the polar (r,θ)-coordinates, where $f_n\colon\mathbb{S}^1\to\mathbb{R}$ is a continuous function. In this case it is not hard to prove the statement. But a general star-shaped set, for example the one on the diagram is problematic.



To do the general case, show that Ω can be presented as a union of a nested sequence of open sets $\Omega_0\subset\Omega_1\subset\dots$ such that each Ω_n can be described by the inequality $r<f_n(\theta)$ in the polar (r,θ)-coordinates with continuous $f_n\colon\mathbb{S}^1\to\mathbb{R}$. We can assume that Ω_0 is a round disc around the origin.

Further construct a sequence of homeomorphisms $\varphi_n\colon\Omega_{n-1}\to\Omega_n$ such that the compositions $\Phi_n=\varphi_n\circ\dots\circ\varphi_1\colon\Omega_0\to\Omega_n$ stabilizes for each $x\in\Omega_0$; that is, $\Phi_n(x)$ is a fixed point for all sufficiently large n. Set

$$\Phi(x)=\lim_{n\to\infty}\Phi_n(x),$$

and show that Φ defines the needed homeomorphism $\Omega_0\leftrightarrow\Omega$.

4.7. Suppose that the sets are $P=\{p_1,p_2,\dots\}$ and $Q=\{q_1,q_2,\dots\}$. Try to construct a sequence of homeomorphisms $\Phi_n\colon\mathbb{R}^2\leftrightarrow\mathbb{R}^2$ such that Φ_n converges to a homeomorphism $\Phi\colon\mathbb{R}^2\leftrightarrow\mathbb{R}^2$ and for any n we have $\Phi_n(\{p_1,\dots,p_n\})\subset Q$ and $\Phi_n^{-1}(\{q_1,\dots,q_n\})\subset P$.

4.9. Apply the definitions.

4.10. Try the maps between two-point spaces with appropriate topologies.

4.11. Construct a nonempty closed nowhere dense set K without isolated points. Map K to 0 and map each component of $\mathbb{R}\setminus K$ by a homeomorphism to $\mathbb{R}$.

Comment. If a function $f\colon\mathbb{R}\to\mathbb{R}$ is closed and open, then it has to be continuous; see the paper by Ivan Baggs [2].

5.3. Apply the observation together with the facts that $(x,y)\mapsto x+y$, $(x,y)\mapsto x{\cdot}y$, and $(x,y)\mapsto\max\{x,y\}=\frac12{\cdot}|x+y|+\frac12{\cdot}|x-y|$ are continuous functions on $\mathbb{R}^2$.

5.4. Check the following function

$$f(x,y)=\begin{cases}0&\text{if}\quad x=0\quad\text{or}\quad y=0,\\ \frac{x}{y}&\text{if}\quad 0<x\leqslant y,\\ \frac{y}{x}&\text{if}\quad 0<y<x,\end{cases}$$

5.6; *only-if part.* Use that a base is a collection of open sets.

If part. Choose an open set $W\subset\mathcal{Y}$. By 5.5,

$$W=\bigcup_\alpha B_\alpha,$$

for some collection $\{B_\alpha\}$ of sets in the base. Then

$$f^{-1}(W) = \bigcup_\alpha f^{-1}(B_\alpha).$$

By the assumption $f^{-1}(B_\alpha)$ is open for any α; it remains to apply 2.1b.

5.7; *if part.* Choose an open set N. For any $x \in N$ choose an element B_x of the base such that $x \in B_x \subset N$. Observe that

$$N = \bigcup_{x \in N} B_x.$$

Only-if part. Suppose that $\mathscr{B}$ is a base. Then

$$N = \bigcup_\alpha B_\alpha,$$

where $B_\alpha \in \mathscr{B}$ for each α. Then for any $x \in N$ there is α such that $B_\alpha \ni x$; in this case, $x \in B_\alpha \subset N$.

5.9. Let $B_1 = \{a_1, a_1 + d_1, a_1 + 2\cdot d_1, \dots\}$ and $B_2 = \{a_2, a_2 + d_2, a_2 + 2\cdot d_2, \dots\}$. Show that if $a \in B_1 \cap B_2$, then

$$\{a, a + d, a + 2\cdot d, \dots\} \subset B_1 \cap B_2,$$

where $d = d_1 \cdot d_2$. Apply 5.8.

Show that the set $\{1\}$ is closed but not open.

Comment. This topology was introduced by Harry Furstenberg [8]; he used it to deduce from the last statement that the set of primes is infinite. The resulting proof is not truly new; it is just the classical proof from Euclid's Elements written in the topological language.

5.10; *only-if part.* Apply that a prebase is a collection of open sets.

If part. Show that for any finite collection of sets $P_1, \dots, P_n$ in the prebase the inverse image $f^{-1}(P_1 \cap \dots \cap P_n)$ is open. Further apply 5.7.

5.11; *(a)* Observe that $(V \times \mathcal{Y}) \cap (\mathcal{X} \times W) = V \times W$. Further, show and use that all the sets $V \times W$ for open subset $V \subset \mathcal{X}$ and $W \subset \mathcal{Y}$ form a base in $\mathcal{X} \times \mathcal{Y}$.

(b). To show that the map F is continuous, apply 5.10 to the prebase described before the exercise. Further, show and use that projection $G\colon (x, f(x)) \to x$ is a continuous left inverse; that is $G(F(x)) = x$ for any x.

5.12. Show that *every two disjoint closed sets of a metric space have disjoint open neighborhoods*; that is, for any two closed sets A and B there are open sets $V \supset A$ and $W \supset B$ such that $V \cap W = \varnothing$. (Topological spaces that share this property are called normal; so you need to show that *any metrizable space is normal.*)

Observe that arithmetic progression is a closed set in the initial topology. Construct two disjoint arithmetic progressions that do not admit disjoint open neighborhoods.

5.13. Check the conditions in 2.1 directly.

5.14+5.15. Spell out the definitions.

5.16. Since f is continuous, $V = f^{-1}(W)$ is open for any open set $W \subset \mathcal{Y}$. It remains to show that if V is open, then so is $W \subset \mathcal{Y}$.

Note that $W = f(V)$. If f is open, then $W = f(V)$ is open as well.

Set $A = \mathcal{X} \setminus V$ and $B = \mathcal{Y} \setminus W$. Since V is open A is closed. Since f is surjective, $B = f(A)$. Since f is a closed map, $B = f(A)$ is closed as well. Therefore, $W = \mathcal{Y} \setminus B$ is open.

5.17. Check the conditions in the definitions of equivalence relation and equivalence class.

5.18. Show that it has three points $a = [0]$, $b = [\frac{1}{2}]$, and $c = [1]$ and the open sets are

$$\varnothing,\ \{b\},\ \{a, b\},\ \{b, c\},\ \text{and}\ \{a, b, c\}.$$

5.19. Note that all positive numbers are in one $\mathbb{R}_+$-orbit. Similarly, all negative numbers are in one orbit and 0 forms another orbit. Thus, $\mathbb{R}/\mathbb{R}_+$ contains three points, say p, n, and z that correspond to positive numbers, negative numbers, and zero. It remains to describe all open subsets in $\{p, n, z\}$.

5.20. Let $f\colon \mathcal{X} \to \mathcal{X}/G$ be the quotient map. Show that for any set $V \subset \mathcal{X}$ we have

$$f^{-1} \circ f(V) = \bigcup_{g \in G} g \cdot V.$$

(a). Apply this formula to show that if V is open, then so is $f^{-1} \circ f(V)$. Finally apply the definition of quotient topology.

(b). Apply this formula to show that if G is finite and V is closed, then so is $f^{-1} \circ f(V)$.

6.2. Use 2.1b and 2.1c.

6.4. Spell the definitions.

6.5. We may assume that the space is nonempty; otherwise there is nothing to prove. Choose a nonempty set V_0 from the cover. Its complement is a finite set, say $\{x_1, \dots, x_n\}$. For each x_i choose a set $V_i \ni x_i$ from the cover. Observe that $\{V_0, \dots, V_0\}$ is a subcover.

6.6. Consider cover of S by intervals $(-c, c)$ for all $c > 0$.

6.7. Choose a point $s \in \bar{S} \setminus S$ and consider the cover by intervals $(-\infty, s - \varepsilon)$ and $(s + \varepsilon, +\infty)$ for all $\varepsilon > 0$.

6.8. Choose a noncompact space $\mathcal{X}$ and consider topology on the union $\mathcal{X} \cup \{a, b\}$ that includes $\mathcal{X} \cup \{a, b\}$, $\mathcal{X} \cup \{a\}$, $\mathcal{X} \cup \{b\}$ and all open sets in $\mathcal{X}$. Observe that the sets $\mathcal{X} \cup \{a\}$ and $\mathcal{X} \cup \{b\}$ are compact, but their intersection is not.

6.11. Apply the finite intersection property.

6.12. Let us write $A \Supset B$ if the interior of A contains closure of B.

Construct a strongly nested sequence of intervals $[0, 1] = [a_0, b_0] \Supset [a_1, b_1] \Supset \dots$ such that $[a_n, b_n] \cap Q_m = \varnothing$ for any $m \leqslant n$. Apply the finite intersection property (6.10) to show that the intersection $X = [a_0, b_0] \cap [a_1, b_1] \cap \dots$ is nonempty. Note that $X \cap Q_m = \varnothing$ for any m.

6.15. Show that $\mathbb{S}^1$ is an image of closed interval under a continuous map, and apply 6.14.

6.17. Apply 6.16 and 6.13.

6.19. Apply 6.14 to the projections $\mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ and $\mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$.

6.21. Apply 5.6.

6.23. Show by example that the obtained collection $\{V_{\alpha_1} \times W_{\alpha_1}, \dots, V_{\alpha_n} \times W_{\alpha_n}, V_{\alpha'_1} \times$ $\times\, W_{\alpha'_1}, \dots, V_{\alpha'_m} \times W_{\alpha'_m}\}$ might not cover the whole $\mathcal{X} \times \mathcal{Y}$.

6.24. By 3.2, it is sufficient to show that any closed set $A \subset \mathcal{K}$ has closed inverse image $B = f^{-1}(A) \subset \mathcal{X}$.

Observe that the set $C = \Gamma \cap (\mathcal{X} \times A)$ is closed, so its complement U can be presented as a union $\bigcup_\alpha V_\alpha \times W_\alpha$.

Suppose B is not closed, choose a point $p \in$ $\in \bar{B} \backslash B$. Note that $\{p\} \times \mathcal{K}$ is a compact set in U. Argue as in 6.18 to prove that there is an open set $N_p \ni p$ such that $N_p \times \mathcal{K} \subset U$. Arrive at a contradiction.

Remark. The following function $f\colon \mathbb{R} \to \mathbb{R}$ has closed graph, but is not continuous:

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It shows that compactness of $\mathcal{K}$ is a necessary assumption.

6.26. Apply the theorem to the prebase from 5.11a.

7.2. Consider an infinite set of points with discrete metric.

7.3. Suppose that a sequence x_n converges to x_∞ in $\mathcal{X}$ and y_n converges to y_∞ in $\mathcal{Y}$. Show and use that (x_n, y_n) converges to (x_∞, y_∞) in $\mathcal{X} \times \mathcal{Y}$ as $n \to \infty$.

7.5. By 7.4, any sequence has a converging subsequence; denote by x the limit of this subsequence. Show that if the sequence is Cauchy, then it converges to x.

8.3. Arguing by contradiction, assume a sequence has two limits x and y. Since the space is Hausdorff we can choose disjoint neighborhoods $V \ni x$ and $W \ni y$. Since the sequence converges to x, the set V contains all but finitely many elements of the sequence. The same holds for W — a contradiction.

8.4. Let V and W be a pair of open sets and $W' = \mathcal{X} \setminus \bar{V}$. Show and use V and W meet 8.1 if and only if V and W' meet 8.1.

8.5. The set Δ is closed if and only if its complement $U = (\mathcal{X} \times \mathcal{X}) \setminus \Delta$ is open. Show and use that the latter means that there is a family $\{(V_\alpha, W_\alpha)\}$ of disjoint pairs of open sets in $\mathcal{X}$ such that

$$U = \bigcup_\alpha V_\alpha \times W_\alpha.$$

8.9. Look at the subsets of a concrete space.

8.11. By 8.10, for any $y \in L$ there is a pair of open sets V_y, W_y such that $V_y \supset K$ and $W_y \ni y$ such that $V_y \cap W_y = \varnothing$. Mimic the proof of 8.10 using these pairs.

8.14. Apply 8.12 to the map $[0, 1] \to \mathbb{S}^1$ defined by $t \mapsto (\cos(2{\cdot}\pi{\cdot}t), \sin(2{\cdot}\pi{\cdot}t))$.

8.15. Apply 8.12 to the map $\mathbb{D} \to \mathbb{R}^3$ that is written from polar to spherical coordinates as

$$(r, \theta) \mapsto (1, \theta, \pi{\cdot}r).$$

8.16. Let $K \subset \mathbb{R}^3$ be a convex body; we can assume that the origin lies in the interior of K.

Show that the boundary ∂K is compact. Show that any half-line that starts from the origin intersects ∂K at a single point. Conclude that $x \mapsto \frac{x}{|x|}$ defines a continuous bijection $\partial K \to \mathbb{S}^2$ and apply 8.13.

9.5. Apply 9.4.

9.7. Show that any open splitting of B splits A as well.

9.9. Apply 9.8, 9.4, and 9.6.

9.10. Connected component is an intersection of clopen sets; in particular it is closed.

Consider the following subspace of real line $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$. Show that the one-point set $\{0\}$ is a connected component in A and it is not open in A.

Remark. A more interesting example is the Cantor set; denote it by K. Each connceted compontnt of K is a one-point set is not open in K.

9.11. Check that being in one connected component defines an equivalence relation on points of topological space.

9.12. Use 9.10 and 9.11.

9.14. Show that $\mathbb{S}^1$ has no cut points, but $[0, 1]$ has.

9.13. Read the note by Solomon Colomb [9]

9.15. Show that $\mathbb{R}^2$ has no cut points, but $\mathbb{R}$ has.

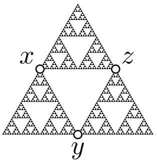
9.16. Count cut points and noncut points for each space.

9.17. Let x the cross pont; let us keep notation x for the corresponding point in Q/H. Observe that any two points distict from x one the circle can be mapped to eachother by a homeomorphism $Q \to Q$. It implies that all these points corresponfd to one point in Q/H; denote it by c. Furthe, let e_1 and e_2 be the endpoint of the segment. Observe that there is a homeomorphsim that sends e_1 to e_2; let e be the corresponding point in Q/H. Observe that any two points on the line segments distict from x, e_1 and e_2 can be mapped to eachother by a homeomorphism $Q \to Q$; denote by i the corresponding point on Q/H.

Argue as in 9.16 to show that x, c, e, and i are distinct points in Q/H. List the open subset of $\{x, c, e, i\}$.

9.18; *(a)* Let T_n be a the union of all sides of the 3^n thriangles after n^{th} iteration. Note that the sequence is nested; that is, $T_0 \subset T_1 \subset \dots$ Use induction to show that each T_i is connected. Conclude that the union $T = T_0 \cup T_1 \cup \dots$ is connected. Finally, show that Sierpiński triangle is the closure of T and apply 9.7.

(b). Denote the Sierpiński triangle by $\triangle$.



Let $\{x, y, z\}$ be a 3-point set in $\triangle$ such that $\triangle \setminus \{x, y, z\}$ has 3 connected components. Show and use that there is a unique choice for the set $\{x, y, z\}$ and it is formed by the midpoints of the original triangle.

10.1. Recall that $\{a\}$ is an open set in $\mathcal{X}$. Show and use that $f\colon [0, 1] \to \mathcal{X}$ defined by

$$f(t) = \begin{cases} a & \text{if } t < 1, \\ b & \text{if } t = 1 \end{cases}$$

is a continuous map.

10.8. Show and use that for any rational numbers a and b, the line $y = a{\cdot}x + b$ lies in $A \cup B$.

10.6. Choose path-connected component V. Assume its complement, say W, is nonempty. Observe that W is a union of path-connected components. Therefore, W is open, and the pair V, W forms an open splitting of the space.

To prove the converse, apply 10.2

10.7. Check the conditions in 5.8. To prove compactness, try to modify the proof of 6.13. Hausdorfness follows from the definition. To prove connectedness, modify the proof of 9.8. To show that the space is not path-connected, use that any path contains at most countable subsets of isolated points. Finally observe that every path-connected component in $[0, 1] \times [0, 1]$ is a vertical segment $\{x\} \times [0, 1]$.

10.3. Show and use that for any continuous map φ and any path f, the composition $\varphi \circ f$ is a path.

10.4. Suppose that f and g are paths in $\mathcal{X}$ and $\mathcal{Y}$ respectively. Show and use that $t \mapsto (f(t), g(t))$ is a path in $\mathcal{X} \times \mathcal{Y}$.

10.5. The flea and the comb.

10.10. Mimic the proof of 10.9.

10.11. Read about pseudo-arc and think.

14.2; *(a).* Suppose there is a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^2$. Choose a point $p \in \mathbb{R}^3$ and let $q = h(p)$. Observe that $\mathbb{R}^3 \setminus \{p\}$ is homeomorphic to $\mathbb{R}^2 \setminus \{q\}$. Show that $\pi_1(\mathbb{R}^3 \setminus \{p\})$ is trivial, but $\pi_1(\mathbb{R}^2 \setminus \{q\})$ is not and arrive at a contradiction.

(b). Apply 11.7, 12.11, 13.4, 14.1.

(c). Apply 11.7, 12.11, and 14.1.

(d). Apply 11.7 and 14.1.

14.4; *(a).* Assume there is a homeomorphism $h\colon K_1 \to K_2$. Argue as in 9.16 and 14.2c to show that (1) h maps interior points of K_1 to ineterior points of K_2, (2) h maps boundary points of K_1 to boundary points of K_2, and (3) h maps the attached segents of K_1 to attached segents of K_2. Try to arrive at a contradiction.

(b). Realize that each products $K_1 \times [0,1]$ and $K_2 \times [0,1]$ is a solid torus with attaced pair of squares. To construct a homeomorphism, whist one solid torus.

15.3. Consider a loop γ with the base point $p_0 \in V \cap W$. Show that there is a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ such that the arc $\gamma|_{[t_{i-1},t_i]}$ lies in V or in W for each i. Denote by γ_i the arc $\gamma|_{[t_{i-1},t_i]}$ reparametrized by $[0,1]$.

Show that we can choose a path β_i in $V \cap W$ from p_0 to $p_i = \gamma(t_i)$ for each i. Set

$$\begin{aligned}
\alpha_1 &= \gamma_1 * \bar{\beta}_1,\\
\alpha_2 &= \beta_1 * \gamma_2 * \beta_2,\\
&\ \vdots\\
\alpha_n &= \beta_{n-1} * \gamma_n.
\end{aligned}$$

Observe that $\alpha_1 * \dots * \alpha_n \sim \gamma$. Observe that each α_i lies entirely in V or W. Conclude that each α_i is null homotopic. Finally, observe that $\alpha_1 * \dots * \alpha_n \sim \gamma$, and conclude that γ is null homotopic.

15.4.

15.6. Show that the functions a and b are continuous. Note that $0 < a(x) < b(x) < 2{\cdot}a(x)$ for any $x \ni \Gamma$. Conclude that f is continuos in the complement of Γ. Further, show that for any $x \in \Gamma$ and $\varepsilon > 0$ there is a neighborhood $V \ni x$ such that $|f(y) - f(x)| < \varepsilon$ for any $y \in V$. Conclude that f is continuos on the entire plane.

For the second part, choose an increasing continuous function $\sigma\colon \mathbb{R} \to (1,2)$ such that $\sigma(x) \to 1$ as $x \to -\infty$ and $\sigma(x) \to 2$ as $x \to +\infty$, say $\sigma(x) = 1 + \frac{1}{2}\cdot\frac{e^x - e^{-x}}{e^x + e^{-x}}$. Consider function $\varphi\colon \Gamma \to (0,1)$ defined by $\sigma \circ h^{-1}$, apply the first part of the exercise and apply σ^{-1} to the obtained function.

15.7. Show and use that the complement of z-axis has unit circle $\{\,(x,y,0) \in \mathbb{R}^3 \,:\, x^2 + y^2 = 1\,\}$ as a homotopy retract.

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